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A study on generalized matrix algebras having generalized Lie derivations

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Abstract. Let $\mathfrak{R}$ be a commutative ring with unity. The $\mathfrak{R}$-algebra $\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B})$ is a generalized matrix algebra defined by the Morita context $(\mathrm{A}, \mathrm{B}, \mathrm{M}, \mathrm{N}, \xi_{\mathrm{MN}}, \Omega_{\mathrm{NM}})$. In this article, we study generalized Lie derivation and show that every generalized Lie derivation on a generalized matrix algebra has the standard form under certain assumptions.

Historical development

There has been a great deal of work concerning characterizations of Lie derivations on rings and algebras. The first characterization of Lie derivations was obtained by Martindale [8] in 1964 who proved that every Lie derivation on the primitive ring can be written as a sum of derivation and an additive mapping of a ring to its center that maps commutators into zero, i.e, Lie derivation has the standard form. Cheung [5] established the structures of commuting maps and Lie derivation on triangular algebras. Benkovic [4] proved that under certain conditions each generalized Lie derivation of a triangular algebra is the sum of a generalized derivation and a central map which vanishes on all commutators. Following the well-established approach and the sophisticated computational method by Cheung [5], several authors studied the different linear mappings on

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generalized matrix algebras for example [3, 6, 9] and the bibliographic content existing therein. Recently, Li and Wei [6] studied the structure of derivations as well as Lie derivations on generalized matrix algebras and proved that it has a standard form. Further, Mokhtari and Vishki [9] presented several sufficient conditions assuring the properness (standard from) of Lie derivations on certain generalized matrix algebras.

The paper is organized as follows. Section 2 is about the basic results and definitions which are subsequently used in the article. Section 3 contains the key results of this article and in this section we compute the structure of generalized Lie derivation on generalized matrix algebras as well as we show that every generalized Lie derivation takes standard form under specific restrictions. Further, Example 1 shows that generalized Lie derivation fails to have the standard form if some specific restrictions do not satisfy. In sections 4 and 5, we consider a classical example of generalized matrix algebras for the direct application of our result and we draw attention to some potential problems for future research respectively.

1. Basic definitions and preliminaries

Let A be an $\mathfrak{R}$-algebra over a commutative ring with unity and $\mathfrak{Z}(\mathrm{A})$ be the center of A. An $\mathfrak{R}$-linear map $\mathrm{L} : \mathrm{A} \to \mathrm{A}$ is called a derivation (resp. Lie derivation) on A if $\mathrm{L}(ab) = \mathrm{L}(a)b + a\mathrm{L}(b)$ (resp. $\mathrm{L}([a, b]) = [\mathrm{L}(a), b] + [a, \mathrm{L}(b)]$) holds for all $a, b \in \mathrm{A}$. An $\mathfrak{R}$-linear map $\mathrm{L}_{\mathfrak{g}} : \mathrm{A} \to \mathrm{A}$ is called a generalized derivation (resp. generalized Lie derivation) on A associated with a derivation (rep. Lie derivation) L on A if $\mathrm{L}_{\mathfrak{g}}(ab) = \mathrm{L}_{\mathfrak{g}}(a)b + a\mathrm{L}(b)$ (resp. $\mathrm{L}_{\mathfrak{g}}([a, b]) = [\mathrm{L}_{\mathfrak{g}}(a), b] + [a, \mathrm{L}(b)]$) holds for all $a, b \in \mathrm{A}$.

A *Morita context* consists of two unital $\mathfrak{R}$-algebras A and B, two bimodules (A, B)-bimodule M and (B, A)-bimodule N, and two bimodule homomorphisms called the bilinear pairings $\xi_{\mathrm{MN}} : \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{N} \longrightarrow \mathrm{A}$ and $\Omega_{\mathrm{NM}} : \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{M} \longrightarrow \mathrm{B}$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{M} & \xrightarrow{\xi_{\mathrm{MN}} \otimes I_{\mathrm{M}}} & \mathrm{A} \underset{\mathrm{A}}{\otimes} \mathrm{M} \\ \Big\downarrow {\scriptstyle I_{\mathrm{M}} \otimes \Omega_{\mathrm{NM}}} & & \Big\downarrow {\scriptstyle \cong} \\ \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{B} & \xrightarrow{\cong} & \mathrm{M} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{N} & \xrightarrow{\Omega_{\mathrm{NM}} \otimes I_{\mathrm{N}}} & \mathrm{B} \underset{\mathrm{B}}{\otimes} \mathrm{N} \\ \Big\downarrow {\scriptstyle I_{\mathrm{N}} \otimes \xi_{\mathrm{MN}}} & & \Big\downarrow {\scriptstyle \cong} \\ \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{A} & \xrightarrow{\cong} & \mathrm{N}\,. \end{array}$$

If $(\mathrm{A}, \mathrm{B}, \mathrm{M}, \mathrm{N}, \xi_{\mathrm{MN}}, \Omega_{\mathrm{NM}})$ is a Morita context, then the set

$$\begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| a \in \mathrm{A}, m \in \mathrm{M}, n \in \mathrm{N}, b \in \mathrm{B} \right\}$$

forms an $\mathfrak{R}$-algebra under matrix addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. Such an $\mathfrak{R}$-algebra is usually called a *generalized matrix algebra* of order 2 and is denoted by

$$\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B}) = \begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix}.$$

This kind of algebra was first introduced by Morita in [10]. All associative algebras with nontrivial idempotents are isomorphic to generalized matrix algebras. Most common examples of generalized matrix algebras are full matrix algebras over a unital algebra and triangular algebras [3, 12]. Also, if the bilinear pairings ξ_{MN} and Ω_{NM} are zero, then $\mathfrak{G}$ is called a trivial generalized matrix algebra and if $\mathrm{N} = 0$, then $\mathfrak{G}$ is called a triangular algebra.

The center of $\mathfrak{G}$ is

$$\mathfrak{Z}(\mathfrak{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| am = mb, na = bn \ \forall\ m \in \mathrm{M}, n \in \mathrm{N} \right\}.$$

Define two natural projections $\pi_{\mathrm{A}} : \mathfrak{G} \to \mathrm{A}$ and $\pi_{\mathrm{B}} : \mathfrak{G} \to \mathrm{B}$ by $\pi_{\mathrm{A}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = a$ and $\pi_{\mathrm{B}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = b$. Moreover, $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathrm{A})$ and $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathrm{B})$ and there exists a unique algebraic isomorphism $\zeta : \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) \to \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$ such that $am = m\zeta(a)$ and $na = \zeta(a)n$ for all $a \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})), m \in \mathrm{M}$ and $n \in \mathrm{N}$.

Let 1_{A} (resp.1_{B}) be the identity of the algebra A (resp.B) and let I be the identity of generalized matrix algebra $\mathfrak{G}$, $e = \begin{bmatrix} 1_{\mathrm{A}} & 0 \\ 0 & 0 \end{bmatrix}$, $f = I - e = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathrm{B}} \end{bmatrix}$ and $\mathfrak{G}_{11} = e\mathfrak{G}e$, $\mathfrak{G}_{12} = e\mathfrak{G}f$, $\mathfrak{G}_{21} = f\mathfrak{G}e$, $\mathfrak{G}_{22} = f\mathfrak{G}f$. Thus $\mathfrak{G} = e\mathfrak{G}e + e\mathfrak{G}f + f\mathfrak{G}e + f\mathfrak{G}f = \mathfrak{G}_{11} + \mathfrak{G}_{12} + \mathfrak{G}_{21} + \mathfrak{G}_{22}$ where $\mathfrak{G}_{11}$ is subalgebra of $\mathfrak{G}$ isomorphic to A, $\mathfrak{G}_{22}$ is subalgebra of $\mathfrak{G}$ isomorphic to B, $\mathfrak{G}_{12}$ is $(\mathfrak{G}_{11}, \mathfrak{G}_{22})$-bimodule isomorphic to M and $\mathfrak{G}_{21}$ is $(\mathfrak{G}_{22}, \mathfrak{G}_{11})$-bimodule isomorphic to N. Also, $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ and $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$ are isomorphic to $e\mathfrak{Z}(\mathfrak{G})e$ and $f\mathfrak{Z}(\mathfrak{G})f$ respectively. Then there is an algebra isomorphisms $\zeta : e\mathfrak{Z}(\mathfrak{G})e \to f\mathfrak{Z}(\mathfrak{G})f$ such that $am = m\zeta(a)$ and $na = \zeta(a)n$ for all

$m \in e\mathfrak{G}f$ and $n \in f\mathfrak{G}e$. Now we should mention some important results which are used subsequently in this article:

Lemma 1 ([6, Propostion 4.1]). *Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B})$ be a generalized matrix algebra over a commutative ring $\mathfrak{R}$. Let L be Lie derivation on $\mathfrak{G}$. Then Lie derivation has the form*

$$\mathrm{L}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0 n + \delta_4(b) & am_0 + \tau_2(m) + \tau_3(n) - m_0 b \\ n_0 a + \nu_2(m) + \nu_3(n) - bn_0 & \mu_1(a) + n_0 m + nm_0 + \mu_4(b) \end{bmatrix},$$

where $a \in \mathrm{A}$; $b \in \mathrm{B}$; $m, m_0 \in \mathrm{M}$; $n, n_0 \in \mathrm{N}$ and $\delta_1 : \mathrm{A} \to \mathrm{A}, \delta_4 : \mathrm{B} \to \mathfrak{Z}(\mathrm{A})$, $\tau_2 : \mathrm{N} \to \mathrm{M}, \tau_3 : \mathrm{N} \to \mathrm{M}, \nu_2 : \mathrm{M} \to \mathrm{N}, \nu_3 : \mathrm{N} \to \mathrm{N}, \mu_1 : \mathrm{A} \to \mathfrak{Z}(\mathrm{B}), \mu_4 : \mathrm{B} \to \mathrm{B}$ are $\mathfrak{R}$-linear maps satisfying the following conditions:

1) *δ_1 is Lie derivation of A and $\delta_1(mn) = -\delta_4(nm) + \tau_2(m)n + m\nu_3(n)$;*
2) *μ_4 is Lie derivation of B and $\mu_4(nm) = \mu_1(mn) + n\tau_2(m) + \nu_3(n)m$;*
3) *$\delta_4([b_1, b_2]) = 0$ for all $b_1, b_2 \in \mathrm{B}$ and $\mu_1([a_1, a_2]) = 0$ for all $a_1, a_2 \in \mathrm{A}$;*
4) *$\tau_2(am) = a\tau_2(m) + \delta_1(a)m - m\mu_1(a)$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b) - \delta_4(b)m$;*
5) *$\nu_3(na) = \nu_3(n)a + n\delta_1(a) - \mu_1(a)n$ and $\nu_3(bn) = \nu_4(b)n - n\delta_4(b) + b\nu_3(n)$;*
6) *$2\tau_3(n) = 0$ and $2\nu_2(m) = 0$.*

Lemma 2. *[9, Proposition 1 (2.3)] Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B})$ be a generalized matrix algebra over a commutative ring $\mathfrak{R}$. A $\mathfrak{R}$-linear map τ on $\mathfrak{G}$ is center valued and vanishes at commutators if and only if it has the following form*

$$\tau\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \mathfrak{l}_1(a) + \mathfrak{p}_4(b) & 0 \\ 0 & \mathfrak{q}_1(a) + \mathfrak{l}_4(b) \end{bmatrix}$$

where $\mathfrak{l}_1 : \mathrm{A} \to \mathfrak{Z}(\mathrm{A})$, $\mathfrak{p}_4 : \mathrm{B} \to \mathfrak{Z}(\mathrm{A})$, $\mathfrak{q}_1 : \mathrm{A} \to \mathfrak{Z}(\mathrm{B})$ and $\mathfrak{l}_4 : \mathrm{B} \to \mathfrak{Z}(\mathrm{B})$ are $\mathfrak{R}$-linear maps vanishing at commutators, having the following properties:

1) *$\mathfrak{l}_1(a) + \mathfrak{q}_1(a) \in \mathfrak{Z}(\mathfrak{G})$ and $\mathfrak{p}_4(b) + \mathfrak{l}_4(b) \in \mathfrak{Z}(\mathfrak{G})$ for all $a \in \mathrm{A}, b \in \mathrm{B}$,*
2) *$\mathfrak{l}_1(mn) = \mathfrak{p}_4(nm)$ and $\mathfrak{q}_1(mn) = \mathfrak{l}_4(nm)$ for all $m \in \mathrm{M}, n \in \mathrm{N}$.*

Lemma 3. *[3, Theorem 2.1] Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B})$ be a generalized matrix algebra over a commutative ring $\mathfrak{R}$. An additive map $\Phi : \mathfrak{G} \to \mathfrak{G}$ is a generalized derivation on $\mathfrak{G}$ if and only if Φ has the following form*

$$\Phi\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \Delta_1(a) - mn_0 - {m_0}'n & am_0 + T_2(m) - {m_0}'b \\ {n_0}'a - bn_0 + V_3(n) & nm_0 + {n_0}'m + U_4(b) \end{bmatrix},$$

where $a \in \mathrm{A}$; $b \in \mathrm{B}$; $m, m_0, m_0{}' \in \mathrm{M}$; $n, n_0, n_0{}' \in \mathrm{N}$ *and* $\Delta_1 : \mathrm{A} \to \mathrm{A}, T_2 : \mathrm{M} \to \mathrm{M}, V_3 : \mathrm{N} \to \mathrm{N}, U_4 : \mathrm{B} \to \mathrm{B}$ *are* $\mathfrak{R}$*-linear maps satisfying the following conditions:*

1) Δ_1 *is generalized derivation of* A *and* $\Delta_1(mn) = T_2(m)n + m\nu_3(n)$;
2) U_4 *is generalized derivation of* B *and* $U_4(nm) = V_3(n)m + n\tau_2(m)$;
3) $T_2(am) = \Delta_1(a)m + a\tau_2(m)$ *and* $T_2(mb) = T_2(m)b + m\mu_4(b)$;
4) $V_3(na) = V_3(n)a + n\delta_1(a)$ *and* $V_3(bn) = U_4(b)n + b\nu_3(n)$.

2. Key results

In this section, we make an attempt to establish the structure of generalized Lie derivation on generalized matrix algebras as follows:

Theorem 1. *Let* $\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B})$ *be a generalized matrix algebra over a commutative ring* $\mathfrak{R}$*. An additive map* $\mathrm{L}_{\mathfrak{g}} : \mathfrak{G} \to \mathfrak{G}$ *is a generalized Lie derivation on* $\mathfrak{G}$ *if and only if* $\mathrm{L}_{\mathfrak{g}}$ *has the following form*

$$\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \mathfrak{p}_1(a) - mn_0 - m_0'n + \mathfrak{p}_4(b) & am_0 + \mathfrak{r}_2(m) + \mathfrak{r}_3(n) - m_0'b \\ n_0'a + \mathfrak{s}_2(m) + \mathfrak{s}_3(n) - bn_0 & \mathfrak{q}_1(a) + n_0'm + nm_0 + \mathfrak{q}_4(b) \end{bmatrix},$$

where $a \in \mathrm{A}$; $b \in \mathrm{B}$; $m, m_0, m_0{}' \in \mathrm{M}$; $n, n_0, n_0{}' \in \mathrm{N}$ *and* $\mathfrak{p}_1 : \mathrm{A} \to \mathrm{A}, \mathfrak{p}_4 : \mathrm{B} \to \mathfrak{Z}(\mathrm{A}), \mathfrak{s}_2 : \mathrm{N} \to \mathrm{M}, \mathfrak{s}_3 : \mathrm{N} \to \mathrm{M}, \mathfrak{r}_2 : \mathrm{M} \to \mathrm{N}, \mathfrak{r}_3 : \mathrm{N} \to \mathrm{N}, \mathfrak{q}_1 : \mathrm{A} \to \mathfrak{Z}(\mathrm{B}), \mathfrak{q}_4 : \mathrm{B} \to \mathrm{B}$ *are* $\mathfrak{R}$*-linear maps satisfying the following conditions:*

1) $\mathfrak{p}_1$ *is generalized Lie derivation of* A *and* $\mathfrak{p}_1(mn) = -\mathfrak{p}_4(nm) + \mathfrak{r}_2(m)n + m\mathfrak{s}_3(n)$;
2) $\mathfrak{q}_4$ *is generalized Lie derivation of* B *and* $\mathfrak{q}_4(nm) = \mathfrak{q}_1(mn) + n\mathfrak{r}_2(m) + \mathfrak{s}_3(n)m$;
3) $\mathfrak{p}_4([b_1, b_2]) = 0$ *for all* $b_1, b_2 \in \mathrm{B}$ *and* $\mathfrak{q}_1([a_1, a_2]) = 0$ *for all* $a_1, a_2 \in \mathrm{A}$;
4) $\mathfrak{r}_2(am) = a\mathfrak{r}_2(m) + \mathfrak{p}_1(a)m - m\mathfrak{q}_1(a)$ *and* $\mathfrak{r}_2(mb) = \mathfrak{r}_2(m)b + m\mathfrak{q}_4(b) - \mathfrak{p}_4(b)m$;
5) $\mathfrak{s}_3(na) = \mathfrak{s}_3(n)a + n\mathfrak{p}_1(a) - \mathfrak{q}_1(a)n$ *and* $\mathfrak{s}_3(bn) = \mathfrak{q}_4(b)n - n\mathfrak{p}_4(b) + b\mathfrak{s}_3(n)$;
6) $2\mathfrak{r}_3(n) = 0$ *and* $2\mathfrak{s}_2(m) = 0$.

Proof. Suppose that generalized Lie derivation takes the following form as

$$\mathrm{L}_{\mathfrak{g}}\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} \mathfrak{p}_1(a)+\mathfrak{p}_2(m)+\mathfrak{p}_3(n)+\mathfrak{p}_4(b) & \mathfrak{r}_1(a)+\mathfrak{r}_2(m)+\mathfrak{r}_3(n)+\mathfrak{r}_4(b) \\ \mathfrak{s}_1(a)+\mathfrak{s}_2(m)+\mathfrak{s}_3(n)+\mathfrak{s}_4(b) & \mathfrak{q}_1(a)+\mathfrak{q}_2(m)+\mathfrak{q}_3(n)+\mathfrak{q}_4(b) \end{bmatrix}$$

where $\mathfrak{p}_1 : \mathrm{A} \to \mathrm{A}, \mathfrak{p}_2 : \mathrm{M} \to \mathrm{A}, \mathfrak{p}_3 : \mathrm{N} \to \mathrm{A}, \mathfrak{p}_4 : \mathrm{B} \to \mathrm{A}$; $\mathfrak{r}_1 : \mathrm{A} \to \mathrm{M}, \mathfrak{r}_2 : \mathrm{M} \to \mathrm{M}, \mathfrak{r}_3 : \mathrm{N} \to \mathrm{M}, \mathfrak{r}_4 : \mathrm{B} \to \mathrm{M}$; $\mathfrak{s}_1 : \mathrm{A} \to \mathrm{N}, \mathfrak{s}_2 : \mathrm{M} \to \mathrm{N}, \mathfrak{s}_3 : \mathrm{N} \to \mathrm{N}, \mathfrak{s}_4 : \mathrm{B} \to \mathrm{N}$ and $\mathfrak{q}_1 : \mathrm{A} \to \mathrm{B}, \mathfrak{q}_2 : \mathrm{M} \to \mathrm{B}, \mathfrak{q}_3 : \mathrm{N} \to \mathrm{B}, \mathfrak{q}_4 : \mathrm{B} \to \mathrm{B}$ are $\mathfrak{R}$-linear maps. As $\mathrm{L}_{\mathfrak{g}}$ is the generalized Lie derivation with Lie derivation L defined by $\mathrm{L}_{\mathfrak{g}}([G_1, G_2]) = [\mathrm{L}_{\mathfrak{g}}(G_1), G_2] + [G_1, \mathrm{L}(G_2)]$ for all $G_1, G_2 \in \mathfrak{G}$.

Now we assume that

$$G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad G_2 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix},$$

then

$$\begin{aligned} &\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} 0 & am \\ 0 & 0 \end{bmatrix}\right) = \left[\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right), \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right] + \left[\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \mathrm{L}\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\right] \\ &\implies \begin{bmatrix} \mathfrak{p}_2(am) & \mathfrak{r}_2(am) \\ \mathfrak{s}_2(am) & \mathfrak{q}_2(am) \end{bmatrix} \\ &\quad = \left[\begin{bmatrix} \mathfrak{p}_1(a) & \mathfrak{r}_1(a) \\ \mathfrak{s}_1(a) & \mathfrak{q}_1(a) \end{bmatrix}, \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right] + \left[\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -mn_0 & \tau_2(m) \\ \nu_2(m) & n_0 m \end{bmatrix}\right] \\ &\quad = \begin{bmatrix} -m\mathfrak{s}_1(a) & \mathfrak{p}_1(a)m - m\mathfrak{q}_1(a) \\ 0 & \mathfrak{s}_1(a)m \end{bmatrix} + \begin{bmatrix} -amn_0 + mn_0 a & a\tau_2(m) \\ -\nu_2(m)a & 0 \end{bmatrix}. \end{aligned}$$

On comparing both sides, we get $\mathfrak{p}_2(am) = -m\mathfrak{s}_1(a) - amn_0 + mn_0 a$, $\mathfrak{r}_2(am) = \mathfrak{p}_1(a)m - m\mathfrak{q}_1(a) + a\tau_2(m), \mathfrak{s}_2(am) = -\nu_2(m)a$ and $\mathfrak{q}_2(am) = \mathfrak{s}_1(a)m$. Now, if we take $a = 1$, then we find that $\mathfrak{p}_2(m) = -mn_0'$, $\mathfrak{r}_2(m) = \mathfrak{p}_1(1)m - m\mathfrak{q}_1(1) + \tau_2(m)$ and $\mathfrak{q}_2(m) = n_0' m$, where $\mathfrak{s}_1(1) = n_0'$. Again, if we consider $G_1 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, then we have $\mathfrak{r}_2(m) = \tau_2(m)$.

Similarly, if $G_1 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, then we have $\mathfrak{p}_2(mb) = -mbn_0$, $\mathfrak{r}_2(mb) = \mathfrak{r}_2(m)b + m\mu_4(b) - \delta_4(b)m$, $\mathfrak{s}_2(mb) = -b\mathfrak{s}_2(m)$ and $\mathfrak{q}_2(mb) = \mathfrak{q}_2(m)b - b\mathfrak{q}_2(m) + bn_0 m$. On substituting $b = 1$, we get $\mathfrak{p}_2(m) = -mn_0$, $m\mu_4(1) = \delta_4(1)m$, $2\mathfrak{s}_2(m) = 0$ and $\mathfrak{q}_2(m) = n_0 m$. Also for

$G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$, we find that $\mathfrak{r}_2(mb) = \tau_2(m)b + m\mathfrak{q}_4(b) - \mathfrak{p}_4(b)m$.

Again, we suppose that $G_1 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, we get

$$\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} 0 & 0 \\ na & 0 \end{bmatrix}\right) = \left[\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right), \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right] + \left[\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}, \mathrm{L}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\right]$$

$$\implies \begin{bmatrix} \mathfrak{p}_3(na) & \mathfrak{r}_3(na) \\ \mathfrak{s}_3(na) & \mathfrak{q}_3(na) \end{bmatrix}$$

$$= \left[\begin{bmatrix} \mathfrak{p}_3(n) & \mathfrak{r}_3(n) \\ \mathfrak{s}_3(n) & \mathfrak{q}_3(n) \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right] + \left[\begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \delta_1(a) & am_0 \\ n_0 a & \mu_1(a) \end{bmatrix}\right]$$

$$= \begin{bmatrix} \mathfrak{p}_3(n)a - a\mathfrak{p}_3(n) & -a\mathfrak{r}_3(n) \\ \mathfrak{s}_3(n)a & 0 \end{bmatrix} + \begin{bmatrix} -am_0 n & 0 \\ n\delta_1(a) - \mu_1(a)n & nam_0 \end{bmatrix}.$$

This leads to $\mathfrak{p}_3(na) = \mathfrak{p}_3(n)a - a\mathfrak{p}_3(n) - am_0 n, \mathfrak{r}_3(na) = -a\mathfrak{r}_3(n)$, $\mathfrak{s}_3(na) = \mathfrak{s}_3(n)a + n\delta_1(a) - \mu_1(a)n, \mathfrak{q}_3(na) = nam_0$. On putting $a = 1$, we have $\mathfrak{p}_3(n) = m_0 n,\ 2\mathfrak{r}_3(n) = 0,\ n\delta_1(a) - \mu_1(a)n = 0,\ \mathfrak{q}_3(n) = nm_0$. For $G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, we have $\mathfrak{s}_3(na) = \nu_3(n)a + n\mathfrak{p}_1(a) - \mathfrak{q}_1(a)n$.

On similar pattern, if $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, then we find that $\mathfrak{p}_3(bn) = \mathfrak{r}_4(b)n$, $\mathfrak{r}_3(bn) = -\tau_3(n)b$, $\mathfrak{s}_3(bn) = \mathfrak{q}_4(b)n - n\mathfrak{p}_4(b) + b\nu_3(n)$, and $\mathfrak{q}_3(bn) = -n\mathfrak{r}_4(b) + bnm_0 - nm_0 b$. Putting $b = 1$ leads to $\mathfrak{p}_3(n) = \mathfrak{r}_4(1)n,\ \mathfrak{r}_3(n) = -\tau_3(n),\ \mathfrak{s}_3(n) = \mathfrak{q}_4(1)n - n\mathfrak{p}_4(1) + \nu_3(n)$, and $\mathfrak{q}_3(n) = nm_0'$, where $\mathfrak{r}_4(1) = -m_0'$. Further with $G_1 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, it follows that $\mathfrak{s}_3(n) = \nu_3(n)$.

On assuming $G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, we obtain that

$$\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left[\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right), \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right] + \left[\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \mathrm{L}\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\right]$$

$$\implies \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \left[\begin{bmatrix} \mathfrak{p}_1(a) & \mathfrak{r}_1(a) \\ \mathfrak{s}_1(a) & \mathfrak{q}_1(a) \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right] + \left[\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \delta_4(b) & -m_0 b \\ -bn_0 & \mu_4(b) \end{bmatrix}\right]$$

$$= \begin{bmatrix} 0 & \mathfrak{r}_1(a)b \\ -b\mathfrak{s}_1(a) & [\mathfrak{q}_1(a), b] \end{bmatrix} + \begin{bmatrix} [a, \delta_4(b)] & -am_o b \\ bn_0 a & 0 \end{bmatrix}.$$

On comparing both sides, we get $[a, \delta_4(b)] = 0$, $\mathfrak{r}_1(a)b = am_0b$, $b\mathfrak{s}_1(a) = bn_0a$, and $[\mathfrak{q}_1(a), b] = 0$. This implies that $\mathfrak{r}_1(a) = am_0$, $\mathfrak{s}_1(a) = n_0a$, and $\mathfrak{q}_1(a) \in \mathfrak{Z}(\mathrm{B})$. Similarly, if $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, then $\mathfrak{p}_4(b) \in \mathfrak{Z}(\mathrm{A})$, $\mathfrak{r}_4(b) = -m_0b$, and $\mathfrak{s}_4(b) = -bn_0$.

Let us consider $G_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned}
&\begin{bmatrix} \mathfrak{p}_1([a_1, a_2]) & \mathfrak{r}_1([a_1, a_2]) \\ \mathfrak{s}_1([a_1, a_2]) & \mathfrak{q}_1([a_1, a_2]) \end{bmatrix} \\
&\quad = \mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} [a_1, a_2] & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\quad = \left[\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}\right), \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}\right] + \left[\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}, \mathrm{L}\left(\begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}\right)\right] \\
&\quad = \left[\begin{bmatrix} \mathfrak{p}_1(a_1) & \mathfrak{r}_1(a_1) \\ \mathfrak{s}_1(a_1) & \mathfrak{q}_1(a_1) \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}\right] + \left[\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \delta_1(a) & am_0 \\ n_0a & \mu_1(a) \end{bmatrix}\right] \\
&\quad = \begin{bmatrix} [\mathfrak{p}_1(a_1), a_2] & -a_2\mathfrak{r}_1(a_1) \\ \mathfrak{s}_1(a_1)a_2 & 0 \end{bmatrix} + \begin{bmatrix} [a_1, \delta_1(a_2)] & a_1a_2m_0 \\ -n_0a_2a_1 & 0 \end{bmatrix}.
\end{aligned}$$

Equating both sides, we find that $\mathfrak{p}_1([a_1, a_2]) = [\mathfrak{p}_1(a_1), a_2] + [a_1, \delta_1(a_2)]$, $\mathfrak{r}_1([a_1, a_2]) = -a_2\mathfrak{r}_1(a_1) + a_1a_2m_0, \mathfrak{s}_1([a_1, a_2]) = \mathfrak{s}_1(a_1)a_2 - n_0a_2a_1$ and $\mathfrak{q}_1([a_1, a_2]) = 0$.

In similar manner, if $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b_1 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b_2 \end{bmatrix}$, then we arrive at $\mathfrak{p}_4([b_1, b_2]) = 0$, $\mathfrak{r}_4([b_1, b_2]) = \mathfrak{r}_4(b_1)b_2 + m_0b_2b_1$, $\mathfrak{s}_4([b_1, b_2]) = -b_2\mathfrak{s}_4(b_1) - b_1b_2n_0$ and $\mathfrak{q}_4([b_1, b_2]) = [\mathfrak{q}_4(b_1), b_2] + [b_1, \mu_4(b_2)]$.

Suppose that $G_1 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, this implies that

$$\begin{aligned}
&\begin{bmatrix} \mathfrak{p}_1(mn) - \mathfrak{p}_4(nm) & \mathfrak{r}_1(mn) - \mathfrak{r}_4(nm) \\ \mathfrak{s}_1(mn) - \mathfrak{s}_4(nm) & \mathfrak{q}_1(mn) - \mathfrak{q}_4(nm) \end{bmatrix} \\
&\quad = \mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} mn & 0 \\ 0 & -nm \end{bmatrix}\right) \\
&\quad = \left[\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right), \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right] + \left[\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}, \mathrm{L}\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right)\right] \\
&\quad = \begin{bmatrix} \mathfrak{r}_2(m)n & 0 \\ \mathfrak{s}_2(m)n - n\mathfrak{p}_2(m) & -n\mathfrak{r}_2(m) \end{bmatrix} + \begin{bmatrix} m\nu_3(n) & mnm_0 + m_0nm \\ 0 & -\nu_3(n)m \end{bmatrix}.
\end{aligned}$$

On comparing both sides we get $\mathfrak{p}_1(mn) - \mathfrak{p}_4(nm) = \mathfrak{r}_2(m)n + m\nu_3(n)$, $\mathfrak{r}_1(mn) - \mathfrak{r}_4(nm) = mnm_0 + m_0nm, \mathfrak{s}_1(mn) - \mathfrak{s}_4(nm) = \mathfrak{s}_2(m)n - n\mathfrak{p}_2(m)$

and $\mathfrak{q}_1(mn) - \mathfrak{q}_4(nm) = -n\mathfrak{r}_2(m) - \nu_3(n)m$. Similarly, if $G_1 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$, then we have $-\mathfrak{p}_1(mn) + \mathfrak{p}_4(nm) = -m\mathfrak{s}_3(n) - \tau_2(m)n$, $-\mathfrak{r}_1(mn) + \mathfrak{r}_4(nm) = \mathfrak{p}_3(n)m - m\mathfrak{p}_3(n)$, $-\mathfrak{s}_1(mn) + \mathfrak{s}_4(nm) = -nmn_0 - n_0mn$ and $-\mathfrak{q}_1(mn) + \mathfrak{q}_4(nm) = \mathfrak{s}_3(n)m + n\tau_2(m)$.

Conversely, suppose that $\mathrm{L}_\mathfrak{g}$ has following form and satisfies condition (1) to (6)

$$\mathrm{L}_\mathfrak{g}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \mathfrak{p}_1(a) - mn_0 - m_0'n + \mathfrak{p}_4(b) & am_0 + \mathfrak{r}_2(m) + \mathfrak{r}_3(n) - m_0'b \\ n_0'a + \mathfrak{s}_2(m) + \mathfrak{s}_3(n) - bn_0 & \mathfrak{q}_1(a) + n_0'm + nm_0 + \mathfrak{q}_4(b) \end{bmatrix}.$$

Then it is easy to prove that $\mathrm{L}_\mathfrak{g}([G_1, G_2]) = [\mathrm{L}_\mathfrak{g}(G_1), G_2] + [G_1, \mathrm{L}(G_2)]$. Hence $\mathrm{L}_\mathfrak{g}$ is generalized Lie derivation on $\mathfrak{G}$. □

If M is faithful but no restriction on N, then the condition $\mathfrak{p}_4([b_1, b_2]) = 0$ and $\mathfrak{q}_1([a_1, a_2]) = 0$ can be dropped.

Theorem 2. *Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B})$ be a 2-torsion free generalized matrix algebra over a commutative ring $\mathfrak{R}$. An additive map $\mathrm{L}_\mathfrak{g} : \mathfrak{G} \to \mathfrak{G}$ is a generalized Lie derivation on $\mathfrak{G}$ if and only if $\mathrm{L}_\mathfrak{g}$ has the following form*

$$\mathrm{L}_\mathfrak{g}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \mathfrak{p}_1(a) - mn_0 - m_0'n + \mathfrak{p}_4(b) & am_0 + \mathfrak{r}_2(m) - m_0'b \\ n_0'a + \mathfrak{s}_3(n) - bn_0 & \mathfrak{q}_1(a) + n_0'm + nm_0 + \mathfrak{q}_4(b) \end{bmatrix},$$

where $a \in \mathrm{A}$; $b \in \mathrm{B}$; $m, m_0, m_0{}' \in \mathrm{M}$; $n, n_0, n_0{}' \in \mathrm{N}$ and $\mathfrak{p}_1 : \mathrm{A} \to \mathrm{A}, \mathfrak{p}_4 : \mathrm{B} \to \mathfrak{Z}(\mathrm{A})$, $\mathfrak{s}_2 : \mathrm{N} \to \mathrm{M}, \mathfrak{s}_3 : \mathrm{N} \to \mathrm{M}, \mathfrak{r}_2 : \mathrm{M} \to \mathrm{N}, \mathfrak{r}_3 : \mathrm{N} \to \mathrm{N}, \mathfrak{q}_1 : \mathrm{A} \to \mathfrak{Z}(\mathrm{B}), \mathfrak{q}_4 : \mathrm{B} \to \mathrm{B}$ are $\mathfrak{R}$-linear maps satisfying the following conditions:

1) *$\mathfrak{p}_1$ is generalized Lie derivation of A and $\mathfrak{p}_1(mn) = -\mathfrak{p}_4(nm) + \mathfrak{r}_2(m)n + m\mathfrak{s}_3(n)$;*
2) *$\mathfrak{q}_4$ is generalized Lie derivation of B and $\mathfrak{q}_4(nm) = \mathfrak{q}_1(mn) + n\mathfrak{r}_2(m) + \mathfrak{s}_3(n)m$;*
3) *$\mathfrak{r}_2(am) = a\mathfrak{r}_2(m) + \mathfrak{p}_1(a)m - m\mathfrak{q}_1(a)$ and $\mathfrak{r}_2(mb) = \mathfrak{r}_2(m)b + m\mathfrak{q}_4(b) - \mathfrak{p}_4(b)m$;*
4) *$\mathfrak{s}_3(na) = \mathfrak{s}_3(n)a + n\mathfrak{p}_1(a) - \mathfrak{q}_1(a)n$ and $\mathfrak{s}_3(bn) = \mathfrak{q}_4(b)n - n\mathfrak{p}_4(b) + b\mathfrak{s}_3(n)$.*

Proof. It is sufficient to show that if M is faithful, then $\mathfrak{p}_4([b_1, b_2]) = 0$ and $\mathfrak{q}_1([a_1, a_2]) = 0$. For any $a_1, a_2 \in \mathrm{A}$, it follows that

$$\mathfrak{r}_2([a_1, a_2]m) \quad = \quad [a_1, a_2]\mathfrak{r}_2(m) + \mathfrak{p}_1([a_1, a_2])m - m\mathfrak{q}_1([a_1, a_2]).$$

On the other hand, we have

$$\begin{aligned}
\mathfrak{r}_2([a_1, a_2]m) &= \mathfrak{r}_2(a_1a_2m - a_2a_1m) \\
&= \mathfrak{r}_2(a_1a_2m) - \mathfrak{r}_2(a_2a_1m) \\
&= a_1\mathfrak{r}_2(a_2m) + \mathfrak{p}_1(a_1)a_2m - a_2m\mathfrak{q}_1(a_1) - a_2\mathfrak{r}_2(a_1m) \\
&\qquad - \mathfrak{p}_1(a_2)a_1m + a_1m\mathfrak{q}_1(a_2) \\
&= a_1\{a_2\mathfrak{r}_2(m) + \mathfrak{p}_1(a_2)m - m\mathfrak{q}_1(a_2)\} + \mathfrak{p}_1(a_1)a_2m - a_2m\mathfrak{q}_1(a_1) \\
&\qquad - a_2\{a_1\mathfrak{r}_2(m) + \mathfrak{p}_1(a_1)m - m\mathfrak{q}_1(a_1)\} - \mathfrak{p}_1(a_2)a_1m + a_1m\mathfrak{q}_1(a_2) \\
&= [a_1, a_2]\mathfrak{r}_2(m) + [\mathfrak{p}_1(a_1), a_2]m + [a_1, \mathfrak{p}_1(a_2)]m.
\end{aligned}$$

It follows that $m\mathfrak{q}_1([a_1, a_2]) = 0$. Note that M is faithful as right B-module, then this leads to $\mathfrak{q}_1([a_1, a_2]) = 0$ for all $a_1, a_2 \in \mathrm{A}$.

Similarly, with $\mathfrak{r}_2(mb) = \mathfrak{r}_2(m)b + m\mathfrak{q}_4(b) - \mathfrak{p}_4(b)m$, we can prove that $\mathfrak{p}_4([b_1, b_2]) = 0$ for all $b_1, b_2 \in \mathrm{B}$. □

The following result provides us a simple way to verify whether a generalized Lie derivation has non standard form.

Theorem 3. *Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B})$ be a 2-torsion free generalized matrix algebra over a commutative ring $\mathfrak{R}$. Then generalized Lie derivation $\mathrm{L}_{\mathfrak{g}}$ has standard form if and only if $\pi_{\mathrm{A}}(\mathrm{L}_{\mathfrak{g}}(\mathrm{B})) \subseteq \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$, $\pi_{\mathrm{B}}(\mathrm{L}_{\mathfrak{g}}(\mathrm{A})) \subseteq \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$ and $\mathfrak{p}_4(nm) + \mathfrak{q}_1(mn) \in \mathfrak{Z}(\mathfrak{G})$ where A and B is treated as subalgebras of $\mathfrak{G}$.*

Proof. Let us assume $\mathrm{L}_{\mathfrak{g}}$ has standard form, i.e., $\mathrm{L}_{\mathfrak{g}} = \Phi + \tau$, where Φ is an additive generalized derivation and $\tau(\mathfrak{G}) \subseteq \mathfrak{Z}(\mathfrak{G})$. By Lemma 2, we know that

$$\tau\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \mathfrak{l}_1(a) + \mathfrak{p}_4(b) & 0 \\ 0 & \mathfrak{q}_1(a) + \mathfrak{l}_4(b) \end{bmatrix}.$$

Also, by Lemma 3

$$\Phi\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \Delta_1(a) - mn_0 - m_0{}'n & am_0 + T_2(m) - m_0{}'b \\ n_0{}'a - bn_0 + V_3(n) & nm_0 + n_0{}'m + U_4(b) \end{bmatrix}.$$

In view of Theorem 2, we obtain that $\mathfrak{p}_1 = \Delta_1 + \mathfrak{l}_1;\ \mathfrak{s}_3 = V_3;\ \mathfrak{r}_2 = T_2$ and $\mathfrak{q}_4 = U_4 + \mathfrak{l}_4$. For any $b \in \mathrm{B}$, we get

$$\mathfrak{p}_4(b) = \pi_{\mathrm{A}}\left(\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\right) \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$$

and for any $a \in \mathrm{A}$, we have

$$\mathfrak{q}_1(a) = \pi_{\mathrm{B}}\left(\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\right) \in \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})).$$

This shows that every generalized Lie derivation on $\mathfrak{G}$ can be written as a sum of generalized derivation and a central mapping. Now it is easy to verify that $\mathfrak{p}_4(nm) + \mathfrak{q}_1(mn) \in \mathfrak{Z}(\mathfrak{G})$ for all $m \in \mathrm{M}$ and $n \in \mathrm{N}$.

Conversely, let us take $\mathfrak{p}_4(\mathrm{B}) \subseteq \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ and $\mathfrak{q}_1(\mathrm{A}) \subseteq \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$. Now we define

$$\Phi\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} \mathfrak{p}_1(a) - mn_0 - m_0'n - \zeta^{-1}(\mathfrak{q}_1(a)) & am_0 + \mathfrak{r}_2(m) - m_0'b \\ n_0'a + \mathfrak{s}_3(n) - bn_0 & n_0'm + nm_0 + \mathfrak{q}_4(b) - \zeta(\mathfrak{p}_4(b)) \end{bmatrix}$$

and

$$\tau\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} \zeta^{-1}(\mathfrak{q}_1(a)) + \mathfrak{p}_4(b) & 0 \\ 0 & \mathfrak{q}_1(a) + \zeta(\mathfrak{p}_4(b)) \end{bmatrix}.$$

It can be easily seen that Φ is a generalized derivation and $\tau(\mathfrak{G}) \subseteq \mathfrak{Z}(\mathfrak{G})$.

Also, define maps $\mathfrak{l}_1 : \mathrm{A} \to \mathfrak{Z}(\mathrm{A})$ and $\mathfrak{l}_4 : \mathrm{B} \to \mathfrak{Z}(\mathrm{B})$ by $\mathfrak{l}_1(a) = \zeta^{-1}(\mathfrak{q}_1(a))$ and $\mathfrak{l}_4(b) = \zeta(\mathfrak{p}_4(b))$ respectively. Now it is obvious that $\mathfrak{l}_1(a) + \mathfrak{q}_1(a) \in \mathfrak{Z}(\mathfrak{G})$ and $\mathfrak{p}_4(b) + \mathfrak{l}_4(b) \in \mathfrak{Z}(\mathfrak{G})$ for all $a \in \mathrm{A}$ and $b \in \mathrm{B}$. Further, we have $\mathfrak{l}_1(mn) = \zeta^{-1}(\mathfrak{q}_1(mn)) = \mathfrak{p}_4(nm)$ and $\mathfrak{l}_4(nm) = \zeta(\mathfrak{p}_4(nm)) = \mathfrak{q}_1(mn)$ for all $m \in \mathrm{M}$ and $n \in \mathrm{N}$. $\square$

The following example shows that the restrictions in Theorem 3 are necessary and cannot be removed.

Example 1. Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B})$ be a generalized matrix algebra where $\mathrm{A} = \mathrm{B} = \left\{ \begin{bmatrix} p & a \\ 0 & p \end{bmatrix} \middle| \, p, a \in \mathfrak{R} \right\}$ and $\mathrm{N} = 0,\ \mathrm{M} = \mathfrak{T}_2(\mathfrak{R})$, the ring of 2×2 upper triangular matrices over $\mathfrak{R}$. Define a map $\mathrm{L}_{\mathfrak{g}} : \mathfrak{G} \to \mathfrak{G}$ such that

$$\mathrm{L}_{\mathfrak{g}}\left(\begin{bmatrix} p & a & u & v \\ 0 & p & 0 & w \\ 0 & 0 & q & b \\ 0 & 0 & 0 & q \end{bmatrix}\right) = \mathrm{L}\left(\begin{bmatrix} p & a & u & v \\ 0 & p & 0 & w \\ 0 & 0 & q & b \\ 0 & 0 & 0 & q \end{bmatrix}\right) = \begin{bmatrix} 0 & b & w & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathrm{L}_{\mathfrak{g}}$ is a generalized Lie derivation associated with Lie derivation $\mathrm{L}:\mathfrak{G}\to\mathfrak{G}$. But $\mathrm{L}_{\mathfrak{g}}$ has no standard form. For any $x,y\in\mathfrak{G}$, it can be easily seen that $\mathrm{L}_{\mathfrak{g}}([x,y])=[\mathrm{L}_{\mathfrak{g}}(x),y]+[x,\mathrm{L}(y)]$. This implies that $\mathrm{L}_{\mathfrak{g}}$ is a generalized Lie derivation with associated Lie derivation L on $\mathfrak{G}$. Note that

$$\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))=\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))=\left\{\begin{bmatrix} t & 0\\ 0 & t\end{bmatrix}\;\middle|\; t\in\mathfrak{R}\right\}.$$

Since $\pi_{\mathrm{A}}(\mathrm{L}_{\mathfrak{g}}(\mathrm{B}))=\begin{bmatrix} 0 & v\\ 0 & 0\end{bmatrix}\notin\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ and hence in view of Theorem 3, $\mathrm{L}_{\mathfrak{g}}$ has no standard form.

3. Applications

In particular, if $\mathrm{N}=0$ in Theorem 3, then we can have following result:

Corollary 1. *Let $\mathfrak{T}=Tri(\mathrm{A},\mathrm{M},\mathrm{B})$ be a 2-torsion free triangular algebra over a commutative ring $\mathfrak{R}$. Then generalized Lie derivation $\mathrm{L}_{\mathfrak{g}}$ has standard form if and only if $\pi_{\mathrm{A}}(\mathrm{L}_{\mathfrak{g}}(\mathrm{B}))\subseteq\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T}))$, and $\pi_{\mathrm{B}}(\mathrm{L}_{\mathfrak{g}}(\mathrm{A}))\subseteq\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{T}))$ where A and B is treated as subalgebras of $\mathfrak{T}$.*

4. For future research

Now we would like to end this article by proposing several potential questions. Given the consideration of Lie derivations and Lie triple derivations, now we discuss a more general class of maps. Define the sequence of polynomials:

$$\begin{aligned}
&\mathfrak{p}_1(x_1)=x_1\\
&\mathfrak{p}_2(x_1,x_2)=[\mathfrak{p}_n(x_1),x_2]=[x_1,x_2]\\
&\mathfrak{p}_3(x_1,x_2,x_3)=[\mathfrak{p}_n(x_1,x_2),x_3]=[[x_1,x_2],x_3]\\
&\vdots\\
&\mathfrak{p}_n(x_1,x_2,x_3,\cdots,x_n)=[\mathfrak{p}_n(x_1,x_2,x_3,\cdots,x_{n-1}),x_n].
\end{aligned}$$

The polynomial $\mathfrak{p}_n(x_1,x_2,x_3,\cdots,x_n)$ is called n-th commutator where $n\geqslant 2$. A map $\mathrm{L}:\mathrm{A}\to\mathrm{A}$ is said to be a Lie n-derivation on A if

$$\begin{aligned}
&\mathrm{L}(\mathfrak{p}_n(x_1,x_2,x_3,\cdots,x_n))\\
&\qquad=\sum_{i=1}^{i=n}\mathfrak{p}_n(x_1,x_2,x_3,\cdots,x_{i-1},\mathrm{L}(x_i),x_{i+1},\cdots,x_n)
\end{aligned}$$

for all $x_1, x_2, x_3, \cdots, x_n \in \mathrm{A}$. The concept of Lie n-derivation was first introduced by Abdullaev [1] on certain von Neumann algebras. Obviously, any Lie 2-derivation is Lie derivation and Lie 3-derivation is Lie triple derivation. Lie derivations, Lie triple derivations and Lie n-derivations are collectively referred to as Lie-type derivations.

Further, A map $\mathrm{G_L} : \mathrm{A} \to \mathrm{A}$ is said to be a generalized Lie n-derivation on A if there exists a Lie n-derivation L such that

$$\begin{aligned} &\mathrm{G_L}(\mathfrak{p}_n(x_1, x_2, x_3, \cdots, x_n)) \\ &\quad = \mathfrak{p}_n(\mathrm{G_L}(x_1), x_2, x_3, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) \\ &\qquad + \sum_{i=2}^{i=n} \mathfrak{p}_n(x_1, x_2, x_3, \cdots, x_{i-1}, \mathrm{L}(x_i), x_{i+1}, \cdots, x_n) \end{aligned}$$

for all $x_1, x_2, x_3, \cdots, x_n \in \mathrm{A}$. Obviously, any generalized Lie 2-derivation is generalized Lie derivation and generalized Lie 3-derivation is generalized Lie triple derivation. These type of Lie derivations, Lie triple derivations and Lie n-derivation collectively known as generalized Lie-type derivations.

Recently, many authors studied Lie n-derivation on various kind of algebras [2, 4, 7, 11, 13] and references therein. In the year 2014, Wang and Wang [13] studied multiplicative Lie n-derivation on generalized matrix algebras and proved that it has standard form under certain assumptions. Furthermore, Qi [11] characterized Lie n-derivation on reflexive algebras and obtained that it has the standard form, i.e, it can be expressed as the sum of derivation and linear functional vanishing at every $(n-1)$-th commutator on reflexive algebras. Lin [7] carried out the study of multiplicative generalized Lie n-derivation on triangular algebras and proved that every multiplicative Lie n-derivation can be written as sum of additive generalized derivation and a central mapping annihilating $(n-1)$-th commutator on triangular algebras under some limitations. Now it is natural to raise a question:

Question 1. *What is the most general form of generalized Lie type derivations on generalized matrix algebras and which constraints are needed to apply on generalized matrix algebras?*

Conclusions

In this article, we realize the structure of generalized Lie derivations on generalized matrix algebras. Further, we demonstrate that generalized Lie derivations has a proper form under specific restrictions on generalized matrix algebras and moreover we come up with an example showing that these specific restrictions are necessary. For an immediate outcome of results, we take a particular case of generalized matrix algebras named as triangular algebras into the consideration. In the end of this article, we try to draw the attention of readers towards the obvious queries related to the theme of article.

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