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A virtually 2-step nilpotent group with polynomial geodesic growth*

A. Bishop and M. Elder

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Abstract. A direct consequence of Gromov's theorem is that if a group has polynomial geodesic growth with respect to some finite generating set then it is virtually nilpotent. However, until now the only examples known were virtually abelian. In this note we furnish an example of a virtually 2-step nilpotent group having polynomial geodesic growth with respect to a certain finite generating set.

Introduction

The *geodesic growth* function for a finitely generated group with respect to a finite (monoid) generating set S is the function which sends n to the number of geodesic words over S of length at most n. It is bounded below by the usual growth function which counts the number of group elements represented by a word over S of length at most n.

Bridson, Burillo, Šunić and the second author [5] investigated groups for which this function is polynomial, building on work of Shapiro [10]. They showed that if a nilpotent group is not virtually cyclic then it has exponential geodesic growth with respect to all finite generating sets. They also gave an example of a virtually $\mathbb{Z}^2$ group having polynomial geodesic

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growth with respect to a certain generating set, and a sufficient condition for when a virtually abelian group has polynomial geodesic growth.

The first author extended this to show for virtually abelian groups, the function is either polynomial or exponential [2] with respect to any finite generating set (in particular, cannot be intermediate).

Here we take the next step, by furnishing a first example of a virtually 2-step nilpotent group having polynomial geodesic growth. The group contains the integral 3-dimensional Heisenberg group of index 2.

Our proof relies on a fact contained in work of Blachère [4] that for the integral 3-dimensional Heisenberg group with respect to the (standard) generating set $\{a, a^{-1}, b, b^{-1}\}$, every element has a geodesic representative which "switches" between $a^{\pm 1}$ letters and $b^{\pm 1}$ letters at most five times (see Lemma 1). We exploit this somewhat surprising fact to construct our example.

Our result opens the door to the intriguing possibility that some construction of a virtually nilpotent group could have intermediate geodesic growth with respect to some generating set. It also raises the question of whether polynomial geodesic growth is restricted to virtually nilpotent groups of step at most 2, or if some construction works for higher steps.

1. Virtually Heisenberg group

Let G be a group with finite generating set X. For each word $w = w_1 w_2 \cdots w_k \in X^*$ we write $|w|_X = k$ for the *word length* of w, and $\overline{w} \in G$ for the element corresponding to the word w. We write $w^R = w_k \cdots w_2 w_1$ for the *reverse* of w. For each element $g \in G$ we write $\ell_X(g) = \min\{|w|_X \mid \overline{w} = g\}$ for the *length* of an element with respect to the generating set X. A word $w \in X^*$ is a *geodesic* if $\ell_X(\overline{w}) = |w|_X$, and $\gamma_X : \mathbb{N} \to \mathbb{N}$ defined by

$$\gamma_X(n) = \{w \in X^* \mid \ell_X(\overline{w}) = |w|_X \leqslant n\}$$

is the *geodesic growth function* of G with respect to X.

Consider the discrete Heisenberg group

$$\mathscr{H} = \langle a, b \mid [a,[a,b]] = [b,[a,b]] = 1 \rangle$$

with generating set $X = \{a, a^{-1}, b, b^{-1}\}$ where $[a,b] = aba^{-1}b^{-1}$. We follow the convention of Blachère [4] and write $(x,y,z) \in \mathscr{H}$ for the element corresponding to the normal form word $[a,b]^z b^y a^x$.

We define the virtually Heisenberg group

$$v\mathscr{H} = \left\langle a, b, t \mid [a,[a,b]] = [b,[a,b]] = t^2 = 1,\ a^t = b \right\rangle. \tag{1}$$

We see that $S = \{a, a^{-1}, t\}$ is a generating set for $v\mathscr{H}$ as after a Tietze transform to remove the generator b we may obtain the presentation

$$v\mathscr{H} = \left\langle a, t \mid [a, [a, a^t]] = [a^t, [a, a^t]] = t^2 = 1 \right\rangle . \tag{2}$$

We provide a partial view of the Cayley graph of $(v\mathscr{H}, S)$ in Figure 1. Informally, one may think of this group as two copies of $\mathscr{H}$ glued with a "twist" by t edges. Our construction is patently inspired by Cannon's virtually-$\mathbb{Z}^2$ example with interesting geodesic behaviour [8, 9].

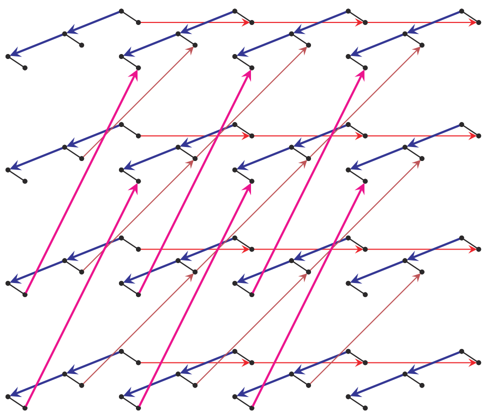

FIGURE 1. Cayley graph for $v\mathscr{H}$ with respect to the generating set S where the undirected edges are labelled by t and directed edges labelled by a.

Our goal is to show that any geodesic of $v\mathscr{H}$ with respect to the generating set S can contain at most 7 instances of the letter t. From this we are able to place a polynomial upper bound on the geodesic growth function of $v\mathscr{H}$. To do this, we first study geodesics of the discrete Heisenberg group with respect to the generating set X.

In [4] Blachère provided explicit formulae for the length of elements in $\mathscr{H}$, with respect the generating set X, by constructing geodesic example. The following lemma is implicit in Blachère's work.

Lemma 1. *Each element $(x, y, z) \in \mathscr{H}$ has a geodesic representative with respect to the generating set $X = \{a, a^{-1}, b, b^{-1}\}$ of the form*

$$a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} a^{\alpha_3} b^{\beta_3} \quad \text{or} \quad b^{\beta_1} a^{\alpha_1} b^{\beta_2} a^{\alpha_2} b^{\beta_3} a^{\alpha_3}$$

where each $\alpha_i, \beta_j \in \mathbb{Z}$.

Proof. We see that our lemma holds in the case of $(0, 0, 0) \in \mathscr{H}$ as the empty word $\varepsilon \in S^*$ is such a geodesic. In the remainder of this proof, we assume that $(x, y, z) \neq (0, 0, 0)$. Following Blachère [4, p. 22] we reduce this proof to the case where $x, z \geqslant 0$ and $-x \leqslant y \leqslant x$ as follows.

Let $\tau\colon X^* \to X^*$ be the monoid isomorphism defined such that $\tau(a^k) = b^k$ and $\tau(b^k) = a^k$ for each $k \in \mathbb{Z}$. If $w \in X^*$ is a word as described in the lemma statement with $\overline{w} = (x, y, z)$, then $w' = \tau(w^R)$ is also in the form described in the lemma statement and $\overline{w'} = (y, x, z)$. Moreover, we see that $\tau(w^R)$ is a geodesic if and only if w is a geodesic. Defining the monoid isomorphisms $\varphi_a, \varphi_b\colon X^* \to X^*$ by $\varphi_a(a^k) = a^{-k}$, $\varphi_a(b^k) = b^k$, and $\varphi_b(a^k) = a^k$, $\varphi_b(b^k) = b^{-k}$ for each $k \in \mathbb{Z}$, we see that if $w \in X^*$ is a geodesic representative for $(x, y, z) \in \mathscr{H}$, then $\varphi_a(w)$, $\varphi_b(w)$ and $\varphi_a(\varphi_b(w))$ are geodesics for $(-x, y, -z)$, $(x, -y, -z)$ and $(-x, -y, z)$, respectively, and each such word is in the form as described in the lemma statement. From application of the above transformations, we may assume without loss of generality that $x, z \geqslant 0$ and $-x \leqslant y \leqslant x$.

Let $h = (x, y, z) \in \mathscr{H}$, then from [4, Theorem 2.2] we have the following formulae for the length $\ell_X(h)$ and (most importantly for us) geodesic representative for h.

I. If $y \geqslant 0$, then we have the following cases.
   - I.1. If $x < \sqrt{z}$, then $\ell_X(h) = 2\lfloor 2\sqrt{z}\rfloor - x - y$ and h has a geodesic representative given by $b^{y-y'} S_z a^{x-x'}$ where x', y' are the values given by $\overline{S_z} = (x', y', z)$ (cf. [4, p. 32]), where S_z is as follows.
     - ∗ If $z = (n+1)^2$ for some $n \in \mathbb{N}$, then $S_z = a^{n+1}b^{n+1}$;
     - ∗ if there exists a $k \in \mathbb{N}$ with $1 \leqslant k \leqslant n$ such that $z = n^2 + k$, then let $S_z = a^k b a^{n-k} b^n$;
     - ∗ otherwise, there exists some $k \in \mathbb{N}$ with $1 \leqslant k \leqslant n$ such that $z = n^2 + n + k$ and we have $S_z = a^k b a^{n+1-k} b^n$.
   - I.2. If $x \geqslant \sqrt{z}$, then we have the following two cases:
     - I.2.1 $xy \geqslant z$, then $\ell_X(h) = x + y$, otherwise
     - I.2.2 $xy \leqslant z$, then $\ell_X(h) = 2\lceil z/x \rceil + x - y$;

     and in both cases, the word $b^{y-u-1} a^v b a^{x-v} b^u$ is a geodesic for h where $0 \leqslant u$, $0 \leqslant v < x$ and $z = ux + v$ (cf. [4, p. 24, 32, 33]).

II. If $y < 0$, then we have the following cases.
   - II.1. If $x \leqslant \sqrt{z - xy}$, then $\ell_X(h) = 2\lceil 2\sqrt{z - xy}\rceil - x + y$. Let $n = \lceil \sqrt{z - xy} \rceil - 1$. Then
     - ∗ there is either some $k \in \mathbb{N}$ with $1 \leqslant k \leqslant n$ such that we have $z - xy = n^2 + k$, and h has $a^{x-n} b^{-n-1} a^k b a^{n-k} b^{n+y}$ as a geodesic representative; or

* there is some $k \in \mathbb{N}$ with $0 \leqslant k \leqslant n$ such that we have $z - xy = (n+1)^2 - k$ and $a^{x-n}b^{-k}a^{-1}b^{k-n-1}a^{n+1}b^{n+1+y}$ is a geodesic representative for h (cf. [4, p. 24][1]).

II.2. If $x \geqslant \sqrt{z-xy}$, then $\ell_X(h) = 2\lceil z/x \rceil + x - y$ and h has a geodesic representative of $b^{y-u-1}a^v b a^{x-v} b^u$ where $u, v \geqslant 0$, From this we obtain the following.

Theorem 1. *The geodesic growth function of* $v\mathscr{H}$ *with respect to* $S = \{a, a^{-1}, t\}$ *is bounded from above by a polynomial of degree* 8.

Proof. From Corollary 1, we see that any geodesic of $v\mathscr{H}$, with respect to the generating set S, must have the form

$$w = a^{m_1} t a^{m_2} t \cdots t a^{m_{k+1}}$$

where $k \leqslant 7$ and each $m_i \in \mathbb{Z}$. Then with k fixed and $r = |w|_S$, we see that there are at most 2^{k+1} choices for the sign of $m_1, m_2, \ldots, m_{k+1}$, and at most $\binom{r}{k}$ choices for the placement of the t's in w. Thus the geodesic growth function $\gamma_S(n)$ has an upper bound given by

$$\gamma_S(n) \leqslant \sum_{r=0}^{n} \sum_{k=0}^{7} 2^{k+1} \binom{r}{k}$$

which give the degree 8 polynomial upper bound. □

2. Open questions and further work

The key to our proof of Theorem 1 is the explicit calculation of geodesic length and consequent explicit form of geodesic words by Blachère. In particular, we exploit the fact that each element of $\mathscr{H}$ has some geodesic of a particularly special form to make our construction of $v\mathscr{H}$. For these reasons, our proof does not immediately appear to generalise to other virtually nilpotent groups (or for that matter to different generating sets of $v\mathscr{H}$). In light of this, we pose the following question.

Question 1 (Characterising polynomial geodesic growth)**.** *For which* $k \in \mathbb{N}$ *is there a virtually* k*-step nilpotent group with polynomial geodesic growth with respect to some finite generating set?*

It follows from [1, Theorem 2] that the usual growth rate of a virtually nilpotent group is polynomial of integer degree. Moreover, from [2] it is known that if a virtually abelian group has polynomial geodesic growth, then it must be of integer degree (since the geodesic growth series is rational in this case). It is not known if there is a virtually nilpotent group with polynomial geodesic growth of a non-integer degree. Based on experimental results we conjecture that the geodesic growth rate of $v\mathscr{H}$ with respect to the generating set S can be bounded from above and below by polynomials of degree six (cf. the usual growth is polynomial of degree four). The corresponding code and first 645 terms of the geodesic growth function are available from [3].

Question 2 (Degree of polynomial geodesic growth)**.** *Is there a group with polynomial geodesic growth which is not equivalent to n^c for $c \in \mathbb{N}$ with respect to some finite generating set?*

The first author showed that the geodesic growth series for virtually abelian groups is D-finite (holonomic) in the exponential case and rational in the polynomial case [2]. It follows from Pólya-Carlson Theorem [6] that a geodesic growth sequence of sub-exponential growth is either rational, or its associated generating function has the unit circle as its natural boundary. (In particular, such a sequence is either rational or is not D-finite.) It was shown by Duchin and Shapiro [7] that the usual growth of $\mathscr{H}$ is rational for all generating sets. Preliminary investigation of the data in [3] leads us to suspect that the geodesic growth sequence for $(v\mathscr{H}, S)$ is not rational, which would mean it is not D-finite.

Finally, we recall a motivating question from [5].

Question 3. *Is there a group with intermediate geodesic growth?*

From [2, 5] it is known that if such a group exists, then it cannot be nilpotent or virtually abelian.

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Contact information

Alex Bishop, Murray Elder

School of Mathematical and Physical Sciences, University of Technology Sydney, Ultimo NSW 2007 Australia

E-Mail(s): alexbishop1234@gmail.com, murray.elder@uts.edu.au

Web-page(s): alexbishop.github.io, sites.google.com/site/melderau

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