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A note on two families of 2-designs arose from Suzuki-Tits ovoid

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Abstract. In this note, we give a precise construction of one of the families of 2-designs arose from studying flag-transitive 2-designs with parameters (v,k,λ) whose replication numbers r are coprime to λ. We show that for a given positive integer $q = 2^{2n+1} \geqslant 8$, there exists a 2-design with parameters $(q^2+1, q, q-1)$ and the replication number q^2 admitting the Suzuki group $\mathsf{Sz}(q)$ as its automorphism group. We also construct a family of 2-designs with parameters $(q^2+1, q(q-1), (q-1)(q^2-q-1))$ and the replication number $q^2(q-1)$ admitting the Suzuki groups $\mathsf{Sz}(q)$ as their automorphism groups.

An *ovoid* in $\mathsf{PG}_3(q)$ with $q > 2$, is a set of q^2+1 points such that no three of which are collinear. The classical example of an ovoid in $\mathsf{PG}_3(q)$ is an elliptic quadric. If q is odd, then all ovoids are elliptic quadratics, see [3, 13], while in even characteristic, there is only one known family of ovoids that are not elliptic quadrics in which $q \geqslant 8$ is an odd power of 2. These were discovered by Tits [17], and are now called the *Suzuki-Tits ovoids* since the Suzuki groups naturally act on these ovoids. The main aim of this note is to introduce two infinite families of 2-designs arose from Suzuki-Tits ovoid whose automorphism groups are the Suzuki groups.

A 2-design $\mathcal{D}$ with parameters (v,k,λ) is a pair $(\mathcal{P},\mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of b blocks such that each block is a k-subset of $\mathcal{P}$ and each two distinct points are contained in λ blocks. The number of blocks

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incident with a given point is a constant number $r := bk/v$ called the *replication number* of $\mathcal{D}$. If $v = b$ (or equivalently, $k = r$), then $\mathcal{D}$ is called *symmetric*. Further definitions and notation can be found in [4, 6, 14].

Our motivation comes from a recent classification of flag-transitive 2-designs whose replication numbers are coprime to λ, see [2]. Excluding eleven sporadic examples, we have found eight possible infinite families of 2-designs with this property, three of which are new and the rest are well-known structures, namely, *point-hyperplane designs*, *Witt-Bose-Shrikhande spaces* [5], *Hermitian Unital spaces* [11] and *Ree Unital spaces* [12]. These new possible 2-designs arose from studying 2-designs admitting finite almost simple exceptional automorphism groups of Lie type, see [1]. Although, we have provided examples of these new 2-designs with smallest possible parameters [1, Section 2], at the time of writing [1], we have not been aware of any generic construction of these incidence structures. In [7], Daneshkhah have constructed two of these infinite families of 2-designs admitting Ree groups as their automorphism groups, and so for the remaining possibility, one may ask the following question:

Question 1. *For a given prime power* $q = 2^{2n+1} \geqslant 8$*, does there exist a* 2*-design with parameters* $(q^2+1, q, q-1)$ *with replication number* q^2 *admitting the Suzuki group* $\mathsf{Sz}(q)$ *as its automorphism group?*

In this note, we aim to give a positive answer to Question 1. Indeed, in Theorem 1 below, we explicitly construct such a 2-design using the natural action of Suzuki groups on Suzuki-Tits ovoids. We would also remark here that these designs can also be constructed geometrically by taking points as ovoids in $\mathsf{PG}_3(q)$ and blocks as pointed conics minus the distinguished points. We also construct a 2-design with parameters $(q^2+1, q(q-1), (q-1)(q^2-q-1))$ admitting the Suzuki group $\mathsf{Sz}(q)$ as its flag-transitive automorphism group. We call such 2-designs *Suzuki-Tits ovoid designs* as they arose from Suzuki-Tits ovoid. We note that Suzuki-Tits ovoid designs have $q(q^2+1)$ number of blocks, and so they are not symmetric. In view of [9, 10], the Suzuki-Tits ovoid designs are non-symmetric 2-designs with doubly transitive automorphism groups on points.

1. Preliminaries

The Suzuki groups were discovered by Suzuki [15], and a geometric construction of these groups was given by Tits [17]. We mainly follow the description of these groups from [8, Section XI.3] with a few exceptions in our notation, see also [15–17].

Let $\mathbb{F} = \mathrm{GF}(q)$ be the finite field of size $q = 2^{2n+1} \geqslant 8$, and let θ be the automorphism of $\mathbb{F}$ mapping α to α^r, where $r = \sqrt{2q} = 2^{n+1}$. Therefore, θ^2 is the Frobenius automorphism $\alpha \mapsto \alpha^2$. Let e be the identity 4×4 matrix, and let e_{ij} be the 4×4 matrix with 1 in the entry ij and 0 elsewhere. For $x, y \in \mathbb{F}$ and $\kappa \in \mathbb{F}^{\times}$, define

$$\begin{aligned} \mathsf{s}(x,y) :=& e + xe_{21} + ye_{31} + x^{\theta}e_{32} + \\ & (x^{1+\theta} + xy + y^{\theta})e_{41} + (x^{1+\theta} + y)e_{42} + xe_{43}; \end{aligned} \tag{1}$$

$$\mathsf{m}(\kappa) := \kappa^{1+2^n}e_{11} + \kappa^{2^n}e_{22} + \kappa^{-2^n}e_{33} + \kappa^{-1-2^n}e_{44}; \tag{2}$$

$$\tau := e_{14} + e_{23} + e_{32} + e_{41}. \tag{3}$$

We know that $\mathsf{s}(x,y) \cdot \mathsf{s}(z,t) = \mathsf{s}(x+z, y+t+x^{\theta}z)$, and hence the set Q of the matrices $\mathsf{s}(x,y)$ is a group of order q^2. The set of matrices $\mathsf{m}(\kappa)$ forms a cyclic group $M \cong \mathbb{F}^{\times}$ of order $q-1$. Since $\mathsf{m}(\kappa)^{-1} \cdot \mathsf{s}(x,y) \cdot \mathsf{m}(\kappa) = (x\kappa, y\kappa^{1+\theta})$, the group H generated by Q and M is a semidirect product of a normal subgroup Q by a complement M, and so it has order $q^2(q-1)$. The Suzuki group $\mathsf{Sz}(q)$ is a subgroup of $\mathsf{GL}_4(q)$ generated by H and the 4×4 matrix τ defined as in (3). In what follows, G will denote the Suzuki group $\mathsf{Sz}(q)$ with $q = 2^{2n+1} \geqslant 8$.

The Suzuki group G naturally acts on the projective space $\mathsf{PG}_3(q)$ via $[w]^x := [w^x]$, for all $x \in G$ and $[w] \in \mathsf{PG}_3(q)$. In fact, G acts as a doubly transitive permutation group of degree q^2+1 on the Suzuki-Tits ovoid

$$\mathcal{P} = \{p(\alpha,\beta) \mid \alpha, \beta \in \mathbb{F}\} \cup \{\infty\} \subseteq \mathsf{PG}_3(q), \tag{4}$$

where $p(\alpha,\beta) = [\alpha^{2+\theta} + \alpha\beta + \beta^{\theta}, \beta, \alpha, 1]$ and $\infty := [1,0,0,0] \in \mathsf{PG}_3(q)$. In particular, the action of the matrices $\mathsf{s}(x,y)$ and $\mathsf{m}(\kappa)$ on the projective points $p(\alpha,\beta)$ of $\mathcal{P} \setminus \{\infty\}$ can be explicitly written as follows

$$p(\alpha,\beta)^{\mathsf{s}(x,y)} = p(\alpha + x, \beta + y + \alpha x^{\theta} + x^{1+\theta}), \tag{5}$$

$$p(\alpha,\beta)^{\mathsf{m}(\kappa)} = p(\alpha\kappa, \beta\kappa^{1+\theta}). \tag{6}$$

Note that H fixes ∞, and hence H is the point-stabilizer G_{∞}. The subgroup H of G acts as a Frobenius group on $\mathcal{P} \setminus \{\infty\}$, where Q is the Frobenius kernel of H acting regularly on $\mathcal{P} \setminus \{\infty\}$, and $M = G_{\infty,\omega}$ is the Frobenius complement of H fixing the second point $\omega := p(0,0) = [0,0,0,1] \in \mathcal{P}$ and acting semiregularly on $\mathcal{P} \setminus \{\infty, \omega\}$. Therefore, the stabilizer of any three points in $\mathcal{P}$ is the trivial subgroup. Moreover, the map $Hg \mapsto \infty^g$ induces a permutational isomorphism between the G-action on the set of right cosets of H in G and the G-action on the Suzuki-Tits avoid $\mathcal{P}$.

We now consider the subgroup Q_0 of Q consisting of all matrices $\mathsf{s}(0, y)$. Note that the matrices of the form $\mathsf{s}(0, y)$ are the only involutions in Q. Thus Q_0 is a normal subgroup of Q of order q. Moreover, $Q_0 = Q' = \mathbf{Z}(Q)$. Let K be the subgroup of H generated by Q_0 and M, that is to say,

$$K = \langle \mathsf{s}(0, y), \mathsf{m}(\kappa) \mid y \in \mathbb{F} \text{ and } \kappa \in \mathbb{F}^{\times} \rangle, \tag{7}$$

where $\mathsf{s}(0, y)$ and $\mathsf{m}(\kappa)$ are as in (1) and (2), respectively. Then K is a Frobenius group of order $q(q-1)$ whose Frobenius kernel and Frobenius complement are Q_0 and M, respectively. Let now

$$\begin{aligned} \Delta_1 &= \{\infty\}, \\ \Delta_2 &= \{p(0, \beta) \mid \beta \in \mathbb{F}\}, \text{ and} \\ \Delta_3 &= \{p(\alpha, \beta) \mid \alpha \in \mathbb{F}^{\times}, \beta \in \mathbb{F}\}. \end{aligned} \tag{8}$$

Then $|\Delta_1| = 1$, $|\Delta_2| = q$ and $|\Delta_3| = q(q-1)$.

2. Existence of Suzuki-Tits ovoid designs

In this section, we prove our main result and construct two infinite families of 2-designs admitting $G = \mathsf{Sz}(q)$ as their automorphism groups.

Theorem 1. *Let $G = \mathsf{Sz}(q)$ with $q = 2^{2n+1} \geqslant 8$. Let also $\mathcal{P}$ be the Suzuki-Tits ovoid defined as in* (4)*, and let Δ_i be as in* (8)*, for $i \in \{2, 3\}$. Then*

(a) *if $\mathcal{B}_2 = \Delta_2^G$, then $(\mathcal{P}, \mathcal{B}_2)$ is a 2-design with parameters $(q^2+1, q, q-1)$ and the replication number q^2;*

(b) *if $\mathcal{B}_3 = \Delta_3^G$, then $(\mathcal{P}, \mathcal{B}_3)$ is a 2-design with parameters $(q^2+1, q(q-1), (q-1)(q^2-q-1))$ and the replication number $q^2(q-1)$.*

Moreover, the Suzuki group $\mathsf{Sz}(q)$ is a flag-transitive automorphism group of the designs in parts (a) *and* (b) *acting primitively on the points set $\mathcal{P}$ but imprimitively on the blocks set $\mathcal{B}_i$.*

Proof. We first prove that the subsets Δ_1, Δ_2 and Δ_3 defined in (8) are the only orbits of K in its action on the ovoid $\mathcal{P}$, where K is as in (7). Since K is a subgroup of H fixing ∞, it follows that Δ_1 is an orbit of K. It is easily followed by (6) that the Δ_i, for $i \in \{2, 3\}$, are M-invariant subsets of $\mathcal{P}$, where $M = \langle \mathsf{m}(\kappa) \mid \kappa \in \mathbb{F}^{\times} \rangle$. Moreover, by (5), we have that

$p(\alpha, \beta)^{\mathsf{s}(0,y)} = p(\alpha, \beta + y)$, and so the Δ_i are also Q_0-invariant. Therefore, by (7), we conclude that the Δ_i are K-invariant subsets of $\mathcal{P}$. Further, if $\alpha \in \mathbb{F}^{\times}$, then it follows from (5) and (6) that $p(0,0)^{\mathsf{s}(0,\beta)} = p(0, \beta)$ and $p(1,0)^{\mathsf{m}(\alpha)\mathsf{s}(0,\beta)} = p(\alpha, \beta)$. This implies that K is transitive on each Δ_i, and since the set of Δ_i, for $i = 1, 2, 3$, forms a K-invariant partition of $\mathcal{P}$, we conclude that the Δ_i are all distinct K-orbits on $\mathcal{P}$.

We claim that K is the setwise-stabilizer G_{Δ_i}, for $i = 2, 3$. We first prove that $G_{\Delta_2} = K$. Obviously, K is a subgroup of G_{Δ_2}. Recall that G is generated by $\mathsf{s}(x, y)$, $\mathsf{m}(\kappa)$ and τ defined as in (1), (2) and (3), respectively. The fact that $p(0, 0)^{\tau} = \infty$ implies that τ does not fix Δ_2. Hence, G_{Δ_2} is a subgroup of $H = \langle \mathsf{s}(x, y), \mathsf{m}(\kappa) \mid x, y \in \mathbb{F} \text{ and } \kappa \in \mathbb{F}^{\times} \rangle$. We have also proved that $K = Q_0M$ fix Δ_2, and so $K \leqslant G_{\Delta_2} \leqslant H$. If a generator $g := \mathsf{s}(x, y)$ fixes Δ_2, then (5) yields $p(x, \beta+y+x^{1+\theta}) = p(0, \beta)^{\mathsf{s}(x,y)} \in \Delta_2$, and so $x = 0$, that is to say, $g = \mathsf{s}(0, y) \in Q_0$. Recall that M fixes Δ_2, and hence we conclude that $G_{\Delta_2} = K$. We finally show that $G_{\Delta_3} = K$. By the same argument as in the previous case, G_{Δ_3} is a subgroup of H containing K. Let $\mathsf{s}(x, y) \in H$ fix Δ_3. Then $p(\alpha, \beta)^{\mathsf{s}(x,y)} \in \Delta_3$ for all $\alpha \neq 0$, and so by (5), this is equivalent to $x \neq \alpha$, for all $\alpha \neq 0$. Thus, $x = 0$, and hence G_{Δ_3} is a subgroup of $K = Q_0M$ implying that $G_{\Delta_3} = K$.

We now prove our main result. Since G is doubly transitive on $\mathcal{P}$ and $\mathcal{B}_i := \Delta_i^G = \{\Delta_i^x \mid x \in G\}$ for $i \in \{2, 3\}$, it follows from [4, Proposition III.4.6] that the incidence structures $\mathcal{D}_i = (\mathcal{P}, \mathcal{B}_i)$ are 2-designs with parameters (v, k_i, λ_i), for $i \in \{2, 3\}$, admitting $G = \mathsf{Sz}(q)$ as their automorphism groups. Recall that $v = |\mathcal{P}| = q^2 + 1$ and $K = G_{\Delta_i}$. Then $|G_{\Delta_i}| = |K| = q(q - 1)$, for $i \in \{2, 3\}$, where K is subgroup of G defined as in (7). Therefore, the designs $\mathcal{D}_i$ in parts (a) and (b) have (the same) number of blocks $b = |G : G_{\Delta_i}| = q(q^2 + 1)$. Note that $k_i = |\Delta_i|$ and $\lambda_i = bk_i(k_i - 1)/v(v - 1)$, and the replication number r_i is equal to bk_i/v, for $i \in \{2, 3\}$. Therefore, $\mathcal{D}_2$ is a 2-design with parameters $(q^2 + 1, q, q - 1)$ as in part (a) and $\mathcal{D}_3$ is a 2-design with parameters $(q^2+1, q(q-1), (q-1)(q^2-q-1))$ as in part (b). Since G is transitive on $\mathcal{B}_i$ and Δ_i is an orbit of $K = G_{\Delta_i}$, we conclude that G is a flag-transitive automorphism group of $\mathcal{D}_i$, for $i \in \{2, 3\}$. The group G is primitive on $\mathcal{P}$ as it is doubly transitive on $\mathcal{P}$, however, G is imprimitive on $\mathcal{B}_i$ as the block-stabilizer $K = G_{\Delta_i}$ is not a maximal subgroup of G. $\square$

We remark here that the design constructed in Theorem 1(a) introduces an infinite family of examples of 2-designs with $\gcd(r, \lambda) = 1$ obtained in [1, Theorem 1.1(a)] and gives a positive answer to Question 1.

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References

[1] S. H. Alavi, *Flag-transitive block designs and finite exceptional simple groups of Lie type*, Graphs and Combinatorics, **36**(4), 2020, pp.1001-1014.

[2] S. H. Alavi, A. Daneshkhah, F. Mouseli, *A classification of flag-transitive block designs*, Submitted.

[3] A. Barlotti, *Un'estensione del teorema di Segre-Kustaanheimo*, Boll. Un. Mat. Ital. (3) **10**, 1955, pp.498-506.

[4] T. Beth, D. Jungnickel, H. Lenz, *Design theory. Vol. I*, vol. 69 of Encyclopedia of Mathematics and its Applications, 2nd ed., Cambridge University Press, Cambridge, 1999.

[5] F. Buekenhout, A. Delandtsheer, J. Doyen, *Finite linear spaces with flag-transitive groups*, Journal of Combinatorial Theory, Series A **49**(2), 1988, pp.268-293.

[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.

[7] A. Daneshkhah, *Ree groups as automorphism groups of block designs*, Submitted.

[8] B. Huppert, N. Blackburn, *Finite Groups III*, Springer Berlin Heidelberg, 1982.

[9] W. M. Kantor, *2-transitive designs*, in: Combinatorics, Springer Netherlands, 1975, pp.365-418.

[10] W. M. Kantor, *Classification of* 2*-transitive symmetric designs*, Graphs Combin. **1**(2), 1985, pp.165-166.

[11] W. M. Kantor, *Homogeneous designs and geometric lattices*, J. Combin. Theory Ser. A **38**(1), 1985, pp.66-74.

[12] H. Lüneburg, *Some remarks concerning the Ree groups of type* (G_2), J. Algebra **3**d, 1966, pp.256-259.

[13] G. Panella, *Caratterizzazione delle quadriche di uno spazio (tridimensionale) lineare sopra un corpo finito*, Boll. Un. Mat. Ital. (3) **10**, 1955, pp.507-513.

[14] C. E. Praeger, C. Schneider, *Permutation groups and Cartesian decompositions*, vol. 449 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2018.

[15] M. Suzuki, *A new type of simple groups of finite order*, Proceedings of the National Academy of Sciences **46**(6), 1960, pp.868-870.

[16] M. Suzuki, *On a class of doubly transitive groups*, Ann. of Math. (2) **75**, 1962, pp.105-145.

[17] J. Tits, *Ovoïdes et groupes de Suzuki*, Arch. Math. **13**, 1962, pp.187-198.

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