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# Coarse structures on groups defined by conjugations

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Abstract. For a group $G$, we denote by $\overset{\leftrightarrow}{G}$ the coarse space on $G$ endowed with the coarse structure with the base $\{\{(x,y)\in G\times G : y\in x^F\} : F\in[G]^{<\omega}\}$, $x^F=\{z^{-1}xz : z\in F\}$. Our goal is to explore interplays between algebraic properties of $G$ and asymptotic properties of $\overset{\leftrightarrow}{G}$. In particular, we show that *asdim* $\overset{\leftrightarrow}{G}=0$ if and only if $G/Z_G$ is locally finite, $Z_G$ is the center of $G$. For an infinite group $G$, the coarse space of subgroups of $G$ is discrete if and only if $G$ is a Dedekind group.

## 1. Introduction

Given a set $X$, a family $\mathcal{E}$ of subsets of $X\times X$ is called a *coarse structure* on $X$ if

- each $E\in\mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X=\{(x,x)\in X : x\in X\}$;
- if $E$, $E'\in\mathcal{E}$ then $E\circ E'\in\mathcal{E}$ and $E^{-1}\in\mathcal{E}$, where $E\circ E'=\{(x,y) : \exists z((x,z)\in E,\ \ (z,y)\in E')\}$, $E^{-1}=\{(y,x) : (x,y)\in E\}$;
- if $E\in\mathcal{E}$ and $\triangle_X\subseteq E'\subseteq E$ then $E'\in\mathcal{E}$;

A subfamily $\mathcal{E}'\subseteq\mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $E\in\mathcal{E}$, there exists $E'\in\mathcal{E}'$ such that $E\subseteq E'$. For $x\in X$, $A\subseteq X$ and $E\in\mathcal{E}$, we denote

$$E[x]=\{y\in X : (x,y)\in E\},\ \ E[A]=\bigcup_{a\in A}E[a],\quad E_A[x]=E[x]\cap A$$

and say that $E[x]$ and $E[A]$ are *balls of radius $E$ around* $x$ and $A$.

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The pair $(X,\mathcal{E})$ is called a *coarse space* [13] or a *ballean* [10], [12].

A coarse space $(X,\mathcal{E})$ is called *finitary,* if for each $E\in\mathcal{E}$, there exists a natural number $n$ such that $|E[x]|<n$ for each $x\in X$.

Let $G$ be a group of permutations of a set $X$. We denote by $X_G$ the set $X$ endowed with the coarse structure with the base

$$\{\{(x,gx): g\in F\}: F\in[G]^{<\omega},\ id\in F\}.$$

By [7, Theorem 1], for every finitary coarse structure $(X,\mathcal{E})$, there exists a group $G$ of permutations of $X$ such that $(X,\mathcal{E})=X_G$. For more general results and applications see [8] and the survey [9].

Let $(X,\mathcal{E})$ be a coarse space. We define an equivalence $\sim$ on $X$ by $x\sim y$ if and only if there exists $E\in\mathcal{E}$ such that $y\in E[x]$, so $X$ is a disjoint union of connected components. If there is only one connected component then $(X,\mathcal{E})$ is called connected.

Now let $G$ be a group. For $x,g\in G$ and $F\subseteq G$, we denote $x^g=g^{-1}xg$, $x^F=\{x^y: y\in F\}$, $F^g=\{y^g: y\in F\}$.

We denote by $\overset{\leftrightarrow}{G}$ the coarse structure on $G$ endowed with the coarse structure with the base $\{\{(x,y)\in G\times G: y\in x^F\}: F\in[G]^{<\omega}\}$. Evidently, each connected component $A$ of $\overset{\leftrightarrow}{G}$ is of the form $a^G$, $a\in A$.

We endow $G$ with the discrete topology and identify the Stone-Čech compactification $\beta G$ of $G$ with the set of all ultrafilters on $G$. For $A\subseteq G$, $\bar{A}$ denotes the set $\{p\in\beta G: A\in p\}$ and the family $\{\bar{A}: A\subseteq G\}$ forms a base for open sets of $\beta G$. The family of all free ultrafilters on $G$ is denoted by $G^*$. By the universal property of $\beta G$, every mapping $f: G\to K$, $K$ is a compact Hausdorff space, can be extended to the continuous mapping $f^\beta:\beta G\to K$.

The action $G$ on $G$ by conjugations extends to the action $G$ on $\beta G$: if $g\in G$, $p\in\beta G$ then $p^g=\{g^{-1}Pg: P\in g\}$. We use this dynamical approach to the conjugacy in groups initiated in [11].

In section 2 and 3, we characterize groups $G$ such that the coarse space $\overset{\leftrightarrow}{G}$ is discrete, $n$-discrete and cellular. In section 4, we show that every finitary coarse space admits an asymorphic embedding to $\overset{\leftrightarrow}{G}$ for an appropriate choice of a group $G$. In section 5, we characterize groups with discrete space of subgroups. We conclude with section 6 on the direct union of connected components of $\overset{\leftrightarrow}{G}$.

## 2. Discreteness

Let $(X, \mathcal{E})$ be a coarse space. We say that a subset $B$ of $X$ is *bounded* if there exist a finite subset $F$ of $X$ and $E \in \mathcal{E}$ such that $B \subseteq E[F]$ and note that the family of all bounded subset of $X$ is a bornology, i.e. an ideal in the Boolean algebra of subsets of $X$ containing all finite subsets.

We say that a subset $A$ of $X$ is

- *discrete* if, for every $E \in \mathcal{E}$, there exists a bounded subset $B$ of $X$ such that $E_A[a] = \{a\}$ for each $a \in A \setminus B$;
- *$n$-discrete*, $n \in \mathbb{N}$ if, for every $E \in \mathcal{E}$, there exists a bounded subset $B$ of $X$ such that $|E_A[a]| \leqslant n$ for each $a \in A \setminus B$.

**Theorem 1.** *For an infinite group $G$, the following conditions are equivalent*

(i) *$G$ is Abelian;*

(ii) *$p^G = \{p\}$ for each $p \in G^*$;*

(iii) *$\overset{\leftrightarrow}{G}$ is discrete.*

*Proof.* The equivalence (i) $\Leftrightarrow$ (ii) is proved in [11, Proposition 1.1], (i) $\Rightarrow$ (iii) is evident.

(iii) $\Rightarrow$ (ii). We assume that $p^x \neq p$ for some $p \in G^*$, $x \in G$ and pick $P \in p$ such that $P^x \cap P = \varnothing$. Let $B$ be a finite subset of $X$. We take $a \in P \setminus B$ and note that $a^x \neq a$ so $\overset{\leftrightarrow}{G}$ is not discrete. □

**Theorem 2.** *For a group $G$, the following conditions are equivalent*

(i) *$p^G$ is finite for each $p \in G^*$;*

(ii) *there exists a natural number $n$ such that $|p^G| \leqslant n$ for each $p \in G^*$;*

(iii) *there exists a natural number $m$ such that $|a^G| \leqslant m$ for each $a \in G^*$;*

(iv) *the commutant $[G, G]$ of $G$ is finite.*

*Proof.* See Theorem 3.1 in [11]. □

**Theorem 3.** *Given a group $G$, the coarse space $\overset{\leftrightarrow}{G}$ is $n$-discrete for some $n \in \mathbb{N}$ if and only if $[G, G]$ is finite.*

*Proof.* We assume that $\overset{\leftrightarrow}{G}$ is $n$-discrete and show that $[G, G]$ is finite. To apply Theorem 2, it suffices to prove that $|p^G| \leqslant n$ for each $p \in G^*$.

We assume the contrary: there exists $p \in G^*$ and $g_1, \dots, g_{n+1} \in G$ such that the ultrafilters $p^{g_1}, \dots, p^{g_{n+1}}$ are distinct. We choose $P \in p$ such that the subsets $P^{g_1}, \dots, P^{g_{n+1}}$ are pairwise disjoint. Given an arbitrary bounded subset $B$ of $G$, we pick $a \in P \setminus B$. Then $a^{g_1}, \dots, a^{g_{n+1}}$ are distinct so $\overset{\leftrightarrow}{G}$ is not $n$-discrete.

On the other hand, if $[G, G]$ is finite then there exists $m \in \mathbb{N}$ such that $|a^G| \leqslant m$ for each $a \in G$, see Theorem 2(iii). □

We recall that $G$ is an *$FC$-group* if the set $a^G$ is finite for each $a \in G$. Clearly, $G$ is an $FC$-group if and only if each connected component of $\overset{\leftrightarrow}{G}$ is bounded.

We note that each connected component of $\overset{\leftrightarrow}{G}$ is discrete if and only if every element $g \in G$ centralizes all but finitely many elements of each conjugacy class.

In the initial version of this paper, we asked whether $G$ is an $FC$-group provided that each connected component of $\overset{\leftrightarrow}{G}$ is discrete? G. Bergman answered this question negatively.

**Theorem 4.** *There exists a group $G$ such that every element of $G$ centralizes all but finitely many element of each conjugacy class and $g^G$ is infinite for each nonindentily element $g \in G$.*

*Proof.* We follow the original Bergman's exposition.

**Claim 1.** Suppose $X$ is a metric space such that, for every $x \in X$ and constant $C > 0$, the number of elements of $X$ within distance $\leqslant C$ of $x$ is finite. Suppose also that $X$ has a group $G$ of distance-preserving permutations each of which moves only finitely many elements. Then every $g \in G$ centralizes all but finitely many elements of each conjugacy class $h^G$.

Given $g, h \in G \setminus \{e\}$, let us choose $C > 0$ such that the finite subset of $X$ consisting of the elements moved by $g$ and the elements moved by $h$ has all elements within distance $\leqslant C$ each other. Since elements of $G$ are distance-preserving, for every conjugate $h^f$, $f \in G$, the elements moved by $h^f$ are also within distance $\leqslant C$ of each other. Hence, if any of the elements moved by $h^f$ has distance $> 2C$ from each element moved by $g$, then the set of elements moved by $h^f$ must be disjoint from the set moved by $g$, so $h^f$ and $g$ commute. So, if $h^f$ and $g$ do not commute, the elements moved by $h^f$ must lie within distance $\leqslant 2C$ of an arbitrary chosen element $x$ moved by $g$. But the number of elements lying within that distance of $x$ if finite, so there are only finitely many posibilities for the permutation $h^f$.

**Claim 2.** For $X$ and $G$ as in Claim 1, if $X$ is infinite and $G$ is transitive on $X$, then every nonidentify element $g \in G$ has infinite conjugacy class $g^G$.

Given finitely many conjugates $g_1, \dots, g_n$ of $g$, we shall find another. Let $Y$ by the finite subset of $X$ consisting of all elements moved by any $g_1, \dots, g_n$, and again choose $C > 0$ such that the distances between the element of $Y$ are all $\leqslant C$. Since $X$ is infinite, the hypothesis of Claim 1 imply that distances among points of $X$ are unbounded, so as $G$ is transitive on $X$, we can find $h \in G$ carries a point moved by $g$ to a point at distance $> 2C$ from point of $Y$. Hence, the set of point moved by $g^h$, namely, the translate by $h$ of the set moved by $g$, is not contained in $Y$, so $g^h \notin \{g_1, \dots, g_n\}$. So, the conjugacy class of $g$ is indeed infinite.

It remains to give an example of $X$ and $G$ with above properties.

Let $X$ be the set of all sequences $(a_1, a_2, \dots)$ of 0's and 1's such that almost all the $a_i$ are 0. Metrize $X$ by letting $d((a_1, a_2, \dots), (b_1, b_2, \dots))$ be the greatest $n$ such that $a_n \neq b_n$, or 0 if $(a_1, a_2, \dots) = (b_1, b_2, \dots)$. That there are only finitely many elements distances $C$ of any element of $X$ is clear.

Let $G$ be the group of all distance-preserving permutations of $X$ which move only finitely many elements. We shall show that $G$ is transitive by constructing, for any $(a_1, a_2, \dots) \in X$ an element $g \in G$ which carries $(0, 0, \dots)$ to $(a_1, a_2, \dots)$. Choose $n$ such that $a_i = 0$ for all $i > n$. Let $g$ carries each element $(b_1, b_2, \dots)$ which likewise has $b_i = 0$ for all $i > n$ to $(b_1 + a_1, b_2 + a_2, \dots)$, while fixing all other elements $(b_1, b_2, \dots)$. The verification of $g \in G$, and that $g$ carries $(0, 0, \dots)$ to $(a_1, a_2, \dots)$ are straightforward. $\square$

G. Bergman noticed that the group $G$ constructed in the proof of Theorem 4 can be described as the direct limit $G_0 \longrightarrow G_1 \longrightarrow \cdots \longrightarrow G_n \longrightarrow \dots$, where $G_0$ is trivial and $G_{n+1} = (G_n \times G_n) \leftthreetimes \mathbb{Z}_2$, with $\mathbb{Z}_2$ acting on $G_n \times G_n$ by interchanging the two coordinates, and with $G_n$ embedded in $G_{n+1}$ by sending $g$ to $((g, e), e)$.

We show that the answer to our question is affirmative provided that $G$ is finitely generated. Let $F$ be a finite subset of $G$ such that $F = F^{-1}$, $e \in F$, $e$ is the identity of $G$ and $F$ generates $G$. We assume that each connected component of $\overset{\leftrightarrow}{G}$ is discrete, take an arbitrary element $g \in G$ and show that $g^G$ is finite. We act on $g$ by conjugations from $x \in F$, write each $g^x$ as a word in $F$ of minimal length, delete duplicates (i.e. words which define the same elements) and get a subset $A_0$. Then we repeat this procedure for each element $g \in A_0$ and get a subset $A_1$, $A_0 \subseteq A_1$. Since

$F$ is finite, by the assumption there exists $n \in \mathbb{N}$ such that $A_{n+1} = A_n$. This means that $g^G = A_n$.

## 3. Cellularity

A coarse space $(X, \mathcal{E})$ is called *cellular* if $\mathcal{E}$ has a base consisting of equivalence relations. By [12, Theorem 3.1.3], $(X, \mathcal{E})$ is cellular if and only if *asdim* $(X, \mathcal{E}) = 0$.

Applying Theorem 3.1.2 from [12] we get

(1) $\overset{\leftrightarrow}{G}$ is cellular if and only if, for every finitely generated subgroup $H$ of $G$, there exists a finite subset $F$ of $G$ such that $g^H \subseteq g^F$ for each $g \in G$.

We recall that a group $G$ is *locally normal* if each finite subset of $G$ is contained in some finite normal subgroup and use the following characterization [2]

(2) $G$ is an FC-group if and only if $G/Z_G$ is locally normal and each element of $G$ is contained in finitely generated normal subgroup, $Z_G$ is the center of $G$.

A group $G$ is called *locally finite* if each finite subset of $G$ generates a finite subgroup.

**Theorem 5.** *For a group $G$, $\overset{\leftrightarrow}{G}$ is cellular if and only if $G/Z_G$ is locally finite.*

*Proof.* We suppose that $\overset{\leftrightarrow}{G}$ is cellular and show

(3) for every element $a \in G$ of infinite order there exists $n \in \mathbb{N}$ such that $a^n \in Z_G$.

We denote by $A$ the subgroup of $G$ generated by $a$ and use (1) to choose a finite subset $F$ of $G$ such that $g^A \subseteq g^F$ for each $g \in G$. Let $|F| = n$. Since $|g^A| \leqslant n$, $a^k g = g a^k$ for some $k \leqslant m$. We put $n = m!$.

By (1), every finitely generated subgroup $H$ of $G$ is an FC-group. By (3), $H/(H \cap Z_G)$ is a torsion group. Applying (2), we conclude that $H/(H \cap Z_G)$ is finite. Hence, $G/Z_G$ is locally finite.

Now let $G/Z_G$ is locally finite. We take an arbitrary finitely generated subgroup $H$ of $G$, choose a set $h_1, \dots, h_n$ of representatives of right cosets of $H$ by $H \cap Z_G$, put $F = \{h_1, \dots, h_n\}$ and note that $g^H = g^F$ for each $g \in G$. Applying (1), we conclude that $\overset{\leftrightarrow}{G}$ is cellular. □

**Remark 1.** Every finitely generated subgroup of a group $G$ is an FC-group if and only if $g^H$ is finite for each $g \in G$ and every finitely generated subgroup $H$. If $G/Z_G$ is locally finite then every finitely generated subgroup $H$ of $G$ is an FC-group. We show that the converse statement does not hold. Let $H = \bigoplus_{i<\omega} H_i$ be the direct sum of $\omega$ copies of $\mathbb{Z}_2$. We partition $\omega$ into consecutive intervals $\{W_i : i < \omega\}$ of length $|W_i| = i+1$. Then we take an automorphism $a$ of $H$ acting on each $\bigoplus\{H_m : m \in W_i\}$ as the cyclic permutations of coordinates, denote by $A$ the cyclic group generated by $A$ and consider the semidirect product $G = H \leftthreetimes A$. Then every finitely generated subgroup of $G$ is an FC-group but $a^n \notin Z_G$ for each $n \in \mathbb{N}$ so $G/Z_G$ is not locally finite.

## 4. Asymorphic embeddings

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \longrightarrow X'$ is called *macro-uniform* if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $f(E[x]) \subseteq E'[f(x)]$ for each $x \in X$. We say that an injective mapping $f : X \longrightarrow X'$ is an *asymorphic embedding* if $f : X \longrightarrow X'$ and $f^{-1} : f(X) \longrightarrow X$ are macro-uniform.

**Theorem 6.** *Every finitary coarse space $(X, \mathcal{E})$ admits an asymorphic embedding to $\overset{\leftrightarrow}{G}$ for an appropriate choice of a group $G$.*

*Proof.* We represent $(X, \mathcal{E})$ as the coarse space $X_H$ for some group $H$ of permutations of $X$, see [7, Theorem 1]. We consider $\{0,1\}^X$ as a group with point-wise addition *mod* 2. For $h \in H$ and $\chi \in \{0,1\}^X$, we put $\chi_h(y) = \chi(h^{-1}y)$. Then we define a semidirect product $G = \{0,1\}^X \leftthreetimes H$ by

$$(\chi, h)(\chi', h') = (\chi + \chi'_h, hh')$$

and note that the mapping $f : X \longrightarrow \{0,1\}^X$, $f(x)$ is the characteristic function of $\{x\}$ is an asymorphic embedding of $(X, \mathcal{E})$ into $\overset{\leftrightarrow}{G}$. □

If a subset $A$ of a coarse space $(X, \mathcal{E})$ is the union of $n$ discrete subsets then $A$ is $n$-discrete.

**Theorem 7.** *Let $G$ be a countable group. Then every $n$-discrete subset $A$ of $\overset{\leftrightarrow}{G}$ can be partitioned into $n$ discrete subsets.*

*Proof.* Use arguments proving this statement in the case of a connected coarse space with a linearly ordered base [6, Theorem 1.2]. □

**Theorem 8.** *There exists a group $G$ such that $\overset{\leftrightarrow}{G}$ has 2-discrete subset which cannot be finitely partitioned into discrete subsets.*

*Proof.* By Theorem 6.3 from [3], there exists 2-discrete finitary coarse space on $\omega$ which cannot be finitely partitioned into discrete subspaces. Apply Theorem 6. □

## 5. The space of subgroups

For a group $G$ we denote by $\mathcal{S}\,(\overset{\leftrightarrow}{G})$ the set $\mathcal{S}(G)$ of all subgroups of $G$ endowed with the coarse structure with the base

$$\{\{(X,Y)\in\mathcal{S}(G)\times\mathcal{S}(G) : Y\in X^F\} : F\in[G]^{<\omega}\},$$

$X^F = \{g^{-1}Xg : g\in F\}$.

We recall that $G$ is a *Dedekind group* if each subgroup of $G$ is normal. A non-abelian Dedekind group is called Hamiltonian. By [1],

(4) $G$ is Hamiltonian if and only if $G$ is isomorphic to $Q_8\times P$, where $Q_8$ is the quaternion group, $P$ is an Abelian group without of elements of order 4.

**Theorem 9.** *For an infinite group $G$, $\mathcal{S}(\overset{\leftrightarrow}{G})$ is discrete if and only if $G$ is a Dedekind group.*

*Proof.* If each subgroup of $G$ is normal then, evidently, $\mathcal{S}(G)$ is discrete.

We assume that $\mathcal{S}(\overset{\leftrightarrow}{G})$ is discrete and consider two cases.

*Case* 1: $G$ has an element of infinite order. First, we show that every infinite cyclic subgroup of $G$ is invariant. We suppose the contrary and choose an infinite cyclic subgroup $A$, $A=\langle a\rangle$ and $z\in G$ such that $z^{-1}az\notin A$. Since $\mathcal{S}\,(\overset{\leftrightarrow}{G})$ is discrete, there exists $m\in\mathbb{N}$ such that $z^{-1}\langle a^n\rangle z=\langle a^n\rangle$ for each $n>m$. By the same reason, there exists $k\in\mathbb{N}$ such that $z^{-1}\langle aa^n\rangle z=\langle aa^n\rangle$ for each $n>k$. We take an arbitrary $n$ such that $n>m$, $n>k$. Then $z^{-1}a^{n+1}z=(z^{-1}az)(z^{-1}a^nz)\in\langle a^{n+1}\rangle$, $z^{-1}a^nz\in\langle a^n\rangle$, so $z^{-1}a^nz\in A$, contradicting the choice of $A$ and $z$.

Second, we take an arbitrary element $a\in G$ of infinite order and show that $a\in Z_G$. Assuming the contrary, we get $z\in G$ such that $z^{-1}az\neq a$. By above paragraph $z^{-1}az=a^{-1}$, so $z^{-2}az^2=a$ and $(a^nz)(a^nz)=a^nz^2z^{-1}a^nz=a^nz^2a^{-n}=z^2$ for each $n\in\mathbb{N}$. Since $\mathcal{S}(\overset{\leftrightarrow}{G})$ is discrete, there exists $m\in\mathbb{N}$ such that

$$z^{-1}(\langle a^nz\rangle\langle z^2\rangle)z=\langle a^nz\rangle\langle z^2\rangle$$

for each $n>m$. Hence,

$$z^{-1}(a^nz)z=a^{-n}z\in\langle a^nz\rangle\langle z^2\rangle$$

and $a^{2n} \in \langle z\rangle$, contradicting $z^{-1}a^{2n}z = a^{-2n}$.

If $b$ is an element of finite order and $a$ is an element of infinite order then $ab$ has an infinite order because $a \in Z_G$, so $ab \in Z_G$, $b \in Z_G$, and $G$ is Abelian.

*Case* 2: Every element of $G$ has a finite order. We prove that $G$ is a Dedekind group provided that the following condition holds

(5) for every finite subset $K$ of $G$ containing the identity $e$, there exists $a \in G$, $a \neq e$ such that $K \cap \langle a\rangle = \{e\}$.

We suppose the contrary and choose $b \in G$, $z \in G$ such that $z^{-1}bz \notin \langle b\rangle$. Since $\mathcal{S}(\overset{\leftrightarrow}{G})$ is discrete, by (5), there exists $a \in G$, $a \neq e$ such that

$$z^{-1}bz\langle b\rangle \cap \langle a\rangle = \{e\},\ z^{-1}\langle a\rangle z = \langle a\rangle,$$

$$b^{-1}\langle a\rangle b = \langle a\rangle,\ z^{-1}\langle b\rangle\langle a\rangle z = \langle b\rangle\langle a\rangle.$$

Then $z^{-1}baz = (z^{-1}bz)(z^{-1}az) \in \langle b\rangle\langle a\rangle$, $z^{-1}bz \in \langle b\rangle\langle a\rangle$ and $z^{-1}bz \in \langle b\rangle$, contradicting the choice of $b$ and $z$.

We denote by $\pi(G)$ the set of all prime divisors of orders of elements of $G$ and put $X_n = \{g \in G : g^n = e\}$. If $G$ is not a Dedekind group, by (5), $\pi(G)$ is finite and $X_p$ is finite for each $p \in \pi(G)$. We prove that $G$ is layer-finite: $X_n$ is finite for each $n \in \mathbb{N}$. It suffices to verify that $X_{p^n}$ is finite for all $p \in \pi(G)$, $n \in \mathbb{N}$. We suppose that $X_{p^m}$ is finite but $X_{p^{m+1}}$ is infinite. Then there exists a sequence $(a_n)_{n\in\omega}$ in $G$ and $a \in G$ such that $|a_n| = p^{m+1}$, $|a| = p^m$ and $\langle a_n\rangle \cap \langle a_k\rangle = \langle a\rangle$ for all distinct $n, k \in \mathbb{N}$. We denote by $H$ the subgroup of $G$ generated by the set $\{a_n : n \in \omega\}$ and put $M = H/\langle a\rangle$. Since $\mathcal{S}\ (\overset{\leftrightarrow}{M})$ is discrete, applying (5) and (4) to $M$, we conclude that $M$ has an infinite Abelian subgroup of exponent $p$. By the Grün's lemma (see [5], p. 398), $H$ has an infinite Abelian subgroup of exponent $p$, so $X_p$ is infinite and we get a contradiction.

Thus, our assumption that $G$ is not a Dedekind group gives $G$ is layer-finite and $\pi(G)$ is finite. Since $G$ is infinite, by the Chernikov's theorem [4], $G$ has a central quasi-cyclic $p$-group $A$, $A = \cup_{n\in\omega}\langle a_n\rangle,\ a^p_{n+1} = a_n$. We take $c, z \in G$ such that $z^{-1}cz \neq \langle c\rangle$, $|c| = q^m$, $q \in \pi(G)$. Since $\mathcal{S}\ (\overset{\leftrightarrow}{G})$ is discrete, there exists $k \in \mathbb{N}$ such that, for each $n > k$, we have

$$z^{-1}\langle a_n c\rangle z = \langle a_n c\rangle,\ \ a_n(z^{-1}cz) \in \langle a_n c\rangle.$$

If $q \neq p$ then $z^{-1}cz \in \langle c\rangle$, contradicting the choice of $c$ and $z$. If $q = p$ and $n > 2m$, $n > k$ then $(a_n c)^{p^m} = a_n^{p^m}$, $|a_n^{p^m}| > p^m$ and $z^{-1}cz \in \langle a_n^{p^m}\rangle$. Since $A$ is central, $z^{-1}cz = c$ and $z^{-1}cz \in \langle c\rangle$, contradicting the choice of $z, c$. The proof is completed. □

**Remark 2.** Let $G$ be a transitive group of permutations of a set $X$, $St(x) = \{g \in G : gx = x\}$, $x \in X$. Then the natural mapping $x \mapsto St(x)$ is an asymorphic embedding of the finitary coarse space $X_G$ into $\mathcal{S}\ (\overset{\leftrightarrow}{G})$.

If $(\overset{\leftrightarrow}{G})$ is cellular then applying (1) we see that $\mathcal{S}\ (\overset{\leftrightarrow}{G})$ is cellular.

**Question 1.** *Is $\overset{\leftrightarrow}{G}$ cellular provided that $\mathcal{S}\ (\overset{\leftrightarrow}{G})$ is cellular?*

## 6. The direct union of connected components

Let $(X, \mathcal{E})$ be a coarse space, $\{X_\alpha : \alpha < \kappa\}$ is the set of all connected components of $(X, \mathcal{E})$. We say that $(X, \mathcal{E})$ is the *direct union* of $\{X_\alpha : \alpha < \kappa\}$ if, for each $E \in \mathcal{E}$, there exists $\alpha_1, \dots, \alpha_n$ such that $E[x] = \{x\}$ for each $x \in X_\alpha$, $\alpha < \kappa$, $\alpha \notin \{\alpha_1, \dots, \alpha_n\}$.

If a group $G$ is either Abelian or the set of conjugacy classes of $G$ is finite then $\overset{\leftrightarrow}{G}$ is the direct union of conjugacy classes.

For every natural number $n$, G. Bergman used $HNN$-extensions to construct a group $G$ such that $G$ has an infinite center (so the number of conjugacy classes of $G$ is infinite) and only $n$ conjugacy classes of $G$ are not singletons. Also, he proved that if $\overset{\leftrightarrow}{G}$ is the direct union of conjugacy classes then all but finely many conjugacy classes are singletons.

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