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RESEARCH ARTICLE

On cofinitely ss-supplemented modules

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Abstract. In this paper, we introduce the concept of (amply) cofinitely ss-supplemented modules as a proper generalization of (amply) ss-supplemented modules, and we provide various properties of these modules. In particular, we prove that arbitrary sum of cofinitely ss-supplemented modules is cofinitely ss-supplemented. Moreover, we show that a ring R is semiperfect and $\mathrm{Rad}(R) \subseteq \mathrm{Soc}({}_R R)$ if and only if every left R-module (amply) cofinitely ss-supplemented.

Introduction

In this paper, all rings have an identity and all modules are left and unital. Our terminology and notation adheres to that of the major references in the theory of rings and modules such as [3] and [12]. Other good references are [5], [13] and [1]. We here highlight a few specific facts, notation and terminology because they have been used in this paper. Let R be such a ring and let M be an R-module. The notation $(K \leqslant M)$ $K < M$ means that K is a (proper) submodule of M. The Socle and Jacobson radical of a module M will be denoted as is customary by $\mathrm{Soc}(M)$ and $\mathrm{Rad}(M)$, respectively. A submodule $K \leqslant M$ is called *small* in M, will be denoted by $K \ll M$, if $M \neq K + L$ for every proper submodule L of M ([12, 19.1]). Let K and L be submodules of M. L is called *a supplement* of K in M if it is minimal with respect to

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$M = K + L$, or equivalently $M = K + L$ and $K \cap L \ll L$. The module M is called *supplemented* if every submodule of M has a supplement in M. A submodule $K \leqslant M$ has ample supplements in M if every submodule L of M such that $M = K + L$ contains a supplement of K in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M ([12, 41]). In [12] and [14] characterized (amply) supplemented modules. A non-zero module M is called *hollow* if every proper submodule of M is small in M and M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. Every local module is hollow and every hollow module is amply supplemented. A ring R is called *local* if ${}_RR$ is a local module.

Supplement submodules plays an important role in ring theory and relative homological algebra. In recent years, types of supplement submodules are extensively studied by many authors. In a series of papers [6], [4], [10], [11], [12], authors have obtained detailed information about types of supplement submodules and related rings. The last defined type of supplement submodules is as follows.

Zhou and Zhang have generalized the notion of $\mathrm{Soc}(M)$ to $\mathrm{Soc}_s(M)$ thereby the class of all simple submodules of M that are small in M in place of the class of all simple submodules of M, that is, $\mathrm{Soc}_s(M) = \sum\{\, N \ll M \mid N \text{ is simple } \}$. Therefore we can be seen easily that $\mathrm{Soc}_s(M) \subseteq \mathrm{Rad}(M)$ and $\mathrm{Soc}_s(M) \subseteq \mathrm{Soc}(M)$. Kaynar et.al. call a module M is *strongly local* if it is local and $\mathrm{Rad}(M) \subseteq \mathrm{Soc}(M)$ [5]. A module M is called *ss-supplemented* if every submodule K of M has a supplement L in M such that $K \cap L$ is semisimple (namely, ss-supplements), and a module M is called *amply ss-supplemented* if every submodule of M has ample ss-supplements in M. Here a submodule K of M has ample ss-supplements in M if for every submodule L of M such that $M = K + L$ contains a ss-supplement K in M [5]. They have given in the same paper the characterization of (amply) ss-supplemented modules via semiperfect ring.

In this paper, we study the various properties of (amply) cofinitely ss-supplemented modules as a proper generalization of ss-supplemented modules. We prove that a module M is cofinitely ss-supplemented if and only if the module $M/\mathrm{Loc}_s(M)$ doesn't contain a maximal submodule where $\mathrm{Loc}_s(M)$ is the sum of all strongly local submodules of M. We also show that every left R-module is amply cofinitely ss-supplemented if and only if every left R-module is the sum of all strongly local submodules. Using the mentioned fact, we give a characterization of semiperfect ring R with $\mathrm{Rad}(R) \subseteq \mathrm{Soc}({}_RR)$.

1. Cofinitely ss-supplemented modules

Definition 1. We call a module M *cofinitely ss-supplemented* if for every cofinite submodule N of M, there exists a submodule V of M such that $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is semisimple. We also call a module M *amply cofinitely ss-supplemented* if every cofinite submodule of M has ample ss-supplements in M.

Lemma 1. *Every homomorphic image of a (an amply) cofinitely ss-supplemented module is (amply) cofinitely ss-supplemented.*

Proof. Let $f : M \longrightarrow N$ be a homomorphism and M be a cofinitely ss-supplemented module. Suppose that K is a cofinite submodule of $f(M)$. Therefore $\frac{M}{f^{-1}(K)} \cong \frac{\frac{M}{\ker(f)}}{\frac{f^{-1}(K)}{\ker(f)}}$ with $\frac{M}{\ker(f)} \cong f(M)$ and $\frac{f^{-1}(K)}{\ker(f)} \cong K$. Thus $\frac{M}{f^{-1}(K)}$ is finitely generated. Since M is cofinitely ss-supplemented, we can write $f^{-1}(K) + V = M$, $f^{-1}(K) \cap V \ll V$ and $f^{-1}(K) \cap V$ is semisimple for some submodule V of M. So, $f(f^{-1}(K)) + f(V) = f(M)$ and since K is a submodule of $f(M)$, $f(f^{-1}(K)) = K$ and so $K + f(V) = f(M)$. In addition to that $f(f^{-1}(K)) \cap f(V) \ll f(V)$ by [12, 41.1(7)]. Therefore $K \cap f(V) \ll f(V)$. Since $f^{-1}(K) \cap V$ is semisimple, $K \cap f(V) = f(f^{-1}(K)) \cap f(V)$ is semisimple by [3, 8.1.5 Corollary (1)]. Thus $f(M)$ is cofinitely ss-supplemented.

By adapting this argument we can show similarly that if M is amply cofinitely supplemented then so too is $f(M)$. □

Corollary 1. *Let M be a (an amply) cofinitely ss-supplemented module and N be any submodule of M. Then $\frac{M}{N}$ is (amply) cofinitely ss-supplemented.*

Proof. Consider the canonical epimorphism $\pi : M \longrightarrow \frac{M}{N}$. Then, by Lemma 1, $\pi(M) = \frac{M}{N}$ is cofinitely ss-supplemented. □

Proposition 1. *If M is a cofinitely ss-supplemented module with a cofinite* $\mathrm{Rad}(M)$*, then* $M/\mathrm{Rad}(M)$ *is semisimple.*

Proof. By Corollary 1, we obtain that the factor module $\frac{M}{\mathrm{Rad}(M)}$ of M is cofinitely ss-supplemented. It follows from the hypothesis that $\frac{M}{\mathrm{Rad}(M)}$ is ss-supplemented. Therefore $\frac{M}{\mathrm{Rad}(M)}$ is supplemented. But $\frac{M}{\mathrm{Rad}(M)}$ has no small submodules thus $\frac{M}{\mathrm{Rad}(M)}$ is semisimple. □

Lemma 2. *Let M be a module and N, U be submodules of M such that N is a cofinitely ss-supplemented submodule and U is cofinite. If $N+U$ has a ss-supplement in M, then U has also a ss-supplement in M.*

Proof. Let X be a ss-supplement of $N+U$ in M. Then

$$\frac{N}{(N\cap(X+U))}\cong\frac{(N+X+U)}{X+U}=\frac{M}{X+U}$$

is finitely generated as a factor module of $\frac{M}{U}$. Since N is cofinitely ss-supplemented $N\cap(X+U)$ has a ss-supplement Y in N such that $N\cap(X+U)\cap Y=(X+U)\cap Y$ is semisimple. Then $M=N+U+X=N\cap(X+U)+Y+U+X=X+U+Y$. In addition to that $Y\cap(X+U)=Y\cap((X+U)\cap N)\ll Y$. Since $Y+U\leqslant N+U$, X is a supplement of $Y+U$. That is $X\cap(Y+U)\ll X$. Since X is a ss-supplement of $N+U$ in M, $X\cap(N+U)$ is semisimple. By [3, 8.1.5 Corollary (1)], $X\cap(U+Y)$ is a semisimple module as a submodule of a semisimple module $X\cap(U+N)$. $U\cap(X+Y)\leqslant X\cap(Y+U)+Y\cap(X+U)\ll Y$ and again appliying [3, 8.1.5 Corollaries (1) and (3)] $U\cap(X+Y)$ is semisimple. Therefore U has a ss-supplement $X+Y$ in M. □

Proposition 2. *Arbitrary sum of cofinitely ss-supplemented submodules of a module M is cofinitely ss-supplemented.*

Proof. Let $\{M_i\}_{i\in I}$ be a collection of cofinitely supplemented submodules of M such that $A=\sum_{i\in I}M_i$. Suppose that N is a cofinite submodule of A. Then $\frac{M}{N}$ has a generating set $\{m_1+N, m_2+N,\ldots,M_r+N\}$ so any m_i with $i\in I$ can be expressed as $m_i=a_{i1}+a_{i2}+\cdots+a_{is(i)}$ with each a_{ik} is from some M_{ik} where $i_k\in I$. Any element $m+N$ from $\frac{A}{N}$ can be represented as: $m+N=r_1m_1+\cdots+r_nm_n+N$. So, $m=r_1(a_{11}+\cdots+a_{1s(1)})+\cdots+r_n(a_{n1}+\cdots+a_{ns(n)})+n$, where $n\in N$. Thus $A=\sum_{j\in J}M_j+N$ with a finite set $J=\{1_1,\ldots,1_{s(1)},2_1,\ldots,n_{s(n)}\}$. Then $A=\sum_{j\in J}M_j+N=M_{11}+\sum_{j\in J-\{11\}}M_j+N$. We know that M_{11} is cofinitely ss-supplemented, and $M_{11}+\sum_{j\in J-\{11\}}M_j+N$ has trivially the ss-supplement 0. Continuing in this way since the set J is finite at the end we can say N has a ss-supplement in A by Lemma 2. □

Recall that a module N is called *M-generated* if there is an epimorphism $f:M^{(I)}\longrightarrow N$ for some index set I.

Corollary 2. *If M is cofinitely ss-supplemented then any M-generated module is cofinitely ss-supplemented.*

Lemma 3. *Let M be a cofinitely ss-supplemented module. Then every cofinite submodule of $\frac{M}{\mathrm{Rad}(M)}$ is a direct summand.*

Proof. Any cofinite submodule of $\frac{M}{\mathrm{Rad}(M)}$ has the form $\frac{N}{\mathrm{Rad}(M)}$ where N is a cofinite submodule of M. Then there exists a submodule K of M such that $M = N + K$, $N \cap K \ll K$ and $N \cap K$ is semisimple. Then $N \cap K \ll M$. Hence $N \cap K \subseteq \mathrm{Rad}(M)$. Thus $\frac{M}{\mathrm{Rad}(M)} = \frac{N}{\mathrm{Rad}(M)} \oplus \frac{K+\mathrm{Rad}(M)}{\mathrm{Rad}(M)}$ as required. □

Let M be any R-module. Then $\mathrm{Loc}_s(M)$ will denote the sum of all strongly local submodules of M and $\mathrm{Cof}_s(M)$ the sum of all cofinitely ss-supplemented submodules of M.

Theorem 1. *The following are equivalent for an R-module M.*

(1) *M is cofinitely ss-supplemented.*

(2) *Every maximal submodule of M has a ss-supplement in M.*

(3) *$\frac{M}{\mathrm{Loc}_s(M)}$ doesn't contain a maximal submodule.*

(4) *$\frac{M}{\mathrm{Cof}_s(M)}$ doesn't contain a maximal submodule.*

Proof. (1) ⇒ (2) Let N be a maximal submodule of M. Then $\frac{M}{N}$ is simple, so by assumption N has a ss-supplement in M.

(2) ⇒ (3) Let N be a maximal submodule of M. Then there exists a submodule K of M such that $M = N + K$, $N \cap K \ll K$ and $N \cap K$ is semisimple. Then $\frac{N+K}{N} = \frac{M}{N}$ and so $\frac{N+K}{N} \cong \frac{K}{N \cap K}$. Therefore $N \cap K$ is a maximal submodule of K. Then $N \cap K = \mathrm{Rad}(K)$. Hence K is a local submodule of M. Since $\mathrm{Rad}(K) = N \cap K \subseteq \mathrm{Soc}(K)$, K is strongly local and so $K \subseteq \mathrm{Loc}_s(M)$. It follows that $\mathrm{Loc}_s(M)$ is not a submodule of N. Hence $\frac{M}{\mathrm{Loc}_s(M)}$ doesn't contain a maximal submodule.

(3) ⇒ (4) Suppose that $\frac{M}{\mathrm{Cof}_s(M)}$ contains a maximal submodule, say $\frac{N}{\mathrm{Cof}_s(M)}$. But then $\theta^{-1}(\frac{N}{\mathrm{Cof}_s(M)})$ is a maximal submodule of $\frac{M}{\mathrm{Cof}_s(M)}$, where $\theta : \frac{M}{\mathrm{Loc}_s(M)} \longrightarrow \frac{M}{\mathrm{Cof}_s(M)}$ is epimorphism. Contradiction. Therefore $\frac{M}{\mathrm{Cof}_s(M)}$ doesn't contain a maximal submodule.

(4) ⇒ (1) Let N be a cofinite submodule of M. Then $N + \mathrm{Cof}_s(M)$ is a cofinite submodule of M and hence by the hypothesis gives that $M = N + \mathrm{Cof}_s(M)$. Since $\frac{M}{N}$ is finitely generated, it follows that $M = N + K_1 + \cdots + K_n$ for some positive integer n and cofinitely ss-supplemented submodules K_i $(1 \leqslant i \leqslant n)$. By Proposition 2 and Lemma 2, N has a ss-supplement in M. It follows that M is cofinitely ss-supplemented. □

Let $P(N)$ denote the collection of maximal submodules of M containing N. In particular, $P(M)$ is the empty set and $P(0)$ is the collection of all maximal submodules of M (which could also be the empty set). Define a relation $\Re$ on the lattice of submodules of M as follows: Given submodules N, L of M then $N \Re L$ if and only if $P(N) = P(L)$. Clearly, $\Re$ is an equivalence relation on the lattice of submodules of M. Note that every maximal submodule has a ss-supplement in M if and only if $M \Re \operatorname{Loc}_s(M)$.

Theorem 2. *The following statements are equivalent for a module M.*

(1) *M is amply cofinitely ss-supplemented.*
(2) *Every submodule N of M with cyclic $\frac{M}{N}$ has ample ss-supplements in M.*
(3) *Every maximal submodule of M has ample ss-supplements in M.*
(4) *$N \Re \operatorname{Loc}_s(M)$ for every submodule N of M.*
(5) *$Rm \Re \operatorname{Loc}_s(M)$ for every $m \in M$ but not in* $\operatorname{Rad}(M)$*.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Let N be any submodule of M such that $\frac{M}{N}$ is finitely generated. If $N = M$ then N has ample ss-supplements in M. Suppose that N doesn't equal to M. Let L be the proper submodule of M containing N such that $\frac{L}{N} = \operatorname{Rad}(\frac{M}{N})$, i.e. L is the intersection of all maximal submodules of M containing N. Note that $\frac{L}{N}$ is small in $\frac{M}{N}$, because $\frac{L}{N}$ is finitely generated. Let P be any maximal submodule of M such that L is contained in P. By assumption there exists a submodule T of M such that $M = P + T$, $P \cap T \ll T$ and $P \cap T$ is semisimple. It follows that $\frac{M}{L} = \frac{P}{L} \oplus \frac{T+L}{L}$ because, $\frac{P}{L} \cap \frac{T+L}{L} \ll \frac{T+L}{L}$ and hence also in $\frac{M}{L}$, i.e. $\frac{P}{L} \cap \frac{T+L}{L} = 0$. Thus, $\frac{M}{L}$ is finitely generated and semisimple. Therefore, L is a finite intersection of maximal submodules of M. By Lemma 2, L has ample ss-supplements in M.

Let K be any submodule of M such that $M = N+K$. Then $M = L+K$. There exists a submodule S of K such that $M = L + S$, $L \cap S \ll S$ and $L \cap S$ is semisimple. Then we have $\frac{M}{N} = \frac{L}{N} + \frac{S+N}{N}$, so that $\frac{M}{N} = \frac{S+N}{N}$ and hence $M = S + N$. Clearly $N \cap S$ is contained in $L \cap S$ so that $N \cap S \ll S$. By [3, 8.1.5 Corollary (1)] N has ample supplements in M.

(3) $\Rightarrow$ (4) Let F be any submodule of M. Let Q be a maximal submodule of M such that F is not contained in Q. Then $M = F + Q$. There exists a submodule T of F such that $M = Q + T$, $Q \cap T \ll T$ and $Q \cap T$ is semisimple. By [5, Proposition 3.1] T is a strongly local submodule of F and T is not contained in Q. Thus $\operatorname{Loc}_s(F)$ is not contained in Q. Therefore $F \Re Loc_s(F)$.

(4) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (3) Let P be any maximal submodule of M and let G be a submodule of M such that $M = P + G$. There exists g in G such that g does not belong to P. Hence Rg is not contained in P, so that $\mathrm{Loc}_s(Rg)$ is not contained in P because $Rg \Re \mathrm{Loc}_s(Rg)$. Let L be a strongly local submodule of Rg and hence also of G, such that L is not contained in P. Then $M = P + L$, $P \cap L \ll L$ and $P \cap L$ is semisimple. Thus L is a ss-supplement of P in M. So P has ample supplements. □

Corollary 3. *Let M be a module such that $N = \mathrm{Loc}_s(N)$ for every submodule N of M. Then every maximal submodule of M has ample supplements in M.*

Lemma 4. *Let M be an R-module and $M = U_1 + U_2$. If the submodules U_1, U_2 have ample ss-supplements in M, then $U_1 \cap U_2$ has also ample ss-supplements in M.*

Proof. Let $V < M$ with $U_1 \cap U_2 + V = M$. Then $M = U_1 + U_2 = U_1 + (U_2 \cap M) = U_1 + (U_2 \cap ((U_1 \cap U_2) + V)) = U_1 + (U_1 \cap U_2) + (U_2 \cap V) = U_1 + (U_2 \cap V)$. $M = U_2 + (U_1 \cap V)$ also holds. Therefore there is a ss-supplement $V_2^{'}$ of U_1 in M with $V_2^{'} < U_2 \cap V$ and a ss-supplement $V_1^{'}$ of U_2 with $V_1^{'} < U_1 \cap V$. By assumption we have, for $V_1^{'} + V_2^{'} < V$, the relations $M = (U_1 \cap U_2) + (V_1^{'} + V_2^{'})$ and $(V_1^{'} + V_2^{'}) \cap (U_1 \cap U_2) = (V_1^{'} \cap U_2) + (V_2^{'} \cap U_1) \ll V_1^{'} + V_2^{'}$. Then we have $(U_1 \cap U_2) \cap (V_1^{'} + V_2^{'}) \leqslant V_1^{'} \cap (U_1 \cap U_2) + V_2^{'} \cap (U_1 \cap U_2) \leqslant V_1^{'} \cap U_2 + V_2^{'} \cap U_1$. Since $V_1^{'} \cap U_2$ and $V_2^{'} \cap U_1$ are semisimple, then by [3, 8.1.5 Corollaries (1) and (3)] $(U_1 \cap U_2) \cap (V_1^{'} + V_2^{'})$ are semisimple. So $V_1^{'} + V_2^{'}$ is a ss-supplement of $U_1 \cap U_2$. Hence $U_1 \cap U_2$ has ample ss-supplements in M. □

Recall that a ring R *left max* if every non-zero left R-module has a maximal submodule. Note that if R is a left max ring, then every left R-module is coatomic.

Lemma 5. *Let R be a ring. Then every left R-module is amply cofinitely ss-supplemented if and only if every left R-module is the sum of all strongly local submodules.*

Proof. If every left R-module M is amply cofinitely ss-supplemented, by [12, 43.9], R is left perfect. This implies that R is a left max ring. By [5, Corollary 3.20], M is the sum of strongly local submodules of M. The converse follows from [5, Theorem 3.19]. □

Recall that an R-module M *semiperfect* if every factor module of M has a projective cover. If the ring R as a left R-module is semiperfect then the ring R is semiperfect.

Theorem 3. *The following statements are equivalent for ring R.*

(1) *${}_RR$ is amply cofinitely ss-supplemented.*
(2) *R is semiperfect and $\mathrm{Rad}(R) \subseteq \mathrm{Soc}({}_RR)$.*
(3) *R is semilocal and $\mathrm{Rad}(R) \subseteq \mathrm{Soc}({}_RR)$.*
(4) *Every projective left R-module is cofinitely ss-supplemented.*
(5) *Every left R-module is (amply) cofinitely ss-supplemented.*
(6) *Every left R-module is the sum of all strongly local submodules.*
(7) *${}_RR$ is a finite sum of strongly local submodules.*
(8) *Every maximal left ideal of R has ample ss-supplement in R.*

Proof. (1) ⇒ (2) ⇒ (3) By [5, Corollary 3.10] and [12, 42.6].

(3) ⇒ (4) Clear by [5, Theorem 3.30].

(4) ⇒ (5) Follows [12, 18.6] and Corollary 1.

(5) ⇒ (6) By Lemma 5.

(6) ⇒ (7) is obvious.

(7) ⇒ (8) By [5, Theorem 3.19].

(8) ⇒ (1) By Theorem 2. □

Proposition 3. *Let M be a π-projective cofinitely ss-supplemented module. Then M is amply cofinitely ss-supplemented.*

Proof. Let N be a cofinite submodule of M and let K be a submodule of M such that $M = N + K$. There exists an endomorphism σ of M such that $\sigma(M) \leqslant N$ and $(1-\sigma)(M) \leqslant K$. Note that $(1-\sigma)(N) \leqslant N$. Let T be a ss-supplement of N in M. Then $M = \sigma(M) + (1-\sigma)(M) = \sigma(M) + (1-\sigma)(N+T) \leqslant N + (1-\sigma)(T) \leqslant M$, so that $M = N + (1-\sigma)(T)$. Note that $(1-\sigma)(T)$ is a submodule of K. Let $y \in N \cap (1-\sigma)(T)$. Then $y \in N$ and $y = (1-\sigma)(x) = x - \sigma(x)$ for some $x \in T$. Then $y + \sigma(x) \in N$, so that $y \in (1-\sigma)(N \cap T)$. But $N \cap T \ll T$ gives that $N \cap (1-\sigma)(T) = (1-\sigma)(N \cap T) \ll (1-\sigma)(T)$ by [12, 19.3]. Since $N \cap T$ is semisimple, $N \cap (1-\sigma)(T) = (1-\sigma)(N \cap T)$ is semisimple. Therefore M is amply cofinitely ss-supplemented. □

Lemma 6. *Let L_i $(1 \leqslant i \leqslant n)$ be a finite collection of strongly local submodules of a module M and let N be a submodule of M such that $N + L_1 + \cdots + L_n$ has a ss-supplement K in M. Then there exists a (possibly empty) subset I of $\{1, 2, \ldots, n\}$ such that $K + \sum_{i \in I} L_i$ is a ss-supplement of N in M.*

Proof. Suppose that $n = 1$. Consider the submodule $H = (N + K) \cap L_1$ of L_1. If $H = L_1$ then 0 is a ss-supplement of H in L_1 and the proof of Lemma 2 shows that $K = K + 0$ is a ss-supplement of N in M. If

$H \neq L_1$, then L_1 is a ss-supplement of U in L_1 and in this case $K + L_1$ is a ss-supplement of N in M, again by the proof of Lemma 1. This proves the result when $n = 1$. Suppose that $n > 1$. By induction on n, there exists a subset I' of $\{2, \dots, n\}$ such that $K + \sum_{i \in I} L_i$ is a ss-supplement of $N + L_1$ in M. Now the case $n = 1$ shows that either $K + \sum_{i \in I'} L_i$ or $K + L_1 + \sum_{i \in I'} L_i$ is a ss-supplement of N in M. □

Theorem 4. *Let R be any ring. The following statements are equivalent for an R-module M.*

(1) *M is amply cofinitely ss-supplemented.*
(2) *Every maximal submodule of M has ample supplements in M.*
(3) *For every cofinite submodule N and submodule L of M such that $M = N + L$, there exists a positive integer n and strongly local submodules L_i $(1 \leqslant i \leqslant n)$ of L such that $M = N + L_1 + \cdots + L_n$.*
(4) *$P(N) = P(\mathrm{Loc}_s(N))$ for every submodule N of M.*
(5) *$P(Rm) = P(\mathrm{Loc}_s(Rm))$ for every element $m \in M \setminus \mathrm{Rad}(M)$.*

Proof. (1) ⇒ (2) Clear.

(3) ⇒ (1) By Lemma 6.

(4) ⇒ (5) Clear.

(2) ⇒ (4) Let N be any submodule of M and let K be a maximal submodule of M such that N is not a submodule of K. Then $M = K + N$. By (2), there exists a submodule L of N such that L is a ss-supplement of K in M. By [5, Proposition 3.1], L is a strongly local submodule of N, follows from $\mathrm{Loc}_s(N)$ is not a submodule of K that (4) holds.

(2) ⇒ (3) Suppose that (2) holds and there exists a cofinite submodule N of M such that $M = N + L$ for some submodule L of M but $M \neq N + K$ for every K of L with K a finite sum of strongly local submodules. Let Ω denote the collection of submodules H of M such that $N \leqslant H$ and $M \neq H + K$ for every submodule K of L with K a finite sum of strongly local submodules. By Zorn's Lemma, Ω contains a maximal element U. Because U is a cofinite submodule of M and $U \neq M$, there exists a maximal submodule X of M such that $U \leqslant X$. Clearly $M = X + L$. By (2) there exists a submodule Y of L such that Y is a ss-supplement in M. Now $M = X + Y$ and $X \cap Y \ll Y$. Note that Y is a strongly local submodule of M by [5, Proposition 3.1]. Clearly Y is not a submodule of X gives that Y is not a submodule of U, i.e., $U \neq U + Y$. By the choice of U, there exists a submodule V of L such that $M = (U + Y) + V$ and V is finite sum of strongly local submodules. But $Y + V$ is a finite sum of strongly local submodules and a submodule of L and $M = U + (Y + V)$, a contradiction. This proves (3).

(5) $\Rightarrow$ (2) Let K be a maximal submodule of M and let H be a submodule of M such that $M = K + H$. There exists $x \in H$ such that $x \notin K$ and hence $M = K + Rx$. Clearly $x \in M \setminus \mathrm{Rad}(M)$ and by the hypothesis $K \notin P(Rx) = P(\mathrm{Loc}_s(Rx))$. By [5, Proposition 3.1], there exists a strongly local submodule L of Rx such that L is not a submodule of K. In this case, $M = K + L$, $K \cap L \ll L$ and $K \cap L$ is semisimple, so that L is a ss-supplement of K in M. Note that L is a submodule of H. Therefore M is amply cofinitely ss-supplemented. □

Corollary 4. *Let M be an R-module such that every cyclic submodule is ss-supplemented. Then M is amply cofinitely ss-supplemented.*

Proof. Let $m \in M$. By hypothesis, we get that Rm is ss-supplemented and so, by Theorem 1, $\frac{Rm}{\mathrm{Loc}_s(Rm)}$ doesn't contain a maximal submodule. Therefore $Rm = \mathrm{Loc}_s(Rm)$. It follows that $Rm = \mathrm{Loc}_s(Rm)$ for all $m \in M$. Hence M is amply cofinitely ss-supplemented according to Theorem 4. □

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