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RESEARCH ARTICLE

Conjugate Laplacian eigenvalues of co-neighbour graphs

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Abstract. Let G be a simple graph of order n. A vertex subset is called *independent* if its elements are pairwise non-adjacent. Two vertices in G are *co-neighbour vertices* if they share the same neighbours. Clearly, if S is a set of pairwise co-neighbour vertices of a graph G, then S is an independent set of G. Let $c = a+b\sqrt{m}$ and $\overline{c} = a-b\sqrt{m}$, where a and b are two nonzero integers and m is a positive integer such that m is not a perfect square. In [M. Lepović, On conjugate adjacency matrices of a graph, *Discrete Mathematics*, **307**, 730–738, 2007], the author defined the matrix $A^c(G) = [c_{ij}]_n$ to be the conjugate adjacency matrix of G, if $c_{ij} = c$ for any two adjacent vertices i and j, $c_{ij} = \overline{c}$ for any two nonadjacent vertices i and j, and $c_{ij} = 0$ if $i = j$. In [S. Paul, Conjugate Laplacian matrices of a graph, *Discrete Mathematics, Algorithms and Applications,* **10**, 1850082, 2018], the author defined the conjugate Laplacian matrix of graphs and described various properties of its eigenvalues and eigenspaces. In this article, we determine certain properties of the conjugate Laplacian eigenvalues and the eigenvectors of a graph with co-neighbour vertices.

1. Introduction and Preliminaries

In this article, we consider only finite simple graphs, i.e, graphs on a finite number of vertices without multiple edges or loops. A graph is denoted by $G = (V(G), E(G))$, where $V(G)$ is its vertex set and $E(G)$ is

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its edge set. The *order* of G is the number $n = |V(G)|$ and its *size* is the number $m = |E(G)|$. The set of vertices adjacent to $v \in V(G)$, denoted by $N_G(v)$, refers to the *neighbourhood* of v. For a subset S of $V(G), G[S]$ denotes the induced subgraph on S (i.e., the maximal subgraph of G on S). For other graph theoretic terms we follow [7]. If $V(G) = \{1, 2, \ldots, n\}$, then the *adjacency matrix* of G, is defined to be $A(G) = [a_{ij}]_n$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The *matrix of vertex degrees* of G is the diagonal matrix $D(G)$ of order n, whose i-th diagonal entry is the *degree* of the vertex i, which is defined as the number of lines incident on i. A vertex with degree 1 is called a *pendent* vertex and the vertex which is adjacent to a pendent vertex is known as *quasi-pendent* vertex. The matrix $L(G) = D(G) - A(G)$, is the *Laplacian matrix* of G. In literature, an extensive study has been done on adjacency and Laplacian matrices of graphs (see [2–6, 8, 13, 15]).

Let $c = a + b\sqrt{m}$, where a and b are two nonzero integers and m is a positive integer such that m is not a perfect square. The number $\overline{c} = a - b\sqrt{m}$ is called the *conjugate number* of c. In [9], Lepović gives the following definition.

Definition 1.1. [9] *If G is a graph of order n, then the matrix $A^c(G) = [c_{ij}]_n$ is called the conjugate adjacency matrix of G, where*

$$c_{ij} = \begin{cases} c, & \textit{if } i \textit{ and } j \textit{ are adjacent,} \\ \overline{c}, & \textit{if } i \textit{ and } j \textit{ are nonadjacent,} \\ 0, & \textit{if } i{=}j. \end{cases}$$

An extensive study on the eigenvalues of $A^c(G)$ has been done by the author in [9–12]. With this inspiration, in [14], we defined the conjugate Laplacian matrix of a graph as follows. Let $d_i^c = \sum_{ij \in E(G)} c + \sum_{ij \notin E(G)} \overline{c}$ to be the *conjugate degree* of a vertex i in a graph G, and let $D^c(G)$ be the diagonal matrix with d_i^c as the i-th diagonal entry. If both c and $\overline{c}$ are positive, then the matrix $L^c(G) = D^c(G) - A^c(G)$ is called the *conjugate Laplacian matrix* of G. Thus with every conjugate Laplacian matrix, there is an associated real constant c such that both c and $\overline{c}$ are positive. Hence for the rest of the paper, whenever we discuss any result involving conjugate Laplacian matrix, it is understood that the respective constant and its conjugate are both positive.

Let $\sigma^c(G)$ denotes the conjugate Laplacian spectrum of G. From now onwards by 'eigenvalues' and 'eigenvectors' we mean conjugate Laplacian eigenvalues and conjugate Laplacian eigenvectors (if not mentioned otherwise). As already observed in [14], since the row sums and the column sums of $L^c(G)$ are zero, $0 \in \sigma^c(G)$ with $\mathbf{1}_n = (\underbrace{1, 1, \ldots, 1}_{n})^T$ as a corresponding eigenvector. Moreover, it was also shown in [14] that 0 is a simple eigenvalue of $L^c(G)$.

Since $L^c(G) = Q^c(G).Q^c(G)^T$, it is a singular positive semidefinite matrix, where $Q^c(G)$ is defined as follows.

Definition 1.2. [14] *Let $\overline{G}$ denotes the complement of G and (i, j) denotes an oriented edge that originates at i and terminates at j. Suppose each edge of G and $\overline{G}$ is assigned an orientation, which is arbitrary but fixed. The vertex edge incidence matrix $Q^c(G) = [q^c_{i,e}]$ is a $n \times \frac{n(n-1)}{2}$ matrix with rows labeled by the vertices of G and columns labeled by the edges in G or $\overline{G}$, satisfying*

$$q^c_{i,e} = \begin{cases} \sqrt{c}, & \textit{if } e = (i, j) \in E(G), \textit{ for some } j; \\ -\sqrt{c}, & \textit{if } e = (j, i) \in E(G), \textit{ for some } j; \\ \sqrt{\overline{c}}, & \textit{if } e = (i, j) \in E(\overline{G}), \textit{ for some } j; \\ -\sqrt{\overline{c}}, & \textit{if } e = (j, i) \in E(\overline{G}), \textit{ for some } j. \end{cases}$$

The following important observation is obtained in [14].

Lemma 1.1. [14] *Let G be a graph and p, q be two non adjacent vertices in it. If $c = a + b\sqrt{m}$, and $b > 0$, then*

$$\rho^c(G + pq) \geqslant \rho^c(G).$$

Remark 1.1. Observe that $L^c(K_n) = cL(K_n)$. Hence by Lemma 1.1 we have $\rho^c(G) \leqslant cn$, when $b > 0$.

Let $0 = \alpha^c_1(G) < \alpha^c_2(G) \leqslant \ldots \leqslant \alpha^c_n(G) = \rho^c(G)$ denotes the eigenvalues of $L^c(G)$. The second smallest eigenvalue of $L(G)$ is called the *algebraic connectivity* and has been studied extensively in literature (see [2, 8, 13]). The following lower bound of the second smallest eigenvalue of $L^c(G)$ has been obtained in [14]

Lemma 1.2. [14] *Let G be a graph and $c = a + b\sqrt{m}$. If $b > 0$, then $\alpha^c_2(G) \geqslant n\overline{c}$ and $\alpha^c_2(\overline{G}) \geqslant n\overline{c}$.*

A vertex subset is called *independent* if its elements are pairwise non-adjacent. Two vertices in $V(G)$ are *co-neighbour vertices* if they share the same neighbours. Clearly, if $S \subset V(G)$ is a set of pairwise co-neighbour vertices of a graph G, then S is an independent set of G. A *cluster* of order k of G is a set S of k pairwise co-neighbour vertices [13]. The *degree of a cluster* is the cardinality of the shared set of neighbours, i.e., the common degree of each vertex in the cluster. An l-cluster is a cluster of degree l. In Section 2, we discuss the conjugate Laplacian spectrum of a graph with k pairwise co-neighbour vertices.

2. Conjugate Laplacian spectrum of a graph with co-neighbour vertices

Here we discuss the conjugate Laplacian spectrum of a graph with co-neighbour vertices. A vector $X \in \mathbb{R}^n$ is called *Faria vector* [13], if there are only two nonzero entries in X which are 1 and -1, respectively. We first observe the following result for graphs with pendent vertices.

Theorem 2.1. *Let p and q, respectively be the number of pendent vertices and the number of quasi-pendent vertices of a graph G. Then $na-(n-2)b\sqrt{m}$ is an eigenvalue of G with multiplicity at least $p-q$.*

Proof. Let $1, 2, \ldots, q$ be the quasi-pendent vertices of G and suppose they are adjacent to $r_1, r_2, \ldots, r_q$ pendent vertices, respectively. Suppose i and j are pendent vertices of G, adjacent to a common quasi-pendent vertex. Let X be a Faria vector with $X(i) = 1 = -X(j)$. Then it can be easily verified that X is an eigenvector of $L^c(G)$, corresponding to the eigenvalue $na-(n-2)b\sqrt{m}$. In this way, we can generate $(r_1-1)+(r_2-1)+\ldots+(r_q-1)$ linearly independent eigenvectors of $L^c(G)$ corresponding to the pendent vertices for the eigenvalue $na-(n-2)b\sqrt{m}$. Hence the multiplicity of $na-(n-2)b\sqrt{m}$ as an eigenvalue of $L^c(G)$ is at least $\sum_{i=1}^{q}(r_i-1) = \sum_{i=1}^{q} r_i - q = p - q$. $\square$

Let k be an integer greater than 1. Then the following result generalizes Theorem 2.1, for graphs with an l-cluster of order k.

Theorem 2.2. *Let G be a graph with an l-cluster S of order k. Then $na-(n-2l)b\sqrt{m}$ is an eigenvalue of G with multiplicity at least $k-1$.*

Proof. Let $S = \{v_1, v_2, \ldots, v_k\}$ be the l-cluster. Assuming $N_G(v_1) = \{v_{k+1}, v_{k+2}, \ldots, v_{k+l}\}$, we have

$$L^c(G) = \left(\begin{array}{c|cc} lcI_k + (n-k-l)\bar{c}I_k + \bar{c}L(K_k) & -c\mathbf{1}_k\mathbf{1}_l^T & -\bar{c}\mathbf{1}_k\mathbf{1}_{n-k-l}^T \\ \hline -c\mathbf{1}_l\mathbf{1}_k^T & & \\ -\bar{c}\mathbf{1}_{n-k-l}\mathbf{1}_k^T & & \mathfrak{B} \end{array}\right), \tag{1}$$

where $\mathfrak{B} = \left(\begin{array}{c|c} kcI_l & \mathbf{0}_{l\times(n-k-l)} \\ \hline \mathbf{0}_{(n-k-l)\times l} & \mathbf{0}_{(n-k-l)\times(n-k-l)} \end{array}\right) + L^c(G-S)$.

It can be verified that the first k rows of the matrix $L^c(G) - \big(na - (n-2l)b\sqrt{m}\big)I$ are equal. Therefore, $\mathrm{rank}\big(L^c(G) - (na - (n-2l)b\sqrt{m})I\big) \leqslant n-(k-1)$. Hence, the null space of $L^c(G) - (na - (n-2l)b\sqrt{m})I$ has dimension not less than $k-1$. Hence $(na - (n-2l)b\sqrt{m}$ is an eigenvalue of $L^c(G)$ with multiplicity at least $k-1$. □

Remark 2.1. Let $X \in \mathbb{R}^k \setminus \{\mathbf{0}_k\}$ with $X^T\mathbf{1}_k = 0$. Then taking into account the labeling of the vertices in Theorem 2.2 and from (1) we have,

$$\begin{aligned} L^c(G)\left(\frac{X}{\mathbf{0}_{n-k}}\right) &= \left(\frac{\Big(lcI_k + (n-k-l)\bar{c}I_k + \bar{c}(kI_k - J_k)\Big)X}{\mathbf{0}_{n-k}}\right) \\ &= \left(\frac{\Big(na - (n-2l)b\sqrt{m}\Big)X}{\mathbf{0}_{n-k}}\right) \\ &= \Big(na - (n-2l)b\sqrt{m}\Big)\left(\frac{X}{\mathbf{0}_{n-k}}\right), \end{aligned} \tag{2}$$

which lists $k-1$ linearly independent eigenvectors of $L^c(G)$ corresponding to $na - (n-2l)b\sqrt{m}$ as desired by Theorem 2.2.

As an immediate application of Theorem 2.2, we consider a complete bipartite graph $K_{r,s}$. Since every pair of vertices in a partition are co-neighbours, we have $s\bar{c} + rc$ is an eigenvalue with multiplicity at least $s-1$ and $r\bar{c} + sc$ is an eigenvalue with multiplicity at least $r-1$. Therefore, taking into account that

1) the trace of $L^c(K_{r,s})$ is $2rsc + (r^2 - r)\bar{c} + (s^2 - s)\bar{c}$,
2) 0 is a simple eigenvalue of $L^c(K_{r,s})$,

the unknown eigenvalue is $(r+s)c$. Thus

$$\sigma^c(K_{r,s}) = \Big\{0, \underbrace{s\bar{c} + rc, \ldots, s\bar{c} + rc}_{s-1}, \underbrace{r\bar{c} + sc, \ldots, r\bar{c} + sc}_{r-1}, (r+s)c\Big\}.$$

Suppose G be a graph and $S \subset V(G)$ be a cluster of order k. Let G^k be the supergraph obtained from G by adding p edges between distinct pairs of vertices in S, where $1 \leqslant p \leqslant \frac{k(k-1)}{2}$. In [1], this operation is denoted by $G^k = G + G_k$, where G_k is the subgraph of G^k induced by S, i.e., $G_k = G^k[S]$. We note that $V(G^k) = V(G)$ and $E(G^k) = E(G) \cup E(G_k)$. In Fig. 1, a graph G with a 4-cluster of order 3, $S = \{1, 2, 3\}$, and a graph $G^3 = G + G_3$, with $G_3 = G^3[S]$ are depicted.

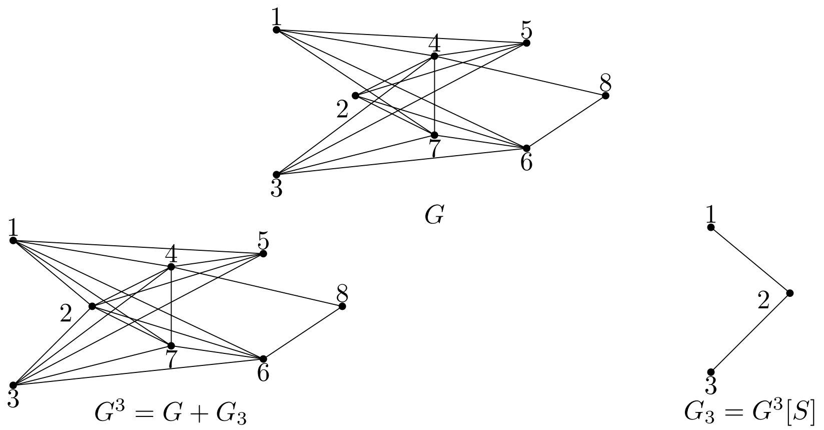

FIGURE 1. A graph G with a 4-cluster of order 3, $S = \{1, 2, 3\}$ and the graphs $G^3 = G + G_3$, and $G_3 = G^3[S]$.

As observed in [1], despite the labeling of S, the operation produces isomorphic graphs. In the following part of this section, we show that at least $n - k + 1$ eigenvalues of $L^c(G)$ are also eigenvalues of $L^c(G^k)$ and that $\sigma^c(G^k)$ is completely characterized from $\sigma^c(G)$ and $\sigma^c(G_k)$. The following lemma will be useful in doing so.

Lemma 2.1. *If*

$$M = \left(\begin{array}{c|c} L^c(G_k) - \bar{c}L(K_k) & \boldsymbol{0}_{k\times(n-k)} \\ \hline \boldsymbol{0}_{(n-k)\times k} & \boldsymbol{0}_{(n-k)\times(n-k)} \end{array}\right), \tag{3}$$

then $L^c(G^k) = L^c(G) + M$ *and* $L^c(G)M = ML^c(G)$.

Proof. The proof follows immediately from the structure of $L^c(G)$ and M, and taking into account that $L^c(G_k)L(K_k) = L(K_k)L^c(G_k)$. □

Corollary 2.1. *Let G be a graph and $S = \{v_1, v_2 \ldots, v_k\} \subset V(G)$ be an l-cluster of order k, sharing the l neighbours $\{v_{k+1}, v_{k+2}, \ldots, v_{k+l}\}$. If G_k is a graph with $V(G_k) = S$ and M be the matrix defined in (3), then reordering the n eigenvalues β_i in $\sigma(M)$, we obtain*

$$\sigma^c(G^k) = \{\alpha_i + \beta_i : i = 1, 2, \ldots, n\}, \tag{4}$$

where $\alpha_1, \ldots, \alpha_n \in \sigma^c(G)$. Moreover,

$$L^c(G^k)L^c(G) = L^c(G)L^c(G^k). \tag{5}$$

We are now able to discuss the dependence of $\sigma^c(G^k)$ on $\sigma^c(G)$ and $\sigma^c(G_k)$.

Theorem 2.3. *Let G be a graph with an l-cluster S of order k. Assume that G_k is a graph with $V(G_k) = S$, $G^k = G + G_k$ and $\Lambda = \Big\{\Big(na - (n-2l)b\sqrt{m}\Big) + \mu - \overline{c}k : \ \mu \in \sigma^c(G_k) \setminus \{0\}\Big\}$ is a multiset. Then $\sigma^c(G^k)$ overlaps $\sigma^c(G)$ in $n-k+1$ places and the elements of Λ are the remaining eigenvalues in $\sigma^c(G^k)$.*

Proof. Let $S = \{v_1, v_2 \dots, v_k\} \subset V(G)$ be the cluster, sharing the l neighbours $\{v_{k+1}, v_{k+2}, \dots, v_{k+l}\}$. Let M be the matrix in (3). Since $L^c(G_k)$ commutes with $L(K_k)$, so $0 \in \sigma(M)$ with multiplicity $n-k+1$, and using (4), it is immediate that $n-k+1$ eigenvalues of $L^c(G)$ overlap the eigenvalues of $L^c(G^k)$. We just need to prove that the elements of Λ are the remaining eigenvalues in $L^c(G^k)$.

By (5), the matrices $L^c(G)$ and $L^c(G^k)$ are simultaneously diagonalizable. Let U be one of the n chosen common eigenvectors of $L^c(G^k)$ and $L^c(G)$ corresponding to eigenvalues λ and λ', respectively. Partitioning U conformally with respect to $L^c(G)$ (or $L^c(G^k)$), we can assume that $U = \left(\begin{array}{c} X \\ \hline Y \\ \hline Z \end{array}\right)$. Now from $L^c(G^k)U = \lambda U$, $L^c(G)U = \lambda' U$ and $L^c(G^k) = L^c(G) + M$, we obtain the following system of equations:

$$lcX + (n-k-l)\overline{c}X + L^c(G_k)X - c(\mathbf{1}_l^T Y)\mathbf{1}_k - \overline{c}(\mathbf{1}_{n-k-l}^T Z)\mathbf{1}_k = \lambda X, \quad (6)$$

$$lcX + (n-k-l)\overline{c}X + \overline{c}L(K_k)X - c(\mathbf{1}_l^T Y)\mathbf{1}_k - \overline{c}(\mathbf{1}_{n-k-l}^T Z)\mathbf{1}_k = \lambda' X, \quad (7)$$

$$\left(\begin{array}{c} -c(\mathbf{1}_k^T X)\mathbf{1}_l + kcY \\ \hline -\overline{c}(\mathbf{1}_k^T X)\mathbf{1}_{n-k-l} \end{array}\right) + L^c(G-S)\left(\begin{array}{c} Y \\ \hline Z \end{array}\right) = \lambda \left(\begin{array}{c} Y \\ \hline Z \end{array}\right), \quad (8)$$

$$\left(\begin{array}{c} -c(\mathbf{1}_k^T X)\mathbf{1}_l + kcY \\ \hline -\overline{c}(\mathbf{1}_k^T X)\mathbf{1}_{n-k-l} \end{array}\right) + L^c(G-S)\left(\begin{array}{c} Y \\ \hline Z \end{array}\right) = \lambda' \left(\begin{array}{c} Y \\ \hline Z \end{array}\right). \quad (9)$$

Subtracting (7) from (6), we obtain

$$\Big(L^c(G_k) - \overline{c}L(K_k)\Big)X = (\lambda - \lambda')X. \quad (10)$$

If $\lambda = \lambda'$, then $\lambda \in \sigma^c(G^k) \cap \sigma^c(G)$ and (10) yields $\Big(L^c(G_k) - \overline{c}L(K_k)\Big)X = \mathbf{0}$. Since $L^c(G_k)$ commutes with $L(K_k)$, so the multiplicity of 0 in $\sigma\Big(L^c(G_k) - \overline{c}L(K_k)\Big)$ is 1. This implies that $X = \gamma\mathbf{1}_k$, where γ is a nonzero scalar.

If $\lambda \neq \lambda'$, then (10) is equivalent to $\Big(L^c(G_k) - \overline{c}L(K_k)\Big)X = \beta X$, with $\beta = \lambda - \lambda' \neq 0$. This implies that $\mathbf{1}_k^T X = 0$. Moreover, (8) and (9) yield $\left(\frac{Y}{Z}\right) = \left(\frac{\mathbf{0}}{\mathbf{0}}\right)$. Thus $X \neq 0$, is an eigenvector of $L^c(G_k) - \overline{c}L(K_k)$ orthogonal to $\mathbf{1}_k$, and $U = \left(\frac{X}{\mathbf{0}}\right)$, where X is an eigenvector of $L^c(G_k) - \overline{c}L(K_k)$ corresponding to eigenvalue $\beta \neq 0$. From (7), it can be seen that $\lambda' = na - (n-2l)b\sqrt{m}$ and so $\lambda = (na - (n-2l)b\sqrt{m}) + \beta$.

From the above analysis, we may conclude that $L^c(G)$ and $L^c(G^k)$ share two types of eigenvectors. The eigenvectors $\begin{pmatrix} \gamma\mathbf{1}_k \\ \hline Y \\ \hline Z \end{pmatrix}$ correspond to the eigenvalues $\lambda \in \sigma^c(G^k) \cap \sigma^c(G)$ and the eigenvectors $\left(\frac{X}{\mathbf{0}}\right)$ correspond to the eigenvalues $(na - (n-2l)b\sqrt{m}) + \beta$ of $L^c(G^k)$, where $(na - (n-2l)b\sqrt{m}) \in \sigma^c(G)$ and $\beta \in \sigma(L^c(G_k) - \overline{c}L(K_k)) \setminus \{0\}$. Thus $\beta = \mu - k\overline{c}$, where $\mu \in \sigma^c(G_k) \setminus \{0\}$. Since $L^c(G_k)$ has $k-1$ non zero eigenvalues μ_i, for $i = 1, 2, \ldots, k-1$, we have $k-1$ shared eigenvectors $\left(\frac{X_i}{\mathbf{0}}\right)$, for $i = 1, 2, \ldots, k-1$. Therefore, by (4) of Corollary 2.1, we have $n-k+1$ shared eigenvectors $\begin{pmatrix} \gamma\mathbf{1}_k \\ \hline Y \\ \hline Z \end{pmatrix}$ which are orthogonal to $\left(\frac{X_i}{\mathbf{0}}\right)$, for $i = 1, 2, \ldots, k-1$. Since both $L^c(G)$ and $L^c(G^k)$ are of order n, so the eigenvalues in $\sigma^c(G^k)$ which can be different from the eigenvalues in $\sigma^c(G)$ are just the $k-1$ elements of Λ. $\square$

Remark 2.2. Let G be a graph with an l-cluster S of order k. Consider two graphs G_k and G'_k defined on S. From the proof of Theorem 2.3, we conclude that $\sigma^c(G^k)$ and $\sigma^c(G'^k)$ overlap in $n-k+1$ places, where $G^k = G + G_k$ and $G'^k = G + G'_k$. Furthermore, the remaining eigenvalues of G^k and G'^k, $(na - (n-2l)b\sqrt{m}) + \mu - \overline{c}k$ (with $\mu \in \sigma^c(G_k) - \{0\}$) and $(na - (n-2l)b\sqrt{m}) + \mu' - \overline{c}k$ (with $\mu' \in \sigma^c(G'_k) - \{0\}$), respectively, replace $k-1$ of the positions of the eigenvalue $(na - (n-2l)b\sqrt{m})$ of G (see Remark 2.1).

If we consider the graph $G = K_{r,s}$ on the vertices $\{v_1, v_2, \ldots, v_{r+s}\}$ and a graph G_r, such that $V(G_r) = S$, where $S = \{v_1, v_2, \ldots, v_r\}$ (an s-cluster of order r in G), then we have the following result.

Theorem 2.4. *If $c = a + b\sqrt{m}$, where $b > 0$, $r \leqslant s$, $G = K_{r,s}$ and G_r is a graph defined on the vertex subset of r pairwise co-neighbours of G, then the graphs G and $G^r = G + G_r$ have the same largest eigenvalue $(r+s)c$ and the same second smallest eigenvalue $s\overline{c} + rc$.*

Proof. Since the degree of the cluster is s, we have

$$na - (n-2s)b\sqrt{m} = (r+s)a - (r-s)b\sqrt{m} = r\overline{c} + sc.$$

By Theorem 2.3, $\sigma^c(G^r) = \Lambda \cup \Big(\sigma^c(G) \setminus \{\underbrace{r\overline{c}+sc, \ldots, r\overline{c}+sc}_{r-1}\}\Big)$, where $\Lambda = \{sc + \mu : \ \mu \in \sigma^c(G_r) \setminus \{0\}\}$, with $r\overline{c} \leqslant \mu \leqslant rc$ (see Remark 1.1 and Lemma 1.2). Since

$$\sigma^c(G) = \Big\{0, \underbrace{s\overline{c}+rc, \ldots, s\overline{c}+rc}_{s-1}, \underbrace{r\overline{c}+sc, \ldots, r\overline{c}+sc}_{r-1}, (r+s)c\Big\},$$

$\sigma^c(G^r) = \Lambda \cup \Big\{0, \underbrace{s\overline{c}+rc, \ldots, s\overline{c}+rc}_{s-1}, (r+s)c\Big\}$. Therefore, the largest eigenvalue of G and G^r is $(r+s)c$. Again, since $s\overline{c}+rc \leqslant r\overline{c}+sc$, the second smallest eigenvalue of G and G^r is $s\overline{c}+rc$. □

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