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A note on modular group algebras with upper Lie nilpotency indices

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Abstract. Let KG be the modular group algebra of an arbitrary group G over a field K of characteristic $p > 0$. In this paper we give some improvements of upper Lie nilpotency index $t^L(KG)$ of the group algebra KG. It can be seen that if KG is Lie nilpotent, then its lower as well as upper Lie nilpotency index is at least $p+1$. In this way the classification of group algebras KG with next upper Lie nilpotency index $t^L(KG)$ upto $9p-7$ have already been classified. Furthermore, we give a complete classification of modular group algebra KG for which the upper Lie nilpotency index is $10p-8$.

1. Introduction

Let KG be the group algebra of a group G over a field K of characteristic $p > 0$. The group algebra KG can be regarded as a associated Lie algebra of KG, via the Lie commutator $[x, y] = xy - yx$, $\forall x, y \in KG$. Set $[x_1, x_2, ...x_n] = [[x_1, x_2, ...x_{n-1}], x_n]$, where $x_1, x_2, ...x_n \in KG$. The n^{th} lower Lie power $KG^{[n]}$ of KG is the associated ideal generated by the Lie commutators $[x_1, x_2, ...x_n]$, where $KG^{[1]} = KG$. By induction, the n^{th} upper Lie power $KG^{(n)}$ of KG is the associated ideal generated by all the Lie commutators $[x, y]$, where $x \in KG^{(n-1)}$, $y \in KG$ and $KG^{(1)} = KG$. KG is said to be upper Lie nilpotent (lower Lie nilpotent) if there exists m

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such that $KG^{(m)} = 0\,(KG^{[m]} = 0)$. The minimal non-negative integer m such that $KG^{(m)} = 0$ and $KG^{[m]} = 0$ is known as the upper Lie nilpotency index and lower Lie nilpotency index of KG, denoted by $t^L(KG)$ and $t_L(KG)$ respectively. It is well known that, if KG is Lie nilpotent, then $p+1 \leqslant t_L(KG) \leqslant |G'|+1$ (see [21, 23]). According to Bhandari and Passi [1], if $p > 3$ then $t^L(KG) = t_L(KG)$. In this direction a recent result can be seen in [17]. The subgroup $D_{(m),K}(G) = G \cap (1 + KG^{(m)}), m \geqslant 1$ is called the m^{th} Lie dimension subgroup of G and by Passi [11], we have

$$D_{(m),K}(G) = \prod_{(i-1)p^j \geqslant m-1} \gamma_i(G)^{p^j}.$$

Let $p^{d_{(m)}} = |D_{(m),K}(G) : D_{(m+1),K}(G)|$, $m \geqslant 2$. If KG is Lie nilpotent such that $|G'| = p^n$, then according to Jenning's theory [20], we have $t^L(KG) = 2 + (p-1)\Sigma_{m\geqslant 1}\, md_{(m+1)}$ and $\Sigma_{m\geqslant 2}\, d_{(m)} = n$. Shalev [19] initiated the study of group algebras with maximum Lie nilpotency index. This problem was completed by [6]. Results on the next smaller Lie nilpotency index can be easily seen in [4–7]. In [3], Bovdi and Kurdics discussed the upper and lower Lie nilpotency index of a modular group algebra of metabelian group G and determine the nilpotency class of the group of units. Recently, we have some results on classification of Lie nilpotent group algebras of Lie nilpotency index upto 14 (see [2, 8, 22, 24]). Furthermore, group algebras with minimal Lie nilpotency index $p+1$ have been classified by Sharma and Bist [21]. A complete description of the Lie nilpotent group algebras with next possible nilpotency indices $2p$, $3p-1$, $4p-2$, $5p-3$, $6p-4$, $7p-5$, $8p-6$ and $9p-7$ is given in [13–16, 18]. In this article, we will classify group algebras with upper Lie nilpotency index $10p-8$. For a prime p and positive integer x, $\vartheta_{p'}(x)$ is the maximal divisor of x which is relatively prime to p. Also $S(n,m)$ denotes the small group number m of order n from the Small Group Library-Gap [9]. We use the following lemma throughout our paper.

2. Preliminaries

Lemma 1. *([19]) Let K be a field with $Char K = p > 0$ and G be a nilpotent group such that $|G'| = p^n$ and $exp(G') = p^l$.*

1) *If $d_{(l+1)} = 0$ for some $l < pm$, then $d_{(pm+1)} \leqslant d_{(m+1)}$.*
2) *If $d_{(m+1)} = 0$, then $d_{(s+1)} = 0$ for all $s \geqslant m$ with $\vartheta_{p'}(s) \geqslant \vartheta_{p'}(m)$ where $\vartheta_{p'}(x)$ is the maximal divisor of x which is relatively prime to p.*

3. Main Results

Theorem 1. *Let G be a group and K be a field of characteristics $p > 0$ such that KG is Lie nilpotent. Then $t^L(KG) = 10p - 8$ if and only if one of the following condition satisfied:*

1) $G' \cong C_{7^2} \times (C_7)^2$ *and* $\gamma_3(G) \subseteq G'^7$;
2) $G' \cong C_{7^2} \times C_7$, $\gamma_3(G) \cong C_7$ *and* $|\gamma_3(G) \cap G'^7| = 1$;
3) $G' \cong C_{7^2} \times C_7$, $\gamma_4(G) \subseteq G'^7 \subseteq \gamma_3(G) \cong (C_7)^2$ *and* $\gamma_5(G) = 1$;
4) $G' \cong C_{5^2} \times (C_5)^4$, $G'^5 \subseteq \gamma_3(G)$ *and* $\gamma_4(G) = 1$;
5) $G' \cong (C_5)^6$, $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_5$ *and* $\gamma_4(G) = 1$;
6) $G' \cong (C_{5^2})^2 \times C_5$ *and* $\gamma_3(G) \subseteq G'^2$;
7) $G' \cong C_{5^2} \times (C_5)^3$, *either* $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_5$ *or* $G'^5 \subseteq \gamma_3(G) \cong (C_5)^2$;
8) $G' \cong (C_5)^5$, $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_5)^2$ *and* $\gamma_4(G) = 1$;
9) G' *is one of the groups* $S(3125, 2)$, $S(3125, 40)$, $S(3125, 41)$, $S(3125, 42)$, $S(3125, 43)$, $S(3125, 44)$, $S(3125, 73)$ *or* $S(3125, 74)$, $G'^5 \subseteq \zeta(G')$, $G'' \subseteq \zeta(G')$, $G'^5 \subseteq \gamma_3(G) \cong (C_5)^2$, $\gamma_4(G) \cong C_5$ *and* $\gamma_5(G) = 1$;
10) $G' \cong C_{5^2} \times (C_5)^2$, *either* $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_5)^2$ *or* $G'^5 = \gamma_3(G) \cong C_5$ *or* $G'^5 \subseteq \gamma_3(G) \cong (C_5)^3$;
11) $G' \cong C_8 \times (C_2)^3$, $\gamma_3(G) \subseteq G'^2$, $\gamma_3(G) \cong C_4$ *and* $\gamma_4(G) = 1$;
12) $G' \cong (C_4)^2 \times (C_2)^2$, $\gamma_3(G) \subseteq G'^2$ *and* $\gamma_4(G) = 1$;
13) $G' \cong C_4 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong C_4$ *and* $\gamma_4(G) = 1$;
14) G' *is one of the groups* $S(64, 199)$ *to* $S(64, 201)$ *or* $S(64, 215)$ *to* $S(64, 245)$, $\gamma_3(G) \subseteq G'^2$ *and* $\gamma_4(G) \cong C_2$;
15) G' *is one of the groups* $S(64, 264)$ *or* $S(64, 265)$, *either* $G'^2 \subseteq \gamma_3(G) \cong C_4$ *or* $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$;
16) G' *is one of the groups* $S(64, 247)$ *or* $S(64, 248)$, $G'^2 = \gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$ *and* $\gamma_5(G) = 1$;
17) $G' \cong S(64, 263)$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$ *and* $\gamma_5(G) = 1$;
18) $G' \cong C_8 \times C_4$, *either* $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$ *or* $\gamma_3(G) \subseteq G'^2$, $\gamma_3(G) \cong C_4$;
19) $G' \cong C_8 \times (C_2)^2$ *and* $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$;
20) $G' \cong (C_4)^2 \times C_2$, *either* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$ *or* $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_3(G) \cong C_4 \times C_2$;
21) $G' \cong C_4 \times (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$ *and* $\gamma_3(G) \cong C_4 \times C_2$;
22) G' *is one of the groups* $S(32, 4)$, $S(32, 5)$ *or* $S(32, 12)$, $\gamma_3(G) \subseteq G'^2 \cong C_4 \times C_2$, $\gamma_4(G) \subseteq G'^4\gamma_3(G)^2 \cong C_2$ *and* $\gamma_5(G) = 1$;

23) *G' is one of the groups $S(32,22)$ to $S(32,26)$, either $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$ or $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \subseteq G'^4\gamma_3(G)^2 \cong C_2$, $\gamma_5(G) = 1$;*
24) *G' is one of the groups $S(32,37)$ or $S(32,38)$, either $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$ or $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \subseteq G'^4\gamma_3(G)^2 \cong C_2$, $\gamma_5(G) = 1$;*
25) *G' is one of the groups $S(32,46)$ to $S(32,48)$, either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_4$ or $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \subseteq G'^4\gamma_3(G)^2 \cong C_2$, $\gamma_5(G) = 1$;*
26) *$G' \cong (C_p)^4$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong (C_p)^2$ and $\gamma_5(G) \cong C_p$ for $p \geqslant 5$;*
27) *$G' \cong C_9 \times (C_3)^2$, either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^2$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$;*
28) *$G' \cong (C_3)^4$, $|G'^3 \cap \gamma_3(G)| = 1$ and $\gamma_3(G) \cong (C_3)^3$;*
29) *$G' \cong C_8 \times C_2$, either $\gamma_3(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$ or $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$;*
30) *$G' \cong C_4 \times (C_2)^2$ and $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$;*
31) *$G' \cong ((C_p \times C_p) \rtimes C_p) \times C_p$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong (C_p)^2$, $\gamma_5(G) \cong C_p$ and $\gamma_6(G) = 1$ for $p \geqslant 5$;*
32) *$G' \cong (C_9)^2$ and $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$;*
33) *$G' \cong C_9 \times (C_3)^2$, either $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^2$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$;*
34) *$G' \cong (C_3)^4$, $|G'^3 \cap \gamma_3(G)| = 1$ and $\gamma_3(G) \cong (C_3)^3$;*
35) *$G' \cong (C_p)^5$, $\gamma_3(G) \cong (C_p)^3$ and $|G'^p \cap \gamma_3(G)| = 1$ for $p \geqslant 5$;*
36) *$G' \cong (C_9)^2 \times C_3$ and $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$;*
37) *$G' \cong C_9 \times (C_3)^3$, either $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^2$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$;*
38) *$G' \cong (C_3)^5$, $|G'^3 \cap \gamma_3(G)| = 1$ and $\gamma_3(G) \cong (C_3)^3$;*
39) *$G' \cong (C_4)^2 \times C_2$, either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$ or $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$;*
40) *$G' \cong C_4 \times (C_2)^3$, either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ or $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$;*
41) *$G' \cong (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 1$ and $\gamma_3(G) \cong (C_2)^3$;*
42) *$G' \cong S(32,2)$, $\gamma_3(G) \subseteq G'^2$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$;*
43) *G' is one of the groups $S(32,22)$ to $S(32,26)$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$, $|G'^2 \cap \gamma_3(G)| = 2$ and $\gamma_3(G) \cong (C_2)^2$;*
44) *G' is one of the groups $S(32,46)$ to $S(32,48)$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$, either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$ or $G'^2 \subseteq \gamma_3(G) \cong (C_3)^3$;*

45) *G' is one of the groups $S(243,2)$, $S(243,33)$, $S(243,34)$ or $S(243,36)$, either $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$ or $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;*
46) *$G' \cong \langle a, b, c, d, e\rangle = \langle c, d\rangle \times \langle a, b\rangle$, where $\langle c, d\rangle \cong C_p \times C_p$ and $\langle a, b, e | = a^p = b^p = e^p = 1, [b, a] = e\rangle$ is abelian group of order p^3 and exponent p, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong C_p$ and $\gamma_5(G) = 1$ for $p \geqslant 5$;*
47) *$G' \cong (C_9)^2 \times (C_3)^2$, $\gamma_3(G) \subseteq G'^3$ and $\gamma_4(G) = 1$;*
48) *$G' \cong C_9 \times (C_3)^4$, either $\gamma_3(G) \cong C_3$, $\gamma_4(G) = 1$, $|G'^3 \cap \gamma_3(G)| = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) = 1$;*
49) *G' is one of the groups $S(729,422)$ or $S(729,502)$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $|G'^3 \cap \gamma_4(G)| = 1$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$;*
50) *G' is one of the groups $S(729,423)$ or $S(729,424)$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$;*
51) *G' is one of the groups $S(729,103)$, $S(729,105)$, $S(729,417)$, $S(729,418)$, $S(729,420)$ or $S(729,421)$, $G'^3 = \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$;*
52) *G' is one of the groups $S(729,416)$, $S(729,419)$, $S(729,499)$ or $S(729,500)$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$;*
53) *$G' \cong C_9 \times (C_3)^6$, $\gamma_3(G) \subseteq G'^3 \cong C_3$ and $\gamma_4(G) = 1$;*
54) *$G' \cong (C_4)^2 \times (C_2)^3$ and $\gamma_3(G) \cong G'^2$;*
55) *$G' \cong C_4 \times (C_2)^5$, either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$;*
56) *$G' \cong C_9 \times (C_3)^5$, either $\gamma_3(G) \cong C_3$, $|G'^3 \cap \gamma_3(G)| = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$;*
57) *$G' \cong (C_p)^7$, $|G'^p \cap \gamma_3(G)| = 1$ and $\gamma_3(G) \cong (C_p)^2$ for $p \geqslant 5$;*
58) *G' is one of the groups $S(128,2157)$ to $S(128,2162)$ or $S(128,2304)$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$;*
59) *G' is one of the groups $S(128,2323)$ to $S(128,2325)$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$;*
60) *G' is one of the groups $S(128,2151)$ to $S(128,2156)$, $S(128,2302)$ or $S(128,2303)$, $G'^2 = \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$;*
61) *G' is one of the groups $S(128,2320)$ to $S(128,2322)$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$;*
62) *G' is one of the groups $S(2187,5874)$, $S(2187,5876)$, $S(2187,9102)$ to $S(2187,9105)$, $G'^3 = \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$;*
63) *G' is one of the groups $S(2187,9100)$, $S(2187,9101)$, $S(2187,9306)$ or $S(2187,9307)$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$;*
64) *G' is one of the groups $S(2187,5867)$, $S(2187,5870)$, $S(2187,5872)$ or $S(2187,9096)$ to $S(2187,9099)$, $G'^3 = \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$;*

65) G' *is one of the groups* $S(2187, 9094)$, $S(2187, 9095)$, $S(2187, 9303)$ *or* $S(2187, 9304)$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ *and* $\gamma_5(G) = 1$;
66) $G' \cong \langle a, b, c, d, e, f, g : a^p = b^p = c^p = d^p = e^p = f^p = g^p = 1, [b, a] = c \rangle$, $\gamma_3(G) \cong (C_p)^2$, $\gamma_4(G) \cong C_p$ *and* $\gamma_5(G) = 1$ *for* $p \geqslant 3$;
67) $G' \cong \langle a, b, c, d, e, f, g : a^p = b^p = c^p = d^p = e^p = f^p = g^p = 1, [b, a] = e, [d, c] = e \rangle$, $\gamma_3(G) \cong (C_p)^2$, $\gamma_4(G) \cong C_p$ *and* $\gamma_5(G) = 1$ *for* $p \geqslant 3$;
68) $G' \cong (C_4)^3$, $\gamma_3(G) \subseteq G'^2$ *and* $\gamma_4(G) = 1$;
69) $G' \cong (C_4)^2 \times (C_2)^2$, *either* $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^2$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$;
70) $G' \cong (C_2)^6$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$ *and* $\gamma_4(G) = 1$;
71) $G' \cong C_9 \times (C_3)^4$, *either* $\gamma_3(G) \cong (C_3)^2$, $|G'^2 \cap \gamma_3(G)| = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$;
72) $G' \cong (C_p)^6$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong C_p$ *and* $\gamma_5(G) = 1$ *for* $p \geqslant 5$;
73) G' *is one of the groups* $S(64, 199)$ *to* $S(64, 201)$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ *and* $\gamma_5(G) = 1$;
74) G' *is one of the groups* $S(64, 264)$ *or* $S(64, 265)$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ *and* $\gamma_5(G) = 1$;
75) G' *is one of the groups* $S(64, 56)$ *to* $S(64, 59)$, $\gamma_3(G) \subseteq G'^2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ *and* $\gamma_5(G) = 1$;
76) G' *is one of the groups* $S(64, 193)$ *to* $S(64, 198)$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ *and* $\gamma_5(G) = 1$;
77) G' *is one of the groups* $S(64, 56)$ *to* $S(64, 59)$, $\gamma_3(G) \subseteq G'^2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$ *and* $\gamma_5(G) = 1$;
78) G' *is one of the groups* $S(64, 193)$ *to* $S(64, 198)$ *and* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$;
79) G' *is one of the groups* $S(729, 103)$ *to* $S(729, 106)$, $S(729, 416)$ *to* $S(729, 420)$, $S(729, 499)$ *or* $S(729, 500)$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$ *and* $\gamma_5(G) = 1$;
80) G' *is one of the groups* $S(729, 103)$, $S(729, 105)$, $S(729, 417)$, $S(729, 418)$, $S(729, 420)$ *or* $S(729, 421)$, $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ *and* $\gamma_5(G) = 1$;
81) G' *is one of the groups* $S(729, 104)$ *or* $S(729, 106)$, $\gamma_3(G) \subseteq G'^2$, $\gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ *and* $\gamma_5(G) = 1$;
82) G' *is one of the groups* $S(729, 416)$, $S(729, 419)$, $S(729, 499)$ *or* $S(729, 500)$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$ *and* $\gamma_5(G) = 1$;
83) $G' \cong \phi_2(1^5) \times (1)$, $\gamma_3(G) \cong (C_p)^3$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_p$ *and* $\gamma_5(G) = 1$ *for* $p \geqslant 5$;

84) $G' \cong (C_4)^2 \times C_2$, *either* $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$ *or* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$;
85) $G' \cong C_4 \times (C_2)^3$, *either* $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$;
86) $G' \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$ *and* $\gamma_3(G) \cong (C_2)^4$;
87) $G' \cong C_9 \times (C_3)^3$, *either* $\gamma_3(G) \cong (C_3)^3$, $|G'^3 \cap \gamma_3(G)| = 1$ *or* $\gamma_3(G) \cong (C_3)^4$, $G'^3 \subseteq \gamma_3(G)$;
88) $G' \cong (C_p)^5$, $\gamma_3(G) \cong (C_p)^4$ *and* $|G'^p \cap \gamma_3(G)| = 1$ *for* $p \geqslant 5$;
89) $G' \cong S(32, 2)$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$ *and* $\gamma_5(G) = 1$;
90) $G' \cong S(243, 32)$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$ *and* $\gamma_4(G) \cong C_3$;
91) $G' \cong (C_p)^{10}$, $\gamma_3(G) = 1$ *and* $|G'^3 \cap \gamma_4(G)| = 1$ *for* $p > 0$;
92) $G' \cong (C_p)^9$, $\gamma_3(G) \cong C_p$, $|G'^p \cap \gamma_3(G)| = 1$ *and* $\gamma_4(G) = 1$ *for* $p \geqslant 3$;
93) $G' \cong C_4 \times (C_2)^7$, $\gamma_3(G) \subseteq G'^2 \cong C_2$ *and* $\gamma_4(G) = 1$;
94) $G' \cong (C_2)^9$, $\gamma_3(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$ *and* $\gamma_4(G) = 1$;
95) $G' \cong (C_p)^8$, $\gamma_3(G) \cong (C_p)^2$, $|G'^p \cap \gamma_3(G)| = 1$ *and* $\gamma_4(G) = 1$ *for* $p \geqslant 3$;
96) $G' \cong (C_2)^8$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$ *and* $\gamma_4(G) = 1$ *for* $p \geqslant 5$;
97) $G' \cong C_4 \times (C_2)^6$, *either* $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$ *for* $p \geqslant 5$;
98) $G' \cong (C_4)^2 \times (C_2)^4$, $\gamma_3(G) \subseteq G'^2$ *and* $\gamma_4(G) = 1$;
99) $G' \cong (C_p)^7$, $\gamma_3(G) \cong C_p \times C_p \times C_p$ *and* $|G'^p \cap \gamma_3(G)| = 1$ *for* $p \geqslant 3$;
100) $G' \cong (C_4)^3 \times C_2$, $\gamma_3(G) \subseteq G'^2$ *and* $\gamma_4(G) = 1$;
101) $G' \cong (C_4)^2 \times (C_2)^3$, *either* $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$ *or* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_2 \times C_2$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$;
102) $G' \cong C_4 \times (C_2)^5$, *either* $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$;
103) $G' \cong (C_2)^7$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$ *and* $\gamma_4(G) = 1$;
104) $G' \cong (C_p)^6$, $|G'^p \cap \gamma_3(G)| = 1$ *and* $\gamma_3(G) \cong (C_p)^4$ *for* $p \geqslant 3$;
105) $G' \cong (C_4)^3$, $\gamma_3(G) \subseteq G'^2$ *and* $\gamma_4(G) = 1$;
106) $G' \cong (C_4)^2 \times (C_2)^2$, *either* $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$ *or* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$;
107) $G' \cong C_4 \times (C_2)^4$, *either* $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$;
108) $G' \cong (C_2)^6$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^4$ *and* $\gamma_4(G) = 1$.

Proof. Let $t^L(KG) = 10p - 8$. Then $l = \frac{t^L(KG)-2}{p-1} = 10$. Thus from [15], $d_{(11)} = 0$, $d_{(10)} = 0$, $d_{(9)} = 0$ and $d_{(8)} \neq 0$ if and only if $p = 7$, $G' \cong$

$C_{7^2} \times (C_7)^2$, $\gamma_3(G) \subseteq G'^7$ or $G' \cong C_{7^2} \times C_7$, $\gamma_3(G) \cong C_7$, $|\gamma_3(G) \cap G'^7| = 1$ or $G' \cong C_{7^2} \times C_7$, $\gamma_4(G) \subseteq G'^7 \subseteq \gamma_3(G) \cong (C_7)^2$, $\gamma_5(G) = 1$.

Now if $\mathbf{d_8 = 0}$, then $d_{(2)} + 2d_{(3)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} + 6d_{(7)} = 10$. If $d_7 \neq 0$, then we have $d_{(7)} = 1$, $d_{(2)} = 4$ or $d_{(7)} = d_{(3)} = 1$, $d_{(2)} = 2$ or $d_{(7)} = d_{(2)} = d_{(4)} = 1$ or $d_{(7)} = 1$, $d_{(3)} = 2$ or $d_{(7)} = d_{(5)} = 1$.

If $\mathbf{d_{(7)} = 1}$, then all the above cases are discarded by Lemma 1.

Now if $\mathbf{d_{(7)} = 0}$, then $d_{(2)} + 2d_{(3)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} = 10$. If $d_{(6)} \neq 0$, then we have the following possibilities: $d_{(6)} = d_{(2)} = d_{(5)} = 1$ or $d_{(6)} = 1$, $d_{(2)} = 5$ or $d_{(6)} = d_{(3)} = 1$, $d_{(2)} = 3$ or $d_{(6)} = d_{(4)} = 1$, $d_{(2)} = 2$ or $d_{(6)} = d_{(2)} = 1$, $d_{(3)} = 2$ or $d_{(6)} = d_{(3)} = d_{(4)} = 1$.

Let $\mathbf{d_{(6)} = d_{(2)} = d_{(5)} = 1}$. Then by Lemma 1(2), $\vartheta_{p'}(4) \geqslant \vartheta_{p'}(2)$, $\forall p > 0$, so $d_{(5)} = 0$.

Let $\mathbf{d_{(6)} = 1, d_{(2)} = 5}$. If $p \neq 5$, then as $d_{(2+1)} = 0$, $\vartheta_{p'}(5) \geqslant \vartheta_{p'}(2)$, hence by Lemma 1(2), $d_{(6)} = 0$. Now if $p = 5$, then $|G'| = 5^6$, $|D_{(6),K}(G)| = |D_{(3),K}(G)| = |D_{(4),K}(G)| = |D_{(5),K}(G)| = 5$. Therefore, G' is abelian and $|G'^5| \leqslant 5$. We have $G' \cong C_{5^2} \times (C_5)^4$ or $(C_5)^6$. If $G' \cong C_{5^2} \times (C_5)^4$, then $G'^5 \subseteq \gamma_3(G)$, $\gamma_4(G) = 1$. If $G' \cong (C_5)^6$, then $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_5$.

Let $\mathbf{d_{(6)} = d_{(3)} = 1, d_{(2)} = 3}$. If $p \neq 5$, then by Lemma 1(2), $d_{(4+1)} = 0$, $\vartheta_{p'}(5) \geqslant \vartheta_{p'}(4)$, so $d_{(6)} = 0$. Now if $p = 5$, then $|G'| = 5^5$, $|D_{(4),K}(G)| = |D_{(5),K}(G)| = |D_{(6),K}(G)| = 5$, $|D_{(3),K}(G)| = 5^2$ and $|G'^5| = 5$. Thus $\gamma_5(G) = 1$, $|\gamma_4(G)| = 5$ and $|\gamma_3(G)| = 5^2$. Let G' be an abelian group, then possible G' are $G' \cong (C_{5^2})^2 \times C_5$ or $C_{5^2} \times (C_5)^3$ or $(C_5)^5$. If $G' \cong (C_{5^2})^2 \times C_5$, then $\gamma_3(G) \subseteq G'^2$. If $G' \cong C_{5^2} \times (C_5)^3$, then either $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_5$ or $G'^5 \subseteq \gamma_3(G) \cong (C_5)^2$. If $G' \cong (C_5)^5$, then $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_5)^2$. Now if G' be a non-abelian group, then $G'' = \gamma_4(G) \cong C_5$, $\gamma_3(G) \cong (C_5)^2$ and $\gamma_3(G) \subseteq \zeta(G')$. Thus $|\zeta(G')| = 5^2$ or 5^3. If $|\zeta(G')| = 5^2$, then $\gamma_3(G) = \zeta(G') \cong (C_5)^2$ but from the Table 1, no such group exists. If $|\zeta(G')| = 5^3$, then $\zeta(G') \cong C_{5^2} \times C_5$ or $(C_5)^3$. Thus possible G' are $S(3125, 2)$, $S(3125, 40)$, $S(3125, 41)$, $S(3125, 42)$, $S(3125, 43)$, $S(3125, 44)$, $S(3125, 73)$ and $S(3125, 74)$ and for all these groups $G'^5 \subseteq \zeta(G')$, $G'' \subseteq \zeta(G')$ and $G'^5 \subseteq \gamma_3(G) \cong (C_5)^2$.

Let $\mathbf{d_{(6)} = d_{(4)} = 1, d_{(2)} = 2}$. Then by Lemma 1, this case is not possible.

Let $\mathbf{d_{(6)} = d_{(2)} = 1, d_{(3)} = 2}$. If $p \neq 5$, then by Lemma 1(2), $d_{(3+1)} = 0$, $\vartheta_{p'}(5) \geqslant \vartheta_{p'}(3)$, so $d_{(6)} = 0$. If $p = 5$, then $|G'| = 5^4$, $|D_{(4),K}(G)| = |D_{(5),K}(G)| = |D_{(6),K}(G)| = 5$, $|D_{(3),K}(G)| = 5^3$ and $|G'^5| \leqslant 5$. Thus $\gamma_5(G) = 1$, $|\gamma_4(G)| = 5$ and $|\gamma_3(G)| = 5^2$ or 5^3. Let G' be an abelian group. Then, possible G' are, $(C_{5^2})^2$ or $C_{5^2} \times (C_5)^2$. If $G' \cong (C_{5^2})^2$, then $|D_{(3),K}(G)| \neq 125$. If $G' \cong C_{5^2} \times (C_5)^2$, then either $|G'^5 \cap \gamma_3(G)| = 1$,

$\gamma_3(G) \cong (C_5)^2$ or $G'^5 = \gamma_3(G) \cong C_5$ or $G'^5 \subseteq \gamma_3(G) \cong (C_5)^3$. Now let G' be a non-abelian group. Then $G'' = \gamma_4(G) \cong C_5$ and $\gamma_3(G) \subseteq \zeta(G')$. If $\gamma_3(G) \cong (C_5)^2$, then $\gamma_3(G) = \zeta(G') \cong (C_5)^2$, but from the Table 2 of [14], no such group exists. If $\gamma_3(G) \cong (C_5)^3$, then $|\zeta(G')| = 125$. Thus G' is abelian in this case.

Let $\mathbf{d_{(6)}} = \mathbf{d_{(3)}} = \mathbf{d_{(4)}} = \mathbf{1}$. Since $d_{(1+1)} = 0$, then by Lemma 1(2), $\vartheta_{p'}(5) \geqslant \vartheta_{p'}(1)$, so $d_{(6)} = 0$, $\forall p > 0$. Thus this case is not possible.

Now let $\mathbf{d_{(6)}} = \mathbf{0}$. If $d_{(5)} \neq 0$, then we have the following possibilities: $d_{(3)} = 1$, $d_{(5)} = 2$ or $d_{(2)} = 2$, $d_{(5)} = 2$ or $d_{(2)} = 6$, $d_{(5)} = 1$ or $d_{(4)} = 2$, $d_{(5)} = 1$ or $d_{(2)} = 4$, $d_{(3)} = d_{(5)} = 1$ or $d_{(2)} = d_{(3)} = 2$, $d_{(5)} = 1$ or $d_{(2)} = d_{(3)} = d_{(4)} = d_{(5)} = 1$ or $d_{(2)} = 3$, $d_{(4)} = d_{(5)} = 1$.

Let $\mathbf{d_{(3)}} = \mathbf{1}, \mathbf{d_{(5)}} = \mathbf{2}$. If $p \neq 2$, then by Lemma 1(2), $\vartheta_{p'}(2) \geqslant \vartheta_{p'}(1)$. So $d_{(3)} = 0$. If $p = 2$, then by Lemma 1(1), $1 = d_{(3)} \leqslant d_{(2)} = 0$, so this case is not possible.

Let $\mathbf{d_{(2)}} = \mathbf{2}, \mathbf{d_{(5)}} = \mathbf{2}$. If $p \neq 2$, then by Lemma(1.1)(2), $d_{(3+1)} = 0$, $\vartheta_{p'}(4) \geqslant \vartheta_{p'}(3)$ and thus $d_{(5)} = 0$. If $p = 2$, then by Lemma 1(1), $d_{(2+1)} = 0$, $d_{(5)} = 0$. Thus this case is not possible.

Let $\mathbf{d_{(2)}} = \mathbf{6}, \mathbf{d_{(5)}} = \mathbf{1}$. If $p \neq 2$, then by Lemma 1(2), $d_{(3)} = 0$, $\vartheta_{p'}(4) \geqslant \vartheta_{p'}(3)$ and so $d_{(5)} = 0$. If $p = 2$, then by Lemma 1(1), $d_{(2+1)} = 0$ and so $d_{(5)} = 0$. Thus this case is not possible.

Let $\mathbf{d_{(4)}} = \mathbf{2}, \mathbf{d_{(5)}} = \mathbf{1}$. Then by Lemma 1, this case is not possible.

Let $\mathbf{d_{(2)}} = \mathbf{4}, \mathbf{d_{(3)}} = \mathbf{d_{(5)}} = \mathbf{1}$. If $p \neq 2$, then by Lemma 1(2), $d_{(3+1)} = 0$, $\vartheta_{p'}(4) \geqslant \vartheta_{p'}(3)$ and so $d_{(5)} = 0$. If $p = 2$, then $|G'| = 2^6$ $|D_{(3),K}(G)| = 2^2$, $|D_{(5),K}(G)| = |D_{(4),K}(G)| = 2$. Since $D_{(6),K}(G) = G'^8\gamma_3(G)^4\gamma_4(G)^2\gamma_6(G) = 1$, thus $\gamma_4(G) \subseteq G'^4\gamma_3(G)^2 \cong C_2$, $\gamma_5(G) = 1$ and $\gamma_3(G) \subseteq \zeta(G')$ and so $G'^2 \neq 1$. First suppose that G' is an abelian group. Then possible G' are $C_8 \times (C_2)^3$ or $(C_4)^2 \times (C_2)^2$ or $C_4 \times (C_2)^4$. If $G' \cong C_8 \times (C_2)^3$, then $\gamma_3(G) \subseteq G'^2$, $\gamma_3(G) \cong C_4$. If $G' \cong (C_4)^2 \times (C_2)^2$, then $\gamma_3(G) \subseteq G'^2 \cong (C_2)^2$. If $G' \cong C_4 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong C_4$. Now let G' is a non-abelian group. Thus $G'^4\gamma_3(G)^2 = \gamma_4(G) = G'' \cong C_2$ and $|\zeta(G')| \leqslant 2^4$. Let $|\zeta(G')| = 4$. Now from the Table 1 of [2] possible G' are $S(64, 199)$ to $S(64, 201)$, $S(64, 215)$ to $S(64, 245)$, $S(64, 264)$ and $S(64, 265)$. If G' is any one of the groups $S(64, 199)$ to $S(64, 201)$ or $S(64, 215)$ to $S(64, 245)$, then $\gamma_3(G) \subseteq G'^2$. If G' is any one of the groups $S(64, 264)$ or $S(64, 265)$, then $G'^2 \subseteq \gamma_3(G) \cong C_4$ or $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$. Let $|\zeta(G')| = 8$. But from the Table 1 of [2] no group exists with $|G''| = 2$. Now let $|\zeta(G')| = 16$. From Table 1 of [2] possible G' are $S(64, 193)$ to $S(64, 198)$, $S(64, 247)$, $S(64, 248)$ and $S(64, 261)$ to $S(64, 263)$. For all these groups $G'' \subseteq G'^2 \subseteq \zeta(G')$. If G' is any one of the groups $S(64, 193)$, $S(64, 194)$, $S(64, 261)$ or $S(64, 262)$, then

$G'^4\gamma_3(G)^2 = 1$. If G' is any one of the groups $S(64, 195)$ to $S(64, 198)$, then $\zeta(G') \cong C_4 \times (C_2)^2$, $G'^2 \cong (C_2)^2$ and $G'^4\gamma_3(G)^2 = 1$. If G' is any one of the groups $S(64, 247)$ to $S(64, 248)$, then $G'^2 = \gamma_3(G) \cong C_4$. If G' is $S(64, 263)$, then $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$.

Let $\mathbf{d_{(2)} = d_{(3)} = 2, d_{(5)} = 1}$. If $p \neq 2$, then by Lemma 1(2), $d_{(3+1)} = 0, \vartheta_{p'}(4) \geqslant \vartheta_{p'}(3)$ and so $d_{(5)} = 0$. If $p = 2$, then $|G'| = 2^5$, $|D_{(3),K(G)}| = 2^3$, $|D_{(4),K}(G)| = |D_{(5),K}(G)| = 2$. Thus $\gamma_5(G) = 1, \gamma_4(G) \subseteq G'^4\gamma_3(G)^2 \cong C_2$. Let G' be abelian. Then possible G' are $C_8 \times C_4$ or $C_8 \times (C_2)^2$ or $(C_4)^2 \times C_2$ or $C_4 \times (C_2)^3$. If $G' \cong C_8 \times C_4$, then either $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$ or $\gamma_3(G) \subseteq G'^2$, $\gamma_3(G) \cong C_4$. If $G' \cong C_8 \times (C_2)^2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$. If $G' \cong (C_4)^2 \times C_2$, then either $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$ or $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_3(G) \cong C_4 \times C_2$. If $G' \cong C_4 \times (C_2)^3$, then $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4 \times C_2$.

Now let G' be a non - abelian. Then $G'^4\gamma_3(G)^2 = \gamma_4(G) = G'' \cong C_2$, $\gamma_3(G) \subseteq \zeta(G')$ and $|\zeta(G')| \leqslant 2^3$. If $|\zeta(G')| = 4$, then from the Table 2 of [22], no group exists with $|G''| = 2$. If $|\zeta(G')| = 8$, then possible G' are $S(32, 2)$, $S(32, 4)$, $S(32, 5)$, $S(32, 12)$, $S(32, 22)$ to $S(32, 26)$, $S(32, 37)$ to $S(32, 38)$ and $S(32, 46)$ to $S(32, 48)$ (see Table 2 of [22]). If G' is $S(32, 2)$, then $G'^4\gamma_3(G)^2 = 1$, which is not possible. If G' is any one of the groups $S(32, 4)$, $S(32, 5)$ or $S(32, 12)$, then $\zeta(G') \cong C_4 \times C_2$ and $\gamma_3(G) \subseteq G'^2 \cong C_4 \times C_2$. If G' is any one of the groups $S(32, 22)$ to $S(32, 26)$, then $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$ or $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$. If G' is any one of the groups $S(32, 37)$ or $S(32, 38)$, then either $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$ or $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$. If G' is any one of the groups $S(32, 46)$ to $S(32, 48)$, then either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_4$ or $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$.

Let $\mathbf{d_{(2)} = d_{(3)} = d_{(4)} = d_{(5)} = 1}$. Then $|G'| = p^4$, $|D_{(3),K}(G)| = p^3$, $|D_{(4),K}(G)| = p^2$, $|D_{(5),K}(G)| = p$ and $D_{(6),K}(G) = 1$, $\forall p > 0$. Let G' is an abelian group. If $p \geqslant 5$, then $G' \cong (C_p)^4$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong (C_p)^2$ and $\gamma_5(G) \cong C_p$. If $p = 3$, then $D_{(6),K}(G) = 1$ leads to $G'^9 = 1$, so either $G' \cong C_9 \times (C_3)^2$ or $(C_3)^4$. If $G' \cong C_9 \times (C_3)^2$, then either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^2$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$. If $G' \cong (C_3)^4$, then $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$. If $p = 2$, then $D_{(6),K}(G) = 1$ leads to $G'^8 = 1$, so $G' \cong C_8 \times C_2$ or $(C_4)^2$ or $C_4 \times (C_2)^2$. If $G' \cong C_8 \times C_2$, then either $\gamma_3(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$ or $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$. Clearly $G' \cong (C_4)^2$ is not possible as $|D_{(3),K}(G)| < 2^3$. If $G' \cong C_4 \times (C_2)^2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$. Let G' be a non-abelian and $p \geqslant 5$. Then $\gamma_6(G) = 1$, $\gamma_5(G) \cong C_p$, $\gamma_4(G) \cong (C_p)^2$, $\gamma_3(G) \cong (C_p)^3$ and $\zeta(G') \cong (C_p)^2$. Thus possible G' is $((C_p \times C_p) \rtimes C_p) \times C_p$, (see [25]). If $p = 3$, then $\gamma_5(G) = 1$, $|\gamma_4(G)| = 3$,

$|\gamma_3(G)| = 3^2$ or 3^3. Now $\gamma_3(G) \subseteq \zeta(G')$ and $|\zeta(G')| \leqslant 3^2$, so $|\gamma_3(G)| \neq 3^3$. Thus $\gamma_3(G) = \zeta(G') \cong C_3 \times C_3$ and $G'' = \gamma_4(G) \cong C_3$. Therefore, from Table 2 of [14] the possibilities for G' are $S(81, 3)$, $S(81, 4)$, $S(81, 12)$ and $S(81, 13)$, but then $|D_{(3),K}(G)| < 3^3$. If $p = 2$, then $\gamma_5(G) = 1$, $G'' = \gamma_4(G) = G'^4\gamma_3(G)^2 \cong C_2$, $\gamma_3(G) \subseteq \zeta(G')$ and $|\zeta(G')| = 2^2$. So possible G' are $S(16, 3)$, $S(16, 4)$ and $S(16, 11)$ to $S(16, 13)$. But for these groups $|D_{(3),K}(G)| \neq 2^3$ (see Table 1 of [14]).

Let $\mathbf{d_{(2)} = 3, d_{(4)} = d_{(5)} = 1}$. If $p \neq 2$, then by Lemma 1(2), $d_{(2+1)} = 0$, $\vartheta_{p'}(4) \geqslant \vartheta_{p'}(2)$ and so $d_{(5)} = 0$. If $p = 2$, then by Lemma 1(1), as $d_{(3)} = 0$, so $d_{(5)} = 0$. Thus this case is not possible.

Now let $\mathbf{d_{(5)} = 0}$. If $d_{(4)} \neq 0$, then we have the following possibilities: $d_{(4)} = 3$, $d_{(2)} = 1$ or $d_{(4)} = d_{(2)} = 2$, $d_{(3)} = 1$ or $d_{(4)} = 2$, $d_{(2)} = 4$ or $d_{(4)} = 1$, $d_{(2)} = 7$ or $d_{(4)} = d_{(3)} = 1$, $d_{(2)} = 5$ or $d_{(4)} = 1$, $d_{(2)} = 3$, $d_{(3)} = 2$ or $d_{(4)} = 1, d_{(2)} = 1$, $d_{(3)} = 3$ or $d_{(4)} = d_{(3)} = 2$.

Let $\mathbf{d_{(4)} = 3, d_{(2)} = 1}$. Now $p \neq 3$ is not possible by Lemma 1. Thus $p = 3$. Now $|G'| = 3^4$, $|D_{(4),K}(G)| = |D_{(3),K}(G)| = 3^3$, $D_{(5),K}(G) = 1$. Let G' be an abelian group, then $G' \cong (C_9)^2$ or $C_9 \times (C_3)^2$ or $(C_3)^4$. If $G' \cong (C_9)^2$, then $G'^3 = \gamma_3(G) \cong (C_3)^3$. If $G' \cong C_9 \times (C_3)^2$, then either $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^2$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$. If $G' \cong (C_3)^4$, then $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$. Now let G' be a non-abelian group. Then $G'' = \gamma_4(G) \cong C_3$, $\gamma_3(G) \subseteq \zeta(G')$. So $\gamma_3(G) = \zeta(G') \cong (C_3)^2$. Hence possible G' are $S(81, 3)$, $S(81, 4)$, $S(81, 12)$ and $S(81, 13)$. But for these groups $|D_{(3),K}(G)| < 3^3$ (see Table 2 of [14]).

Let $\mathbf{d_{(4)} = d_{(2)} = 2, d_{(3)} = 1}$. Then $|G'| = p^5$, $|D_{(3),K}(G)| = p^3$ and $|D_{(4),K}(G)| = p^2$, $\forall p > 0$. Let G' be an abelian group and $p \geqslant 5$. Now $D_{(5),K}(G) = 1$ leads to $G'^p = 1$, so $G' \cong (C_p)^5$, $\gamma_3(G) \cong (C_p)^3$ and $|G'^p \cap \gamma_3(G)| = 1$. If $p = 3$, then $G' \cong (C_9)^2 \times C_3$ or $C_9 \times (C_3)^3$ or $(C_3)^5$. If $G' \cong (C_9)^2 \times C_3$, then either $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$ or $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong (C_3)^2$ or $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3$. If $G' \cong C_9 \times (C_3)^3$, then either $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^2$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$. If $G' \cong (C_3)^5$, then $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$. If $p = 2$, then $G' \cong (C_4)^2 \times C_2$ or $C_4 \times (C_2)^3$ or $(C_2)^5$. If $G' \cong (C_4)^2 \times C_2$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$ or $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$. If $G' \cong C_4 \times (C_2)^3$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ or $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$. If $G' \cong (C_2)^5$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$. Let G' be a non - abelian group. If $p = 2$, then $D_{(5),K}(G) = G'^4\gamma_3(G)^2\gamma_5(G) = 1$ leads to $|\gamma_4(G)| = 2$ or 4 and $|\gamma_3(G)| = 4$ or 8. First, let $|\gamma_4(G)| = 2$. Then $G'' = \gamma_4(G) \cong C_2$, so $\gamma_3(G) \cong (C_2)^2$ or $(C_2)^3$ and $|\zeta(G')| = 2^2$ or 2^3. If $|\zeta(G')| = 4$, then from the Table 2 of [22], $|G''| \neq 2$. If $|\zeta(G')| = 8$, then $\zeta(G') \cong C_4 \times C_2$ or $(C_2)^3$.

Therefore possible G' are $S(32,2)$, $S(32,4)$, $S(32,5)$, $S(32,12)$, $S(32,22)$ to $S(32,26)$, $S(32,37)$ and $S(32,46)$ to $S(32,48)$ (see table 2 of [22]). If G' is any one of the groups $S(32,4)$, $S(32,5)$, $S(32,12)$ or $S(32,37)$, then $G'^4 \neq 1$. If G' is $S(32,2)$, then $\gamma_3(G) \subseteq G'^2$. If G' is any one of the groups $S(32,22)$ to $S(32,26)$, then either $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$. If G' is any one of the groups $S(32,46)$ to $S(32,48)$, then either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$. Now for $|\gamma_4(G)| = 4$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_3(G) = \zeta(G') \cong (C_2)^3$. But from the Table 2 of [22], no such group exists with $|G''| = 4$. Let $p = 3$, then $D_{(5),K}(G) = 1$ leads to $G'^9 = 1$ and $|D_{(4),K}(G)| = 3^2$ leads to $|\gamma_4(G)| = 3$ or 3^2. So $G'' = \gamma_4(G) \cong C_3$ and $\gamma_3(G) \cong (C_3)^2$ or $(C_3)^3$. Thus $|\zeta(G')| = 9$ or 27. First let $|\zeta(G')| = 9$, then from the Table 5 of [14], no such group exists. If $|\zeta(G')| = 27$, then possible G' are $S(243,2)$, $S(243,32)$ to $S(243,36)$ and $S(243,62)$ to $S(243,64)$ (see Table 5 of [14]). Now $\gamma_4(G) \subseteq G'^3\gamma_3(G)^3 \cong (C_3)^2$ and $\gamma_3(G)^3 = 1$. Hence $|G'^3| = 9$. If G' is one of the group from $S(243,32)$, $S(243,35)$ or $S(243,62)$ to $S(243,64)$, then $|G'^3| \neq 9$. If G' is any one of the groups $S(243,2)$, $S(243,33)$, $S(243,34)$ or $S(243,36)$, then either $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$ or $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong (C_3)^2$. For $|\gamma_4(G)| = 9$, $G'' = \gamma_4(G) \cong (C_3)^2$ and $\gamma_3(G) = \zeta(G') \cong (C_3)^3$. But from the Table 3 no such group exists. For $p \geqslant 5$, $\gamma_3(G) \cong (C_p)^3$ or $(C_p)^2$, $\gamma_4(G) \cong (C_p)^2$ or C_p and $\gamma_5(G) = 1$. If $G'' = \gamma_4(G) \cong (C_p)^2$ and $\gamma_3(G) \cong (C_p)^3$, then $\gamma_3(G) \subseteq \zeta(G')$ and hence $\gamma_3(G) = \zeta(G') \cong (C_p)^3$. But from [12] no such group exists. If $G'' = \gamma_4(G) \cong C_p$, $\gamma_3(G) \cong (C_p)^3$ or $(C_p)^2$ and $\gamma_5(G) = 1$, then $|\zeta(G')| = p^2$ or p^3. Let $\zeta(G') \cong (C_p)^2$, then from [12] no such group exists. Now let $\gamma_3(G) = \zeta(G') \cong (C_p)^3$, then from [12], $G' \cong \langle a,b,c,d,e \rangle = \langle c,d \rangle \times \langle a,b \rangle$, where $\langle c,d \rangle \cong C_p \times C_p$ and $\langle a,b,e | = a^p = b^p = e^p = 1, [b,a] = e \rangle$ is a non-abelian group of order p^3 and exponent p.

Let $\mathbf{d_{(4)} = 2, d_{(2)} = 4}$. If $p \neq 3$ and $d_{(3)} = 0$, then by Lemma 1(2), $\vartheta_{p'}(3) \geqslant \vartheta_{p'}(2)$, so $d_{(4)} = 0$. If $p = 3$, then $|G'| = 3^6$, $|D_{(3),K}(G)| = |D_{(4),K}(G)| = 3^2$, $D_{(5),K}(G) = 1$ and $G'^3 \neq 1$. Let G' be an abelian group. So possible G' are $(C_9)^2 \times (C_3)^2$ or $C_9 \times (C_3)^4$. If $G' \cong (C_9)^2 \times (C_3)^2$, then $\gamma_3(G) \subseteq G'^3$. If $G' \cong C_9 \times (C_3)^4$, then either $\gamma_3(G) \cong C_3$, $|G'^3 \cap \gamma_3(G)| = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$. Let G' be a non - abelian group. Now $G'' = \gamma_4(G) \cong C_3$, $\gamma_3(G) \cong (C_3)^2$ and $G'^3 \subseteq \gamma_3(G)$. Hence either $G'^3 = \gamma_3(G) \cong (C_3)^2$ or $G'^3 \cong C_3$ and $|G'^3 \cap \gamma_4(G)| = 1$. As $\gamma_3(G) \subseteq \zeta(G')$, so $|\zeta(G')| = 3^2$ or 3^3 or 3^4. If $|\zeta(G')| = 3^2$, then $\gamma_3(G) = \zeta(G') \cong (C_3)^2$. Hence from the Table 6 of [14], possible G' are $S(729,422)$ to $S(729,424)$ and $S(729,502)$. If G' is any one of the groups $S(729,422)$ or $S(729,502)$, then $G'^3 \cong C_3$, $|G'^3 \cap \gamma_4(G)| = 1$. If G' is any one of the groups $S(729,423)$ or $S(729,424)$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$. If $|\zeta(G')| = 3^3$, then from the Table 6 of [14], no such group exists. Now if $|\zeta(G')| = 3^4$, then possible G' are $S(729, 103)$, $S(729, 105)$, $S(729, 416)$ to $S(729, 421)$ and $S(729, 499)$ to $S(729, 500)$. If G' is any one of the groups $S(729, 103)$, $S(729, 105)$, $S(729, 417)$, $S(729, 418)$, $S(729, 420)$ or $S(729, 421)$, then $G'^3 = \gamma_3(G) \cong (C_3)^2$. If G' is any one of the groups $S(729, 416)$, $S(729, 419)$, $S(729, 499)$ or $S(729, 500)$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$ (see Table 6 of [14]).

Let $\mathbf{d_{(4)} = 1, d_{(2)} = 7}$. If $p \neq 3$ and $d_{(2+1)} = 0$, then by Lemma 1(2), $\vartheta_{p'}(3) \geqslant \vartheta_{p'}(2)$, so $d_{(4)} = 0$. If $p = 3$, then $|G'| = 3^8$, $|D_{(3),K}(G)| = |D_{(4),K}(G)| = 3$ and $\gamma_4(G) = 1$. Thus G' is abelian in this case. Now $|D_{(4),K}(G)| = 3$ leads to $|G'^3| = 3$. So only possible G' is $C_9 \times (C_3)^6$, $\gamma_3(G) \subseteq G'^3 \cong C_3$.

Let $\mathbf{d_{(4)} = d_{(3)} = 1, d_{(2)} = 5}$. Now $|G'| = p^7$, $|D_{(3),K}(G)| = p^2$, $|D_{(4),K}(G)| = p$, $|D_{(5),K}(G)| = 1$, for all $p > 0$. Let G' be an abelian group. If $p = 2$, then $|G'^2| = 2$ or 4. So possible G' are $(C_4)^2 \times (C_2)^3$ or $C_4 \times (C_2)^5$. If $G' \cong (C_4)^2 \times (C_2)^3$, then $\gamma_3(G) \subseteq G'^2$. If $G' \cong C_4 \times (C_2)^5$, then either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$. If $p = 3$, then $|D_{(4),K}(G)| = 3$ leads to $|G'^3| = 3$. So $G' \cong C_9 \times (C_3)^5$, either $\gamma_3(G) \cong C_3$, $|G'^3 \cap \gamma_3(G)| = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$. If $p \geqslant 5$, then $|D_{(4),K}(G)| = p$ leads to $G'^p = 1$ and $G' \cong (C_p)^7$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^2$. Let G' be a non - abelian group. Then for $p = 2$, $G'' = \gamma_4(G) \cong C_2$, $\gamma_3(G) \cong (C_2)^2$ and $G'^2 \subseteq \gamma_3(G)$. Since $\gamma_3(G) \subseteq \zeta(G')$, therefore $|\zeta(G')| \geqslant 4$. If $|\zeta(G')| = 4$ or 16, then from the Table 4 of [22] no such group exists. If $|\zeta(G')| = 8$, then possible G' are $S(128, 2157)$ to $S(128, 2162)$, $S(128, 2304)$ and $S(128, 2323)$ to $S(128, 2325)$ (see table 4 of [22]). If G' is any one of the groups $S(128, 2157)$ to $S(128, 2162)$ or $S(128, 2304)$, then $G'^2 = \gamma_3(G) \cong (C_2)^2$. If G' is any one of the groups $S(128, 2323)$ to $S(128, 2325)$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$. Let $|\zeta(G')| = 32$, then from the Table 4 of [22], possible G' are $S(128, 2151)$ to $S(128, 2156)$, $S(128, 2302)$, $S(128, 2303)$ and $S(128, 2320)$ to $S(128, 2322)$. If G' is any one of the groups $S(128, 2151)$ to $S(128, 2156)$, $S(128, 2302)$ or $S(128, 2303)$, then $G'^2 = \gamma_3(G) \cong (C_2)^2$. If G' is any one of the groups $S(128, 2320)$ to $S(128, 2322)$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$. If $p = 3$, then $D_{(5),K}(G) = G'^9\gamma_3(G)^3\gamma_5(G) = 1$ leads to $\gamma_5(G) = 1$. Now $G'' = \gamma_4(G) \cong C_3$, $\gamma_3(G) \cong (C_3)^2$ and $|\zeta(G')| \geqslant 3^2$. First let $exp(G') = 9$. If $|\zeta(G')| = 3^2$ and 3^4, then from the Table 2, no such group exists. If $|\zeta(G')| = 3^3$, then possible G' are $S(2187, 5874)$, $S(2187, 5876)$, $S(2187, 9100)$ to $S(2187, 9105)$ and $S(2187, 9306)$ to $S(2187, 9307)$ (see Table 2). If G' is any one of the groups $S(2187, 5874)$, $S(2187, 5876)$, $S(2187, 9102)$ to $S(2187, 9103)$, $S(2187, 9104)$ or $S(2187, 9105)$, then $G'^3 = \gamma_3(G) \cong (C_3)^2$. If G' is any one of the groups $S(2187,9100)$ to $S(2187,9101)$,$S(2187,9306)$ or $S(2187,9307)$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$. Now let $|\zeta(G')| = 3^5$. So possible G' are $S(2187,5867)$, $S(2187,5870)$, $S(2187,5872)$, $S(2187,9094)$ to $S(2187,9099)$ and $S(2187,9303)$ to $S(2187,9304)$ (see Table 2). If G' is any one of the groups $S(2187,5867)$, $S(2187,5870)$, $S(2187,5872)$ or $S(2187,9096)$ to $S(2187,9099)$, then $G'^3 = \gamma_3(G) \cong (C_3)^2$. If G' is any one of the groups $S(2187,9094)$ to $S(2187,9095)$, $S(2187,9303)$ or $S(2187,9304)$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$. For $p \geqslant 5$, $D_{(5),K}(G) = G'^p\gamma_3(G)^p\gamma_5(G) = 1$ leads to $exp(G') = p$. Now let $exp(G') = p$, for $p \geqslant 3$. Therefore $G'' = \gamma_4(G) \cong C_p$ and $\gamma_3(G) \cong (C_p)^2$. Therefore possible G' are $\langle a,b,c,d,e,f,g : a^p = b^p = c^p = d^p = e^p = f^p = g^p = 1, [b,a] = c\rangle$ and $\langle a,b,c,d,e,f,g : a^p = b^p = c^p = d^p = e^p = f^p = g^p = 1, [b,a] = e, [d,c] = e\rangle$ and for these groups, we have $\gamma_3(G) \cong (C_p)^2$ (see[26]).

Let $\mathbf{d_{(4)} = 1, d_{(2)} = 3, d_{(3)} = 2}$. Now $|G'| = p^6$, $|D_{(4),K}(G)| = p$, $|D_{(3),K}(G)| = p^3$, for all $p > 0$. Let G' be an abelian group. For $p = 2$, $G' \cong (C_4)^3$ or $(C_4)^2 \times (C_2)^2$ or $C_4 \times (C_2)^4$ or $(C_2)^6$. If $G' \cong (C_4)^3$, then $\gamma_3(G) \subseteq G'^2$. If $G' \cong (C_4)^2 \times (C_2)^2$, then either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$. If $G' \cong (C_2)^6$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$. For $p = 3$, $|D_{(4),K}(G)| = 3$ leads to $|G'^3| = 3$. Hence $G' \cong C_9 \times (C_3)^4$. For this group, either $\gamma_3(G) \cong (C_3)^2$, $|G'^2 \cap \gamma_3(G)| = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$. For $p \geqslant 5$, $D_{(5),K}(G) = 1$ leads to $G'^p = 1$ and $G' \cong (C_p)^6$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$. Let G' be a non-abelian group. For $p = 2$, $G'' = \gamma_4(G) \cong C_2$ and $\gamma_3(G) \cong (C_2)^2$ or $(C_2)^3$. If $\gamma_3(G) \cong (C_2)^2$, then $|\zeta(G')| = 4$, 8 or 16. Let $|\zeta(G')| = 4$. Then $\gamma_3(G) = \zeta(G') \cong (C_2)^2$. Therefore, possible G' are $S(64,199)$ to $S(64,201)$ and $S(64,264)$ to $S(64,265)$ (see Table 1 of [2]). If G' is any one of the groups $S(64,199)$ to $S(64,201)$, then $|G'^2 \cap \gamma_3(G)| = 2$. If G' is any one of the groups $S(64,264)$ or $S(64,265)$, then $|G'^2 \cap \gamma_3(G)| = 1$. Let $|\zeta(G')| = 8$, therefore from the Table 1 of [2], no such group exists. Let $|\zeta(G')| = 16$. Then for $|G'^2| = 8$, possible G' are $S(64,56)$ to $S(64,59)$. For these groups, $\gamma_3(G) \subseteq G'^2$, $\gamma_3(G) \cong (C_2)^2$. For $|G'^2| = 4$, possible G' are $S(64,193)$ to $S(64,198)$ and for these groups $|G'^2 \cap \gamma_3(G)| = 2$. For $|G'^2| = 2$, possible G' are $S(64,261)$ to $S(64,263)$. For these groups $2 = |G'' \cap G'^2| \leqslant |G'^2 \cap \gamma_3(G)| = 1$. If $|\gamma_3(G)| = 8$, then $|\zeta(G')| = 8$ or 16. Let $|\zeta(G')| = 8$, then from the Table 1 of [2] no such group exists. Let $|\zeta(G')| = 16$. Now for $|G'^2| = 8$, possible G' are $S(64,56)$ to $S(64,59)$ and for these groups $\gamma_3(G) \subseteq G'^2$. For $|G'^2| = 4$, possible G' are $S(64,193)$ to $S(64,198)$, and for these groups $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$. For $|G'^2| = 2$, no group exists (see Table 1 of [2]). For $p = 3$, $G'' = \gamma_4(G) \cong C_3$. So $\gamma_3(G) \cong (C_3)^2$ or $(C_3)^3$

and $|\zeta(G')| \geqslant 3^2$. Let $\gamma_3(G) \cong (C_3)^2$ and $|\zeta(G')| = 9$. Then possible G' are $S(729,422)$ to $S(729,424)$ and $S(729,502)$ (see Table 6 of [14]). But then $|D_{(3),K}(G)| \neq 3^3$. If $|\zeta(G')| = 3^3$, then no group exists (see Table 6 of [14]). Let $|\zeta(G')| = 3^4$. If $|\gamma_3(G)| = 3^3$, then G' is any one of the groups $S(729,103)$ to $S(729,106)$, $S(729,416)$ to $S(729,420)$ or $S(729,499)$ to $S(729,500)$. For all these groups $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$. Let $|\gamma_3(G)| = 3^2$, then possible G' are $S(729,103)$ to $S(729,106)$, $S(729,416)$ to $S(729,421)$ and $S(729,499)$ to $S(729,500)$. If G' is any one of the groups $S(729,103)$, $S(729,105)$, $S(729,417)$, $S(729,418)$, $S(729,420)$ or $S(729,421)$, then $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong (C_3)^2$. If G' is any one of the groups $S(729,104)$ or $S(729,106)$, then $\gamma_3(G) \subseteq G'^2$. If G' is any one of the groups $S(729,416)$, $S(729,419)$, $S(729,499)$ or $S(729,500)$, then $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^2$ (see Table 6 of [14]). For $p \geqslant 5$, $|D_{(5),K}(G)| = 1$ leads to $G'^p = 1$, $\gamma_3(G) \cong (C_p)^3$, $G'' = \gamma_4(G) \cong C_p$, $|\zeta(G')| = p^3$ or p^4. If $|\zeta(G')| = p^3$, then no group exists (see [10]). If $|\zeta(G')| = p^4$. Then $G' \cong \phi_2(1^5) \times (1)$, $\gamma_3(G) \cong (C_p)^3$, $\zeta(G') \cong (C_p)^4$ and $|G'^p \cap \gamma_3(G)| = 1$ (see [10]).

Let $\mathbf{d_{(4)} = d_{(2)} = 1, d_{(3)} = 3}$. Then $|G'| = p^5$, $|D_{(4),K}(G)| = p$, $|D_{(3),K}(G)| = p^4$. Let G' be an abelian group. For $p = 2$, $G' \cong (C_4)^2 \times C_2$ or $C_4 \times (C_2)^3$ or $(C_2)^4$. If $G' \cong (C_4)^2 \times C_2$, then either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$. If $G' \cong C_4 \times (C_2)^3$, then either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$. If $G' \cong (C_2)^4$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^4$. Now for $p = 3$, $|D_{(4),K}(G)| = 3$ leads to $|G'^3| = 3$. So $G' \cong C_9 \times (C_3)^3$, then either $\gamma_3(G) \cong (C_3)^3$, $|G'^3 \cap \gamma_3(G)| = 1$ or $\gamma_3(G) \cong (C_3)^4$, $G'^3 \subseteq \gamma_3(G)$. Now for $p \geqslant 5$, $D_{(5),K}(G) = 1$ leads to $G'^p = 1$. So $G' \cong (C_p)^5$, $\gamma_3(G) \cong (C_p)^4$ and $|G'^p \cap \gamma_3(G)| = 1$. Let G' be a non-abelian group. For $p = 2$, $G'' = \gamma_4(G) \cong C_2$ and $|\gamma_3(G)| \geqslant 4$. As $\gamma_3(G) \subseteq \zeta(G')$, hence $|\zeta(G')| = 4$ or 8. If $|\zeta(G')| = 4$, then from the Table 2 of [22], $|G''| \neq 2$. If $|\zeta(G')| = 8$, then $\gamma_3(G) = \zeta(G') \cong (C_2)^3$. Therefore only possible G' is $S(32,2)$ and for this group $|G'^2 \cap \gamma_3(G)| = 4$ (see Table 2 of [22]). For $p = 3$, $G'' = \gamma_4(G) \cong C_3$, $G'^3 \subseteq \gamma_4(G) \cong C_3$ and $|\gamma_3(G)| \geqslant 3^2$. If $|\gamma_3(G)| = 3^2$ and $|\zeta(G')| = 3^2$ or 3^3, then from the Table 5 of [14] no such group exists. If $|\gamma_3(G)| = 3^3$, then $\gamma_3(G) = \zeta(G') \cong (C_3)^3$, so only possible G' is $S(243,32)$ and for this group $|G'^3 \cap \gamma_3(G)| = 1$. For $p \geqslant 5$, $D_{(5),K}(G) = G'^p\gamma_3(G)^p\gamma_5(G) = 1$ leads to $G'^p = 1$. Now $G'' = \gamma_4(G) \cong C_p$, $\gamma_3(G) \cong (C_p)^4$ and $\gamma_3(G) = \zeta(G') \cong (C_p)^4$. Thus G' is abelian in this case.

Let $\mathbf{d_{(4)} = d_{(3)} = 2}$. Since $d_{(1+1)} = 0$, therefore by Lemma 1(2), $\vartheta_{p'}(2) \geqslant \vartheta_{p'}(1)$ for all $p > 0$ and so $d_{(3)} = 0$.

TABLE 1.

G'	G'^5	$\exp(G')$	$\zeta(G')$	G''	$G''\cap G'^5$	$G''\cap\zeta(G')$	$G'^5\cap\zeta(G')$
S(3125,2)	$C_5\times C_5$	25	$C_5\times C_5\times C_5$	C_5	1	C_5	$C_5\times C_5$
S(3125,16)	$C_{25}\times C_5$	125	$C_{25}\times C_5$	C_5	C_5	C_5	$C_{25}\times C_5$
S(3125,17)	C_{25}	125	$C_{25}\times C_5$	C_5	1	C_5	C_{25}
S(3125,26)	$C_{25}\times C_5$	125	$C_{25}\times C_5$	C_5	C_5	C_5	$C_{25}\times C_5$
S(3125,29)	C_{125}	625	C_{125}	C_5	C_5	C_5	C_{125}
S(3125,40)	C_5	25	$C_5\times C_5\times C_5$	C_5	1	C_5	C_5
S(3125,41)	$C_5\times C_5$	25	$C_5\times C_5\times C_5$	C_5	C_5	C_5	$C_5\times C_5$
S(3125,42)	$C_5\times C_5$	25	$C_{25}\times C_5$	C_5	C_5	C_5	$C_5\times C_5$
S(3125,43)	C_5	25	$C_{25}\times C_5$	C_5	1	C_5	C_5
S(3125,44)	$C_5\times C_5$	25	$C_{25}\times C_5$	C_5	C_5	C_5	$C_5\times C_5$
S(3125,59)	C_{25}	125	$C_{25}\times C_5$	C_5	C_5	C_5	C_{25}
S(3125,60)	C_{25}	125	C_{125}	C_5	C_5	C_5	C_{25}
S(3125,72)	1	5	$C_5\times C_5\times C_5$	C_5	1	C_5	1
S(3125,73)	C_5	25	$C_5\times C_5\times C_5$	C_5	C_5	C_5	C_5
S(3125,74)	C_5	25	$C_{25}\times C_5$	C_5	C_5	C_5	C_5
S(3125,75)	1	5	C_5	C_5	1	C_5	1
S(3125,76)	C_5	25	C_5	C_5	C_5	C_5	C_5

Let $\mathbf{d_{(4)}}=\mathbf{0}$. Then we have the following possibilities: $d_{(2)}=10$ or $d_{(2)}=8$, $d_{(3)}=1$ or $d_{(2)}=6$, $d_{(3)}=2$ or $d_{(2)}=4$, $d_{(3)}=3$ or $d_{(2)}=2$, $d_{(3)}=4$ or $d_{(3)}=5$.

Let $\mathbf{d_{(2)}}=\mathbf{10}$. Then $|G'|=p^{10}$, $|D_{(3),K}(G)|=1$ and hence $G'^p=\gamma_3(G)=1$, for all $p>0$. Thus G' is abelian and $G'\cong(C_p)^{10}$, $\gamma_3(G)=1$.

Let $\mathbf{d_{(2)}}=\mathbf{8},\mathbf{d_{(3)}}=\mathbf{1}$. Thus $|G'|=p^9$, $|D_{(3),K}(G)|=p$ and G' is abelian for all $p>0$. For $p\geqslant 3$, $G'^p=1$ and hence $G'\cong(C_p)^9$, $\gamma_3(G)\cong C_p$, $|G'^p\cap\gamma_3(G)|=1$. For $p=2$, $|D_{(3),K}(G)|=2$ leads to $|G'^2|\leqslant 2$. So $G'\cong C_4\times(C_2)^7$ or $(C_2)^9$. If $G'\cong C_4\times(C_2)^7$, then $\gamma_3(G)\subseteq G'^2\cong C_2$. If $G'\cong(C_2)^9$, then $\gamma_3(G)\cong C_2$, $|G'^2\cap\gamma_3(G)|=1$.

Let $\mathbf{d_{(2)}}=\mathbf{6},\mathbf{d_{(3)}}=\mathbf{2}$. Thus $|G'|=p^8$, $|D_{(3),K}(G)|=p^2$ and G' is abelian for all $p>0$. For $p\geqslant 3$, $G'^p=1$, hence $G'\cong(C_p)^8$, $\gamma_3(G)\cong(C_p)^2$ and $|G'^p\cap\gamma_3(G)|=1$. For $p=2$, $|D_{(3),K}(G)|=2^2$ leads to $|G'^2|\leqslant 4$. So $G'\cong(C_2)^8$ or $C_4\times(C_2)^6$ or $(C_4)^2\times(C_2)^4$. If $G'\cong(C_2)^8$, then $|G'^2\cap\gamma_3(G)|=1$, $\gamma_3(G)\cong(C_2)^2$. If $G'\cong C_4\times(C_2)^6$, then either $|G'^2\cap\gamma_3(G)|=1$, $\gamma_3(G)\cong C_2$ or $G'^2\subseteq\gamma_3(G)\cong(C_2)^2$. If $G'\cong(C_4)^2\times(C_2)^4$, then $\gamma_3(G)\subseteq G'^2$.

Let $\mathbf{d_{(2)}}=\mathbf{4},\mathbf{d_{(3)}}=\mathbf{3}$. Thus $|G'|=p^7$, $|D_{(3),K}(G)|=p^3$ and G' is abelian, for all $p>0$. If $p\geqslant 3$, then $G'\cong(C_p)^7$ and $\gamma_3(G)\cong(C_p)^3$,

Table 2.

G'	G'^3	$\exp(G')$	$\zeta(G')$	G''	$G''\cap G'^3$	$G'^3\cap\zeta(G')$
S(2187,5867)	$C_3\times C_3$	9	$C_9\times C_9\times C_3$	C_3	1	$C_3\times C_3$
S(2187,5868)	$C_3\times C_3\times C_3$	9	$C_9\times C_9\times C_3$	C_3	C_3	$C_3\times C_3\times C_3$
S(2187,5869)	$C_3\times C_3\times C_3$	9	$C_9\times C_9\times C_3$	C_3	C_3	$C_3\times C_3\times C_3$
S(2187,5870)	$C_3\times C_3$	9	$C_9\times C_3\times C_3\times C_3$	C_3	1	$C_3\times C_3$
S(2187,5871)	$C_3\times C_3\times C_3$	9	$C_9\times C_3\times C_3\times C_3$	C_3	C_3	$C_3\times C_3\times C_3$
S(2187,5872)	$C_3\times C_3$	9	$C_3\times C_3\times C_3\times C_3\times C_3$	C_3	1	$C_3\times C_3$
S(2187,5873)	$C_3\times C_3\times C_3$	9	$C_9\times C_3\times C_3\times C_3$	C_3	C_3	$C_3\times C_3\times C_3$
S(2187,5874)	$C_3\times C_3$	9	$C_3\times C_3\times C_3$	C_3	1	$C_3\times C_3$
S(2187,5875)	$C_3\times C_3\times C_3$	9	$C_3\times C_3\times C_3$	C_3	C_3	$C_3\times C_3\times C_3$
S(2187,5876)	$C_3\times C_3$	9	$C_3\times C_3\times C_3$	C_3	1	$C_3\times C_3$
S(2187,5877)	$C_3\times C_3\times C_3$	9	$C_3\times C_3\times C_3$	C_3	C_3	$C_3\times C_3\times C_3$
S(2187,9094)	C_3	9	$C_3\times C_3\times C_3\times C_3\times C_3$	C_3	1	C_3
S(2187,9095)	C_3	9	$C_9\times C_3\times C_3\times C_3$	C_3	1	C_3
S(2187,9096)	$C_3\times C_3$	9	$C_9\times C_9\times C_3$	C_3	C_3	$C_3\times C_3$
S(2187,9097)	$C_3\times C_3$	9	$C_9\times C_3\times C_3\times C_3$	C_3	C_3	$C_3\times C_3$
S(2187,9098)	$C_3\times C_3$	9	$C_9\times C_3\times C_3\times C_3$	C_3	C_3	$C_3\times C_3$
S(2187,9099)	$C_3\times C_3$	9	$C_3\times C_3\times C_3\times C_3\times C_3$	C_3	C_3	$C_3\times C_3$
S(2187,9100)	C_3	9	$C_3\times C_3\times C_3$	C_3	1	C_3
S(2187,9101)	C_3	9	$C_9\times C_3$	C_3	1	C_3
S(2187,9102)	$C_3\times C_3$	9	$C_9\times C_3$	C_3	C_3	$C_3\times C_3$
S(2187,9103)	$C_3\times C_3$	9	$C_9\times C_3$	C_3	C_3	$C_3\times C_3$
S(2187,9104)	$C_3\times C_3$	9	$C_3\times C_3\times C_3$	C_3	C_3	$C_3\times C_3$
S(2187,9105)	$C_3\times C_3$	9	$C_3\times C_3\times C_3$	C_3	C_3	$C_3\times C_3$
S(2187,9303)	C_3	9	$C_3\times C_3\times C_3\times C_3\times C_3$	C_3	C_3	C_3
S(2187,9304)	C_3	9	$C_9\times C_3\times C_3\times C_3$	C_3	C_3	C_3
S(2187,9306)	C_3	9	$C_3\times C_3\times C_3$	C_3	C_3	C_3
S(2187,9307)	C_3	9	$C_9\times C_3$	C_3	C_3	C_3
S(2187,9309)	C_3	9	C_3	C_3	C_3	C_3

$|G'^p\cap\gamma_3(G)|=1$. For $p=2$, $|D_{(3),K}(G)|=2^3$ leads to $|G'^2|\leqslant 8$. So $G'\cong(C_4)^3\times C_2$ or $(C_4)^2\times(C_2)^3$ or $C_4\times(C_2)^5$ or $(C_2)^7$. If $G'\cong(C_4)^3\times C_2$, then $\gamma_3(G)\subseteq G'^2$. If $G'\cong(C_4)^2\times(C_2)^3$, then either $|G'^2\cap\gamma_3(G)|=1$, $\gamma_3(G)\cong C_2$ or $|G'^2\cap\gamma_3(G)|=2$, $\gamma_3(G)\cong C_2\times C_2$ or $G'^2\subseteq\gamma_3(G)\cong(C_2)^3$. If $G'\cong C_4\times(C_2)^5$, then either $|G'^2\cap\gamma_3(G)|=1$, $\gamma_3(G)\cong(C_2)^2$ or $G'^2\subseteq\gamma_3(G)\cong(C_2)^3$. If $G'\cong(C_2)^7$, then $|G'^2\cap\gamma_3(G)|=1$, $\gamma_3(G)\cong(C_2)^3$.

Let $\mathbf{d_{(2)}=2,d_{(3)}=4}$. Thus $|G'|=p^6$, $|D_{(3),K}(G)|=p^4$ and G' is abelian for all $p>0$. For $p\geqslant 3$, $G'^p=1$, so $G'\cong(C_p)^6$, $|G'^p\cap\gamma_3(G)|=1$ and $\gamma_3(G)\cong(C_p)^4$. For $p=2$, $|D_{(3),K}(G)|=2^4$ leads to $|G'^2|\leqslant 8$.

TABLE 3.

G'	G'^3	$\exp(G')$	$\zeta(G')$	G''	$G'' \cap G'^3$	$G'^3 \cap \zeta(G')$
S(243,13)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	$C_3 \times C_3$
S(243,14)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	$C_3 \times C_3$
S(243,15)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	$C_3 \times C_3$
S(243,16)	C_9	27	C_9	$C_3 \times C_3$	C_3	C_9
S(243,17)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	$C_3 \times C_3$
S(243,18)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	$C_3 \times C_3$
S(243,19)	C_9	27	C_9	$C_3 \times C_3$	C_3	C_9
S(243,20)	C_9	27	C_9	$C_3 \times C_3$	C_3	C_9
S(243,22)	$C_9 \times C_3$	27	C_3	C_9	C_9	C_3
S(243,37)	1	3	$C_3 \times C_3$	$C_3 \times C_3$	1	1
S(243,38)	C_3	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	C_3
S(243,39)	C_3	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	C_3
S(243,40)	C_3	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	C_3
S(243,41)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$
S(243,42)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$
S(243,43)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$
S(243,44)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$
S(243,45)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$
S(243,46)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$
S(243,47)	$C_3 \times C_3$	9	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3$
S(243,51)	C_3	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	C_3
S(243,52)	C_3	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	C_3
S(243,53)	C_3	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	C_3
S(243,54)	C_3	9	$C_3 \times C_3$	$C_3 \times C_3$	C_3	C_3
S(243,55)	C_3	9	C_9	$C_3 \times C_3$	C_3	C_3
S(243,56)	C_3	9	C_3	$C_3 \times C_3$	C_3	C_3
S(243,57)	C_3	9	C_3	$C_3 \times C_3$	C_3	C_3
S(243,58)	C_3	9	C_3	$C_3 \times C_3$	C_3	C_3
S(243,59)	C_3	9	C_3	$C_3 \times C_3$	C_3	C_3
S(243,60)	C_3	9	C_3	$C_3 \times C_3$	C_3	C_3

So $G' \cong (C_4)^3$ or $(C_4)^2 \times (C_2)^2$ or $C_4 \times (C_2)^4$ or $(C_2)^6$. If $G' \cong (C_4)^3$, then $\gamma_3(G) \subseteq G'^2$. If $G' \cong (C_4)^2 \times (C_2)^2$, then either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$. If $G' \cong C_4 \times (C_2)^4$, then either $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$. If $G' \cong (C_2)^6$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^4$.

Let $\mathbf{d_{(3)} = 5}$. Since $d_{(1+1)} = 0$, therefore by Lemma 1(2), $\vartheta_{p'}(2) \geqslant \vartheta_{p'}(1)$ for all $p > 0$ and so $d_{(3)} = 0$.

Converse can be easily done by computing $d_{(m)}$'s in each case. □

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