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On weakly s-normal subgroups of finite groups

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Abstract. In this paper, we present some sufficient conditions for a finite group to be p-nilpotent and p-supersoluble under assumption that some subgroups are weakly s-normal. Some earlier results are improved and extended.

Introduction

All groups considered in this article are finite. G stands for a finite group, $|G|$ is the order of G and p denotes a prime.

A subgroup H of G is said to be s-permutable [9] in G if H permutes with every Sylow subgroup of G. A subgroup H of G is called weakly s-permutable [17] in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G.

A subgroup H of G is said to be s-semipermutable [5] in G if H permutes with every Sylow p-subgroup G_p of G with $(|H|, p) = 1$. A subgroup H of G is called weakly s-semipermutable [12] in G if there are a subnormal subgroup T of G and an s-semipermutable subgroup H_{ssG} of G contained in H such that $G = HT$ and $H \cap T \leq H_{ssG}$.

A subgroup H of G is said to be s-permutably embedded [4] in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. A subgroup H of G is called weakly s-permutably embedded [11] in G if there are a subnormal

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subgroup T of G and an s-permutably embedded subgroup H_{seG} of G contained in H such that $G = HT$ and $H \cap T \leq H_{seG}$.

In [13], Li and Qiao introduced the following concept which covers the above mentioned subgroups:

Definition 1. *A subgroup H of G is called weakly s-normal in G if there are a subnormal subgroup T of G and a subgroup H_* of H such that $G = HT$ and $H \cap T \leq H_*$, where H_* is a subgroup of H which is either s-permutably embedded or s-semipermutable in G.*

In [13], the authors obtained some results on p-nilpotency and supersolubility by using the notion of weakly s-normal subgroup. In this paper, we continue to investigate this concept and we give some criteria of p-nilpotency and p-supersolvability. Some recent results are generalized.

1. Preliminaries

Lemma 1 ([13, Lemma 2.5])**.** *Let U be a weakly s-normal subgroup of G and N a normal subgroup of G. Then*

(1) *If $U \leq H \leq G$, then U is weakly s-normal in H.*

(2) *Suppose that U is a p-group for some prime p. If $N \leq U$, then U/N is weakly s-normal in G/N.*

(3) *Suppose that U is a p-group for some prime p and N is a p'-subgroup. Then UN/N is weakly s-normal in G/N.*

(4) *Suppose that U is a p-group for some prime p and U is neither s-semipermutable nor s-permutably embedded in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = MU$.*

(5) *If $U \leq O_p(G)$ for some prime p, then U is weakly s-permutable in G.*

Lemma 2. *Let P be a normal p-subgroup of G, where p is a prime dividing $|G|$. Suppose that there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with $|H| = |D|$ (and order 4 if $|D| = 2$) is weakly s-permutable in G. Then every G-chief factor of P is cyclic.*

Proof. It is a corollary of [19, Theorem]. □

Lemma 3 ([1, Theorem 2.1.6])**.** *If G is p-supersoluble and $O_{p'}(G) = 1$, then the Sylow p-subgroup of G is normal in G.*

Lemma 4 ([6, Theorem 8.3.1])**.** *Let P be a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. Then G is p-nilpotent if and only if $N_G(Z(J(P)))$ is p-nilpotent, where $J(P)$ is the Thompson subgroup of P.*

Lemma 5 ([14, Lemma 2.3])**.** *Suppose that H is s-permutable in G, and let P be a Sylow p-subgroup of H. If $H_G = 1$, then P is s-permutable in G.*

$F^*(G)$ is the generalized Fitting subgroup of G, i.e., the product of all normal quasinilpotent subgroups of G.

Lemma 6 ([18, Theorem C])**.** *Let E be a normal subgroup of G. If every G-chief factor of $F^*(E)$ is cyclic, then every G-chief factor of E is also cyclic.*

$F_p^*(G)$ is the generalized p-Fitting subgroup of G, i.e., the normal subgroup of G such that $F^*(G/O_{p'}(G)) = F_p^*(G)/O_{p'}(G)$.

Lemma 7 ([2, Lemma 2.10])**.** *Let p be a prime and G a group.*
(1) *$Soc(G) \leq F_p^*(G)$.*
(2) *$O_{p'}(G) \leq F_p^*(G)$.*
In fact, $F^*(G/O_{p'}(G)) = F_p^*(G/O_{p'}(G)) = F_p^*(G)/O_{p'}(G)$.
(3) *If $F_p^*(G)$ is p-soluble, then $F_p^*(G) = F_p(G)$.*

Lemma 8. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with $|H| = |D|$ (and 4 if $|D| = 2$) is weakly s-normal in G, then G is p-nilpotent.*

Proof. It is a corollary of [13, Theorem 3.2]. □

Lemma 9 ([10, Lemma 2.4])**.** *Let p be a prime dividing $|G|$ with $(|G|, p-1) = 1$. If G is p-supersoluble, then G is p-nilpotent.*

2. Main Results

Theorem 1. *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with $|H| = |D|$ is weakly s-normal in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and G is a counter-example with minimal order. We will derive a contradiction in several steps.

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then we can consider the factor group $\overline{G} = G/O_{p'}(G)$. Since $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$, it follows that $N_{\overline{G}}(\overline{P})$ is p-nilpotent. For every subgroup $\overline{H}$ of Sylow p-subgroup $PO_{p'}(G)/O_{p'}(G)$ of $G/O_{p'}(G)$ with order $|D|$, we can write $\overline{H} = HO_{p'}(G)/O_{p'}(G)$, where H is a subgroup of P with order $|D|$. Then H is weakly s-normal in G by hypothesis. Hence $\overline{H}$ is weakly s-normal in $\overline{G}$ by Lemma 1(3). Therefore, $\overline{G}$ satisfies the hypothesis of our theorem. The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent and so G is p-nilpotent, a contradiction.

(2) G is not p-supersoluble.

If G is p-supersoluble, then P is normal in G by Lemma 3. By hypothesis, $G = N_G(P)$ is p-nilpotent, a contradiction.

(3) If E is a proper subgroup of G with $P \leq E$, then E is p-nilpotent.

By Lemma 1(1), every subgroup H of P with $|H| = |D|$ is weakly s-normal in E. Since $N_E(P) \leq N_G(P)$ and $N_G(P)$ is p-nilpotent, it follows that $N_E(P)$ is p-nilpotent. Hence E satisfies the hypothesis of the theorem and so E is p-nilpotent by the minimal choice of G.

(4) $O_p(G) \neq 1$.

Consider the group $Z(J(P))$, where $J(P)$ is the Thompson subgroup of P. If $N_G(Z(J(P))) < G$, then $N_G(Z(J(P))$ is p-nilpotent by step (3). Then G is p-nilpotent by Lemma 4, a contradiction. Hence $N_G(Z(J(P))) = G$ and $1 < Z(J(P)) \leq O_p(G) < P$.

(5) $G/O_p(G)$ is p-nilpotent. In particular, $G/O_p(G)$ is p-supersoluble.

Let $\overline{G} = G/O_p(G)$, $\overline{P} = P/O_p(G)$, $\overline{K} = Z(J(\overline{P}))$ and $G_1/O_p(G) = N_{\overline{G}}(Z(J(\overline{P})))$. Since $O_p(\overline{G}) = 1$, we have $N_{\overline{G}}(Z(J(\overline{P})) < \overline{G}$. Thus $G_1 < G$. By step (3), we have G_1 is p-nilpotent. Then $N_{\overline{G}}(Z(J(\overline{P}))$ is p-nilpotent. Thus $\overline{G}$ is p-nilpotent by Lemma 4.

(6) Every G-chief factor of $O_p(G)$ is not cyclic.

If not, then G is p-supersolvable since $G/O_p(G)$ is p-supersolvable, contrary to step (2).

(7) $|P| > pD|$.

Suppose that $|P| = p|D|$. Obviously, G is p-solvable. Let N be a minimal normal group of G. From (1), $N \leq O_p(G)$. It is easy to see that G/N satisfies the hypothesis of the theorem. Hence the choice of G yields that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. Consequently, G has a maximal subgroup M such that

$G = MN$. Clearly, $P = P \cap NM = N(P \cap M)$. Since $P \cap M < P$, we may take a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Consequently, $P = NP_1$ and $N \not\leq P_1$. By hypothesis, P_1 is weakly s-normal in G. Then there are a subnormal subgroup T of G and a subgroup $(P_1)_*$ of P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_*$, where $(P_1)_*$ is a subgroup of P_1 which is either s-permutably embedded or s-semipermutable in G. Since $|G : T|$ is a power of p, we have $N \leq O^p(G) \leq T$. It follows that $P_1 \cap N = (P_1)_* \cap N$.

Firstly, we have $N \cap P_1 \neq 1$. If not, $|N : P_1 \cap N| = |NP_1 : P_1| = |P : P_1| = p$ and so $P_1 \cap N$ is a maximal of N. Therefore $|N| = p$, which contradicts step (6).

Secondly, $N \cap P_1$ is not normal in G. If not, then $N \cap P_1 = N$ since $N \cap P_1 \neq 1$. This contradicts the fact $N \not\leq P_1$.

Thirdly, $(P_1)_*$ is s-semipermutable in G. Assume that $(P_1)_*$ is s-permutably embedded in G. Then there is an s-permutable subgroup K of G such that $(P_1)_*$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ since N is the unique minimal normal subgroup of G. It follows that $N \leq (P_1)_* \leq P_1$, a contradiction. If $K_G = 1$, then, by Lemma 5, we have $(P_1)_*$ is s-permutable in G and so $(P_1)_*$ is s-semipermutable in G.

Now, $(P_1)_*Q = Q(P_1)_*$ for any Sylow q-subgroup Q of G, $q \neq p$. Then, there holds $[P_1 \cap N, Q] \leq N \cap (P_1)_*Q = N \cap (P_1)_* = N \cap P_1$. Obviously, $N \cap P_1$ is normalized by P. Therefore $N \cap P_1$ is normal in G, a contradiction.

(8) $O_p(G)$ is not a maximal subgroup of P.

If not, then, by step (7), we have $1 < |D| < |O_p(G)|$. In view of Lemma 1(5), every subgroup H of $O_p(G)$ with $|H| = |D|$ is weakly s-permutable in G. Applying Lemma 2, every G-chief factor of $O_p(G)$ is cyclic, contrary to step (6).

(9) G has no normal subgroup of index p.

Assume that G has a normal subgroup M such that $|G : M| = p$. Clearly, $P \cap M$ is a Sylow p-subgroup of M. Since $|P : P \cap M| = = |PM : M| = |G : M| = p$, we have $P \cap M$ is a maximal subgroup of P. If $N_G(P \cap M) < G$, then, by step (3), $N_G(P \cap M)$ is p-nilpotent and so is $N_M(P \cap M)$. By step (7), $1 < |D| < |P \cap M|$. In view of Lemma 1(1), every subgroup H of $P \cap M$ with $|H| = |D|$ is weakly s-normal in M. Hence M satisfies the hypothesis of the theorem. The the minimal choice of G shows that M is p-nilpotent. It follows that G is also p-nilpotent. This contradiction shows that $N_G(P \cap M) = G$, namely, $P \cap M$ is a normal p-subgroup of G. Hence $P \cap M \leq O_p(G)$. Since $O_p(G) < P$, we have $P \cap M = O_p(G)$, which contradicts step (8).

(10) Final contradiction.

Since $G/O_p(G)$ is p-nilpotent, we can assume that $G/O_p(G)$ has a normal Hall p'-subgroup, say $L/O_p(G)$. Clearly, L is also normal in G and G/L is p-group. Then there exists a normal subgroup M of G such that $L \leq M$ and $|G : M| = p$, contrary to step (9). □

Corollary 1. *Let p be an odd prime divisor of $|G|$. Suppose that G has a normal subgroup N such that G/N is p-nilpotent and P is a Sylow p-subgroup of N. Suppose that $N_G(P)$ is p-nilpotent and there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with $|H| = |D|$ is weakly s-normal in G. Then G is p-nilpotent.*

Proof. By Lemma 1(1), every subgroup H of P with $|H| = |D|$ is weakly s-normal in N and $N_N(P)$ is p-nilpotent. Applying Theorem 1, N is p-nilpotent. It follows that $O_{p'}(N)$ is the normal Hall p'-subgroup of N. Clearly, $O_{p'}(N) \lhd G$. If $O_{p'}(N) \neq 1$, then it is easy to see that $G/O_{p'}(N)$ satisfies the hypothesis of the theorem by virtue of Lemma 1(3). Hence $G/O_{p'}(N)$ is p-nilpotent by induction. It follows that G is p-nilpotent. Now we assume that $O_{p'}(N) = 1$, i.e. $N = P$. Then $G = N_G(P)$ is p-nilpotent by hypothesis. □

Theorem 2. *Let N be a p-soluble normal subgroup of G such that G/N is p-supersoluble, where p is a prime divisor of $|G|$. Suppose that for a Sylow p-subgroup P of $F_p(N)$, there exists a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ (and order 4 if $|D| = 2$) is weakly s-normal in G. Then G is p-supersoluble.*

Proof. Suppose that this theorem is false and consider a counterexample (G, N) for which $|G||N|$ is minimal. In fact, $F_p(N) = O_{p'p}(N)$ and $F_p(N) = PO_{p'}(N)$.

(1) $O_{p'}(N) = 1$.

Assume that $O_{p'}(N) \neq 1$. We consider the factor group $G/O_{p'}(N)$. Obviously, $(G/O_{p'}(N))/(N/O_{p'}(N)) \cong G/N$ is p-supersoluble. Since $O_{p'}(N/O_{p'}(N)) = 1$, we have

$$F_p(N/O_{p'}(N)) = O_p(N/O_{p'}(N)) = F_p(N)/O_{p'}(N) = PO_{p'}(N)/O_{p'}(N).$$

In view of Lemma 1(3), every subgroup of $F_p(N/O_{p'}(N))$ with the order $|D|$ (and order 4 if $|D| = 2$) is weakly s-normal in $G/O_{p'}(N)$. Thus $G/O_{p'}(N)$ satisfies the hypothesis of the theorem. By the minimal choice of (G, N), we have $G/O_{p'}(N)$ is p-supersoluble and so is G, a contradiction.

(2) Every G-chief factor of N is cyclic.

By (1), $F_p(N) = F(N) = O_p(N) = P$. In view of Lemma 1(5), every subgroup H of $O_p(N)$ with $|H| = |D|$ (and order 4 if $|D| = 2$) is weakly s-permutable in G. Applying Lemma 2, every G-chief factor of $O_p(N)$ is cyclic. Since N is p-soluble, we have $F^*(N) = F^*_p(N) = F_p(N) = O_p(N) = P$ by Lemma 7 and so every G-chief factor of $F^*(N)$ is cyclic. Applying Lemma 6, every G-chief factor of N is cyclic.

(3) The final contradiction.

Since G/N is p-supersoluble by hypothesis, we have G is p-supersoluble, a contradiction. □

Corollary 2. *Let G be a p-soluble and P a Sylow p-subgroup of $F_p(G)$, where p is a prime divisor of $|G|$. If there exists a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ (and order 4 if $|D| = 2$) is weakly s-normal in G, then G is p-supersoluble.*

Theorem 3. *Let N be a normal subgroup of G such that G/N is p-supersoluble, where p is a prime divisor of $|N|$ with $(|N|, p-1) = 1$. Suppose that for a Sylow p-subgroup P of N, there exists a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ (and 4 if $|D| = 2$) is weakly s-normal in G. Then G is p-supersoluble.*

Proof. Suppose that this theorem is false and consider a counterexample (G, N) for which $|G||N|$ is minimal.

(1) N is p-nilpotent.

By Lemma 1(1), it is easy to see that every subgroup H of P with $|H| = |D|$ (and order 4 if $|D| = 2$) is is weakly s-normal in N. Applying Lemma 8, N is p-nilpotent.

(2) $P = N$.

By (1), we know $O_{p'}(N)$ is the normal Hall p'-subgroup of N. Assume that $O_{p'}(N) \neq 1$. In view of Lemma 1(3), the hypothesis holds for $(G/O_{p'}(N), N/O_{p'}(N))$. Hence, by the minimal choice of (G, N), the theorem is true for $(G/O_{p'}(N), L/O_{p'}(N))$ and so $G/O_{p'}(N)$ is p-supersoluble. Consequently, G is p-supersoluble. This contradiction shows that $O_{p'}(N) = 1$. Hence N is a normal p-subgroup of G.

(3) Every G-chief factor of N is cyclic.

In view of Lemma 1(5), every subgroup H of N with $|H| = |D|$ (and order 4 if $|D| = 2$) is weakly s-permutable in G. Applying Lemma 2, every G-chief factor of N is cyclic.

(4) The final contradiction.

Since G/N is p-supersoluble by hypothesis, we have G is p-supersoluble. □

In [3] and [12], we can find the following Theorem:

Theorem 4. *Let p be a prime dividing $|G|$ and P a Sylow p-subgroup of G. If there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with $|H| = |D|$ is s-permutably embedded (or weakly s-semipermutable) in G, then G is p-supersoluble.*

However, above Theorem does not hold if we replace s-permutably embedded (or weakly s-semipermutable) subgroup by weakly s-normal subgroup. For example, let $G = Z_5 \times A_5$, where A_5 it the alternating group of degree 5. It is not hard to see that each subgroup of G with order 5 is weakly s-normal subgroup in G. But, G is not 5-supersoluble.

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