© Algebra and Discrete Mathematics
Volume **33** (2022). Number 2, pp. 29–59
DOI:10.12958/adm1663

RESEARCH ARTICLE

On some relationships between snake graphs and Brauer configuration algebras*

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Communicated by V. M. Futorny

Abstract. In this paper, suitable Brauer configuration algebras are used to give an explicit formula for the number of perfect matchings of a snake graph. Some relationships between Brauer configuration algebras with path problems as the Lindström problem are described as well.

Introduction

Snake graphs are combinatorial objects arising from the research of cluster algebras. They allowed to Çanakçi and Schiffler to compute the Laurent expansions of the cluster variables in cluster algebras of surface type. The terms in the Laurent polynomial of such variables are parametrized by the perfect matchings of the associated snake graph [3–8, 15]. Such graphs were studied by Propp [15] in the context of the investigation of the Laurent phenomenon, which is a problem of paramount importance in the theory of cluster algebras. Propp proved that two examples of rational recurrences, the two-dimensional frieze patterns of Conway and Coxeter and the tree of Markoff numbers-relate to one another and to the Laurent phenomenon. In the program of Propp perfect matchings of snake graphs derived from triangulations of polygons are linked with frieze patterns of Conway and Coxeter.

*J. L. Ramírez was partially supported by Universidad Nacional de Colombia, Project No. 46240. P. F. F. Espinosa was partially supported by COLCIENCIAS, Doctorados nacionales No. 785.

2020 MSC: 16G20, 16G60, 16G30.

Key words and phrases: Brauer configuration algebra, Fibonacci number, perfect matching, snake graph.

Propp in [15] also reported an interesting connection between snake graphs and continued fractions, according to him, work of Benjamin and Quinn in the context of the strip tiling model, shows how combinatorial models can illuminate facts about continued fractions. In [3–7] Çanakçi and Schiffler go beyond Propp by proving that each snake graph $\mathscr{G}$ has associated a unique continued fraction whose numerator is given by the number of perfect matchings of $\mathscr{G}$. They report that snake graphs provide a new combinatorial model for continued fractions allowing to interpret the numerators and denominators of positive continued fractions as cardinals of combinatorially defined sets.

Regarding applications of the theory of snake graphs we recall that recently Çanakçi and Schroll [8] defined abstract string modules associating to each of such modules a suitable snake graph, whose lattice of perfect matchings is in bijective correspondence with the lattice of submodules of such abstract module. In this work, the number of perfect matchings of a snake graph is interpreted as the message of a labeled Brauer configuration. The same is done for the number of k-paths connecting two fixed points u and v in an acyclic finite digraph.

The following is a list of our main results:

1) It is proved that the number of perfect matchings of an arbitrary snake graph is given by messages of some suitable labeled Brauer configurations (see Theorem 17). Particular cases of this result are given in Theorem 14 and Corollaries 15 and 16.
2) An interpretation of the Lindström theorem (regarding the number of tuples of non-intersecting lattice paths) is given based on the message of a labeled Brauer configuration (see Theorem 23 and Corollary 24),
3) Properties of the Brauer configuration algebra $\Lambda_{\mathscr{D}(k)}$ induced by a suitable Brauer configuration $\mathscr{D}(k)$ are given and proved in Theorem 26 and Corollary 27. Such Brauer configuration $\mathscr{D}(k)$ allows to enunciate Theorem 20 which, together with Corollary 21 states Corollary 22 obtaining in this fashion an alternative interpretation of the formula for perfect matchings given in Theorem 17. Specializations of $\mathscr{D}(k)$ are used to give the number of vertices in a suitable system of k disjoint paths $(A_0, A_1, \ldots, A_{k-1})$ in Corollary 25.
4) Integer sequences $a(n, k)$ arising from perfect matchings of some suitable snake graphs and their relationships with the Fibonomial array are described in Section 2.2. Actually, we pose the following conjecture.

Conjecture.

$$\sum_{n\geqslant 2} a(n,k)x^n = \frac{p_k(x)}{\sum_{k=0}^{n}(-1)^{\binom{k+1}{2}}\left[\begin{smallmatrix} n\\ k\end{smallmatrix}\right]_F x^k},$$

where $a(n,k) = \mathrm{Match}(\mathcal{G}_f(\underbrace{n,n,\ldots,n}_{k \text{ times}}))$ (i.e., the number of perfect matchings of a snake graph whose rows and columns have n tiles), $p_k(x)$ is a polynomial of degree k, and $\left[\begin{smallmatrix} n\\ k\end{smallmatrix}\right]_F$ is the (n,k)-th entry of the Fibonomial triangle (array A010048 in the OEIS).

The paper is organised as follows. In section 2, we recall main notation and definitions regarding snake graphs and Brauer configuration algebras. In particular, we introduce the notion of a labeled Brauer configuration. In section 3, we give the number of perfect matchings of snake graphs via suitable labeled Brauer configuration algebras, the Lindström theorem is enunciated based on the message of a suitable Brauer configuration algebra and some interesting sequences in the On-line Encyclopedia of Integer Sequences (OEIS) arising from these computations are described as well.

1. Preliminaries

In this section, we recall main definitions and notation to be used throughout the paper [3–7].

1.1. Snake graphs

A *tile* G is a square in the plane whose sides are parallel or orthogonal to the elements in the standard orthonormal basis of the plane (as in [3] in this work a tile G is considered as a graph with four vertices and four edges in the obvious way).

A *snake graph* $\mathcal{G}$ is a connected planar graph consisting of a finite sequence of tiles $G_1, G_2, \ldots, G_d$, such that G_i and G_{i+1} share exactly one edge e_i and this edge is either the north edge of G_i and the south edge of G_{i+1} or the east edge of G_i and the west edge of G_{i+1}, [3–7]. Denote by $\mathrm{Int}(\mathcal{G}) = \{e_1, e_2, \ldots, e_{d-1}\}$ the set of interior edges of the snake graph $\mathcal{G}$. We will use the natural ordering of the set of interior edges of $\mathcal{G}$, so that e_i is the edge shared by tiles G_i and G_{i+1}. A snake graph is called *straight* if all its tiles lie in one column or one row, and a snake graph is called *zigzag* if no three consecutive tiles are straight. Two snake graphs are *isomorphic* if they are isomorphic as graphs.

For positive integers $n_1, n_2, \ldots, n_k$, we let $\mathscr{G}_f(n_1, n_2, \ldots, n_k)$ denote a snake graph, with $n_1 \geqslant 2$ tiles in the first row, $n_2 \geqslant 2$ tiles in the first column, $n_3 \geqslant 2$ tiles in the second row and so on, up to $n_k \geqslant 2$. In this case the last tile in a given row is the first tile in the next column (if it exists) vice versa the last tile in a given column coincides with the first tile in the next row. As an example, in Figure 1, it is shown the snake graph $\mathscr{G}_f(5, 3, 3, 2, 5, 4, 2)$.

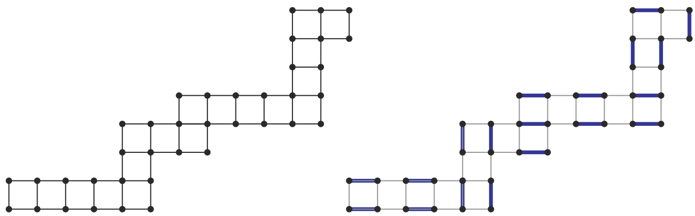


FIGURE 1. Snake graph $\mathscr{G}_f(5, 3, 3, 2, 5, 4, 2)$ (left) and an example of its perfect matchings.

A *perfect matching* P of a graph G is a subset of the edges of G such that every vertex of G is incident to exactly one edge in P. We denote by $\text{Match}(G)$ the set of perfect matchings of G.

A *sign function* f of a snake graph $\mathscr{G}$ is a map f from the set of edges of $\mathscr{G}$ to the set of signs $\{+, -\}$, such that on every tile in $\mathscr{G}$ the north and the west edge have the same sign and the sign on the north edge is opposite to the sign on the south edge. For example, in Figure 2 we show a labeling of the snake graph $\mathscr{G}_f(5, 3, 3, 2, 5, 4, 2)$.

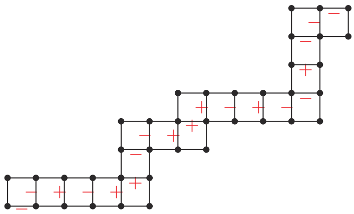


FIGURE 2. Example of a sign function defined on the set of edges of a snake graph.

Note that, on every snake graph there are exactly two sign functions. A snake graph is determined up to symmetry by its sequence of tiles together with a sign function on its interior edges $\{e_1, e_2, \ldots, e_{d-1}\}$. Henceforth, it will be assumed the notation $e_0 = sw(\mathscr{G})$ (the edge at the southwest of the first tile).

If $e_d \in ne(\mathscr{G})$ (the edge at the northeast of the last tile) then sign function can be extended in a unique way to all edges in $\mathscr{G}$ and it is obtained a sign sequence
$\operatorname{sgn}(\mathscr{G}) = \{f(e_0), f(e_1), f(e_2), \ldots, f(e_{d-1}), f(e_d)\}$ actually this sequence uniquely determines the snake graph and a choice of a north east edge $e_d \in ne(\mathscr{G})$.

A positive finite *continued fraction* is a function

$$[a_1, a_2, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}$$

on n variables $a_1, a_2, \ldots, a_n$, $a_i \in \mathbb{Z}_{\geqslant 1}$. Now let $[a_1, a_2, \ldots, a_n]$ be a positive continued fraction and let $d = a_1 + a_2 + \cdots + a_n - 1$ and consider the sign sequence:

$$(\underbrace{-\varepsilon, \ldots, -\varepsilon}_{a_1}, \underbrace{\varepsilon, \ldots, \varepsilon}_{a_2}, \ldots, \underbrace{\pm\varepsilon, \ldots, \pm\varepsilon}_{a_n}), \tag{1}$$

where $\varepsilon \in \{+, -\}$,

$$-\varepsilon = \begin{cases} + & \text{if } \varepsilon = -; \\ - & \text{if } \varepsilon = +; \end{cases} \quad \text{and} \quad \operatorname{sgn}(a_i) = \begin{cases} -\varepsilon & \text{if } i \text{ is odd;} \\ \varepsilon & \text{if } i \text{ is even;} \end{cases}$$

Thus each integer a_i corresponds to a maximal subsequence of constant sign $\operatorname{sgn}(a_i)$ in the sequence (1).

According to Çanackçi and Schiffler [3], the snake graph $\mathscr{G}[a_1, a_2, \ldots, a_n]$ of the positive continued fraction $[a_1, a_2, \ldots, a_n]$ is the snake graph with d tiles determined by the sign sequence (1). In particular, $\mathscr{G}[1]$ is a single edge and the continued fraction of the graph in Figure 2 is $[2, 1, 1, 2, 2, 3, 1, 1, 2, 1, 3]$. Figure 3 shows examples of the chosen notations for snake graphs.

Çanakçi and Schiffler report the following results regarding snake graphs and their relationships with continued fractions:

Theorem 1 ([3], Corollary 4.3)**.** *The number of snake graphs with exactly N perfect matchings is $\phi(N)$, where ϕ is the totient Euler function.*

Theorem 2 ([3], Theorem 3.4)**.**

1) *The number of perfect matchings of $\mathscr{G}[a_1, a_2, \ldots, a_n]$ is equal to the numerator of the continued fraction $[a_1, a_2, \ldots, a_n]$.*

FIGURE 3. $\mathcal{G}_f(2,3) = \mathcal{G}[3,2]$ and $\mathcal{G}[2,3] = \mathcal{G}_f(3,2)$.

2) *The number of perfect matchings of* $\mathcal{G}[a_2, a_3, \ldots, a_n]$ *is equal to the denominator of the continued fraction* $[a_1, a_2, \ldots, a_n]$.
3) *If* $\mathrm{Match}(\mathcal{G})$ *denotes the number of perfect matchings of the snake graph* $\mathcal{G}$ *then* $[a_1, a_2, \ldots, a_n] = \frac{\mathrm{Match}(\mathcal{G}[a_1,a_2,\ldots,a_n])}{\mathrm{Match}(\mathcal{G}[a_2,a_3,\ldots,a_n])}$.

For instance, the snake graph

$$\mathcal{G}[2,1,1,2,2,3,1,1,2,1,3] = \mathcal{G}_f(5,3,3,2,5,4,2)$$

shown in Figure 2 has 3221 perfect matchings. For $[a_1, a_2, \ldots, a_n] = [1, 1, \ldots, 1]$, we will prove more ahead (see Theorem 14) that the straight snake graph $\mathcal{G}[1, 1, \ldots, 1] = \mathcal{G}_f(n-1)$ with $n-1$ tiles has F_{n+1} perfect matchings, where F_{n+1} denotes the $(n+1)$-th Fibonacci number, this fact is also reported by Çanakçi and Schiffler in [4].

A continued fraction $[a_1, a_2, \ldots, a_n]$ is said to be of *even length* if n is even. It is called *palindromic* if the sequences $(a_1, a_2, \ldots, a_n)$ and $(a_n, a_{n-1}, \ldots, a_2, a_1)$ are equal. A snake graph $\mathcal{G}$ is called palindromic if it is the snake graph of a palindromic continued fraction. Given a snake graph $\mathcal{G}$, we can construct a palindromic snake graph of even length $\mathcal{G}_{\leftrightarrow}$ by glueing two copies of $\mathcal{G}$ to a new center tile. This graph is called the *palindromification* of $\mathcal{G}$. More precisely, if $\mathcal{G} = \mathcal{G}[a_1, a_2, \ldots, a_n]$, then its palindromification is the snake graph $\mathcal{G}_{\leftrightarrow} = \mathcal{G}[a_n, \ldots, a_2, a_1, a_1, a_2, \ldots, a_n]$.

Theorem 3 ([4], Theorem 3.10). *Let* $\mathcal{G} = \mathcal{G}[a_1, a_2, \ldots, a_n]$ *be a snake graph and* $\mathcal{G}_{\leftrightarrow}$ *its palindromification. Let* $\mathcal{G}' = \mathcal{G}[a_2, a_3, \ldots, a_n]$ *then* $\mathrm{Match}(\mathcal{G}_{\leftrightarrow}) = (\mathrm{Match}(\mathcal{G}))^2 + (\mathrm{Match}(\mathcal{G}'))^2$.

Corollary 4 ([4], Corollary 3.14).

1) *If* $N = p^2 + q^2$ *with* $(p, q) = 1$ *(i.e., N is a sum of two relatively prime squares), then there exists a palindromic snake graph of even length, such that* $\mathrm{Match}(\mathcal{G}) = N$.
2) *For each positive integer* N*, the number of ways we can write* N *as a sum of two relatively prime numbers is equal to one half of the number of palindromic snake graphs of even length with* N *perfect matchings.*

3) *For each positive integer N, the number of ways one can write N as a sum of two relatively prime squares is equal to one half of the number of palindromic continued fractions of even length with numerator N.*

Remark 5. Hereinafter, we assume notation $\mathscr{G}_f(n_1, n_2, \ldots, n_k)$ for snake graphs.

1.2. Brauer configuration algebras

Brauer configuration algebras were introduced by Green and Schroll in [12] as a generalization of Brauer graph algebras which are biserial algebras of tame representation type and whose representation theory is encoded by some combinatorial data based on graphs. Actually, underlying every Brauer graph algebra is a finite graph with acyclic orientation of the edges at every vertex and a multiplicity function [8]. The construction of a Brauer graph algebra is a special case of the construction of a Brauer configuration algebra in the sense that every Brauer graph is a Brauer configuration with the restriction that every polygon is a set with two vertices. In the sequel, we give precise definitions of a Brauer configuration and a Brauer configuration algebra.

A *Brauer configuration* Γ is a quadruple of the form $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ where:

(B1) Γ_0 is a finite set whose elements are called *vertices*,

(B2) Γ_1 is a finite collection of multisets called *polygons*. In this case, if $V \in \Gamma_1$ then the elements of V are vertices possibly with repetitions, $\mathrm{occ}(\alpha, V)$ denotes the frequency of the vertex α in the polygon V and the *valency* of α denoted $\mathrm{val}(\alpha)$ is defined in such a way that:

$$\mathrm{val}(\alpha) = \sum_{V \in \Gamma_1} \mathrm{occ}(\alpha, V).$$

(B3) μ is an integer valued function such that $\mu : \Gamma_0 \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers, it is called the *multiplicity function*,

(B4) $\mathcal{O}$ denotes an orientation defined on Γ_1 which is a choice, for each vertex $\alpha \in \Gamma_0$, of a cyclic ordering of the polygons in which α occurs as a vertex, including repetitions, we denote S_α such collection of polygons. More specifically, if $S_\alpha = \{V_1^{(\alpha_1)}, V_2^{(\alpha_2)}, \ldots, V_t^{(\alpha_t)}\}$ is the collection of polygons where the vertex α occurs with $\alpha_i = \mathrm{occ}(\alpha, V_i)$ and $V_i^{(\alpha_i)}$ meaning that S_α has α_i copies of V_i then an orientation $\mathcal{O}$ is obtained by endowing a linear order $<$ to S_α and adding a relation $V_t < V_1$, if $V_1 = \min S_\alpha$ and $V_t = \max S_\alpha$,

(B5) Every vertex in Γ_0 is a vertex in at least one polygon in Γ_1,
(B6) Every polygon has at least two vertices,
(B7) Every polygon in Γ_1 has at least one vertex α such that $\mu(\alpha)\mathrm{val}(\alpha)>1$.

The set $(S_\alpha, <)$ is called the *successor sequence* at the vertex α.

A vertex $\alpha \in \Gamma_0$ is said to be *truncated* if $\mathrm{val}(\alpha)\mu(\alpha) = 1$, that is, α is truncated if it occurs exactly once in exactly one $V \in \Gamma_1$ and $\mu(\alpha) = 1$. A vertex is *non-truncated* if it is not truncated.

The quiver of a Brauer configuration algebra. The quiver $Q_\Gamma = ((Q_\Gamma)_0, (Q_\Gamma)_1)$ of a Brauer configuration algebra is defined in such a way that the vertex set $(Q_\Gamma)_0 = \{v_1, v_2, \ldots, v_m\}$ of Q_Γ is in correspondence with the set of polygons $\{V_1, V_2, \ldots, V_m\}$ in Γ_1, noting that there is one vertex in $(Q_\Gamma)_0$ for every polygon in Γ_1.

Arrows in Q_Γ are defined by the successor sequences. That is, there is an arrow $v_i \xrightarrow{s_i} v_{i+1} \in (Q_\Gamma)_1$ provided that $V_i < V_{i+1}$ in $(S_\alpha, <) \cup \{V_t < V_1\}$ for some non-truncated vertex $\alpha \in \Gamma_0$. In other words, for each non-truncated vertex $\alpha \in \Gamma_0$ and each successor V' of V at α, there is an arrow from v to v' in Q_Γ where v and v' are the vertices in Q_Γ associated to the polygons V and V' in Γ_1, respectively.

The ideal of relations and definition of a Brauer configuration algebra. Fix a polygon $V \in \Gamma_1$ and suppose that $\mathrm{occ}(\alpha, V) = t \geqslant 1$ then there are t indices $i_1, \ldots, i_t$ such that $V = V_{i_j}$. Then the *special α-cycles* at v are the cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_t}$ where v is the vertex in the quiver of Q_Γ associated to the polygon V. If α occurs only once in V and $\mu(\alpha) = 1$ then there is only one special α-cycle at v.

Let $\mathbb{F}$ be a field and Γ a Brauer configuration. The *Brauer configuration algebra associated to* Γ is defined to be the bounded path algebra $\Lambda_\Gamma = \mathbb{F}Q_\Gamma/I_\Gamma$, where Q_Γ is the quiver associated to Γ and I_Γ is the *ideal* in $\mathbb{F}Q_\Gamma$ generated by the following set of relations ρ_Γ of type I, II and III. Henceforth, if there is no confusion, we will assume notations, Λ, I and ρ instead of Λ_Γ, I_Γ and ρ_Γ for a Brauer configuration algebra, the ideal and set of relations, respectively defined by a given Brauer configuration Γ.

1) **Relations of type I**. For each polygon $V = \{\alpha_1, \ldots, \alpha_m\} \in \Gamma_1$ and each pair of non-truncated vertices α_i and α_j in V, the set of relations ρ contains all relations of the form $C^{\mu(\alpha_i)} - C'^{\mu(\alpha_j)}$ where C is a special α_i-cycle and C' is a special α_j-cycle.
2) **Relations of type II**. Relations of type II are all paths of the form $C^{\mu(\alpha)}a$ where C is a special α-cycle and a is the first arrow in C.

3) **Relations of type III**. These relations are quadratic monomial relations of the form ab in $\mathbb{F}Q_\Gamma$ where ab is not a subpath of any special cycle unless $a = b$ and a is a loop associated to a vertex of valency 1 and $\mu(\alpha) > 1$.

Let $\Lambda = \mathbb{F}Q_\Gamma/I$ be the Brauer configuration configuration algebra associated to a reduced Brauer configuration Γ (i.e., truncated vertices $\alpha \in \Gamma_0$ occur only in polygons with two vertices). Denote by $\pi : \mathbb{F}Q_\Gamma \to \Lambda$ the canonical surjection then $\pi(x)$ is denoted by $\overline{x}$, for $x \in \mathbb{F}Q_\Gamma$.

The following results describe the structure of a Brauer configuration algebra. In this case, $\operatorname{rad} M$ $(\operatorname{soc} M)$ denotes the radical (socle) of a module M, $\operatorname{rad} M$ is the intersection of all the maximal submodules of M, whereas $\operatorname{soc} M$ is generated by all simple modules of M.

Theorem 6 ([12], Theorem B, Proposition 2.7, Theorem 3.10, Corollary 3.12)**.** *Let Λ be a Brauer configuration algebra with Brauer configuration Γ.*

1) *There is a bijective correspondence between the set of indecomposable projective Λ-modules and the polygons in Γ.*
2) *If P is an indecomposable projective Λ-module corresponding to a polygon V in Γ. Then* $\operatorname{rad} P$ *is a sum of r indecomposable uniserial modules, where r is the number of (non-truncated) vertices of V and where the intersection of any two of the uniserial modules is a simple Λ-module.*
3) *A Brauer configuration algebra is a multiserial algebra.*
4) *The number of summands in the heart $ht(P) = \operatorname{rad} P/\operatorname{soc} P$ of an indecomposable projective Λ-module P such that $\operatorname{rad}^2 P \neq 0$ equals the number of non-truncated vertices of the polygons in Γ corresponding to P counting repetitions.*
5) *If Λ' is a Brauer configuration algebra obtained from Λ by removing a truncated vertex of a polygon in Γ_1 with $d \geqslant 3$ vertices then Λ is isomorphic to Λ'.*

Proposition 7 ([12], Proposition 3.3)**.** *Let Λ be the Brauer configuration algebra associated to the Brauer configuration Γ. For each $V \in \Gamma_1$ choose a non-truncated vertex α and exactly one special α-cycle C_V at V,*

$$
\begin{aligned}
A &= \{\overline{p} \mid p \text{ is a proper prefix of some } C^{\mu(\alpha)} \\
&\qquad\qquad \text{where } C \text{ is a special } \alpha\text{-cycle}\}, \\
B &= \{\overline{C_V^{\mu(\alpha)}} \mid V \in \Gamma_1\}.
\end{aligned}
$$

Then $A \cup B$ is a $\mathbb{F}$-basis of Λ.

Proposition 8 ([12], Proposition 3.13). *Let Λ be a Brauer configuration algebra associated to the Brauer configuration Γ and let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be a full set of equivalence class representatives of special cycles. Assume that for $i = 1, \ldots, t$, C_i is a special α_i-cycle where α_i is a non-truncated vertex in Γ. Then*

$$\dim_{\mathbb{F}} \Lambda = 2|Q_0| + \textstyle\sum_{C_i \in \mathcal{C}} |C_i|(n_i|C_i| - 1),$$

where $|Q_0|$ denotes the number of vertices of Q, $|C_i|$ denotes the number of arrows in the α_i-cycle C_i and $n_i = \mu(\alpha_i)$.

Proposition 9 ([12], Proposition 3.6). *Let Λ be the Brauer configuration algebra associated to a connected Brauer configuration Γ. The algebra Λ has a length grading induced from the path algebra $\mathbb{F}Q$ if and only if there is an $N \in \mathbb{Z}_{>0}$ such that for each non-truncated vertex $\alpha \in \Gamma_0$ $\mathrm{val}(\alpha)\mu(\alpha) = N$.*

The following result regards the center of a Brauer configuration algebra.

Theorem 10 ([16], Theorem 4.9). *Let Γ be a reduced and connected Brauer configuration and let Q be its induced quiver and let Λ be the induced Brauer configuration algebra such that $\mathrm{rad}^2 \Lambda \neq 0$ then the dimension of the center of Λ denoted $\dim_{\mathbb{F}} Z(\Lambda)$ is given by the formula:*

$$\dim_{\mathbb{F}} Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#(\mathrm{Loops}\, Q) - |\mathscr{C}_\Gamma|.$$

where $|\mathscr{C}_\Gamma| = \{\alpha \in \Gamma_0 \mid \mathrm{val}(\alpha) = 1,$ and $\mu(\alpha) > 1\}$.

Example 11. As an example consider a configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ such that:

1) $\Gamma_0 = \{1, 2, 3, 4\}$,
2) $\Gamma_1 = \{U = \{1, 1, 4\}, V = \{1, 2, 2\}, W = \{2, 3, 3\}, X = \{3, 4, 4\}\}$,
3) At vertex 1, it holds that; $U < U < V$, $\mathrm{val}(1) = 3$,
4) At vertex 2, it holds that; $V < V < W$, $\mathrm{val}(2) = 3$,
5) At vertex 3, it holds that; $W < W < X$, $\mathrm{val}(3) = 3$,
6) At vertex 4, it holds that; $X < X < U$, $\mathrm{val}(4) = 3$,
7) $\mu(\alpha) = 1$ for any vertex α.

The ideal I of the corresponding Brauer configuration algebra Λ_Γ is generated by the following relations (see Figure 4), for which it is assumed the following notation for the special cycles:

$$\begin{aligned} C_1^{U,1} &= a_1^1a_2^1a_3^1, & C_1^{U,2} &= a_2^1a_3^1a_1^1, & C_1^{V,1} &= a_3^1a_1^1a_2^1,\\ C_2^{V,1} &= a_1^2a_2^2a_3^2, & C_2^{V,2} &= a_2^2a_3^2a_1^2, & C_2^{W,1} &= a_3^2a_1^2a_2^2,\\ C_3^{W,1} &= a_1^3a_2^3a_3^3, & C_3^{W,2} &= a_2^3a_3^3a_1^3, & C_3^{X,1} &= a_3^3a_1^3a_2^3,\\ C_4^{X,1} &= a_1^4a_2^4a_3^4, & C_4^{X,2} &= a_2^4a_3^4a_1^4, & C_4^{U,1} &= a_3^4a_1^4a_2^4. \end{aligned} \tag{2}$$

1) $a_i^h a_r^s$, if $h \neq s$, for all possible values of i and r,
2) $(a_1^1)^2$, $(a_1^2)^2$, $(a_1^3)^2$, $(a_1^4)^2$, $a_3^1a_2^1$, $a_3^2a_2^2$, $a_3^3a_2^3$, $a_3^4a_2^4$,
3) $C_j^{U,i} - C_l^{U,k}$, for all possible values of i, j, k and l,
4) $C_j^{V,i} - C_l^{V,k}$, for all possible values of i, j, k and l,
5) $C_j^{W,i} - C_l^{W,k}$, for all possible values of i, j, k and l,
6) $C_j^{X,i} - C_l^{X,k}$, for all possible values of i, j, k and l,
7) $C_i^{U,j}a$ $(C_i^{V,j}a')$, with a (a') being the first arrow of $C_i^{U,j}$ $(C_i^{V,j})$ for all i, j.
8) $C_i^{W,j}a$ $(C_i^{X,j}a')$, with a (a') being the first arrow of $C_i^{W,j}$ $(C_i^{X,j})$ for all i, j.

Figures 4–7 show the quiver Q_Γ associated to this configuration, indecomposable projective modules P_U, P_V, P_W, and P_X over Λ_Γ together with their corresponding hearts and radical square.

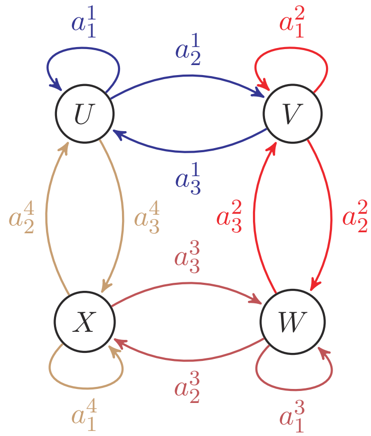


FIGURE 4. The quiver Q_Γ defined by the configuration Γ. Colors denote special cycles at a given vertex. Color blue (red, purple, brown) is associated to a special cycle at vertex 1 (2, 3, 4 respectively).

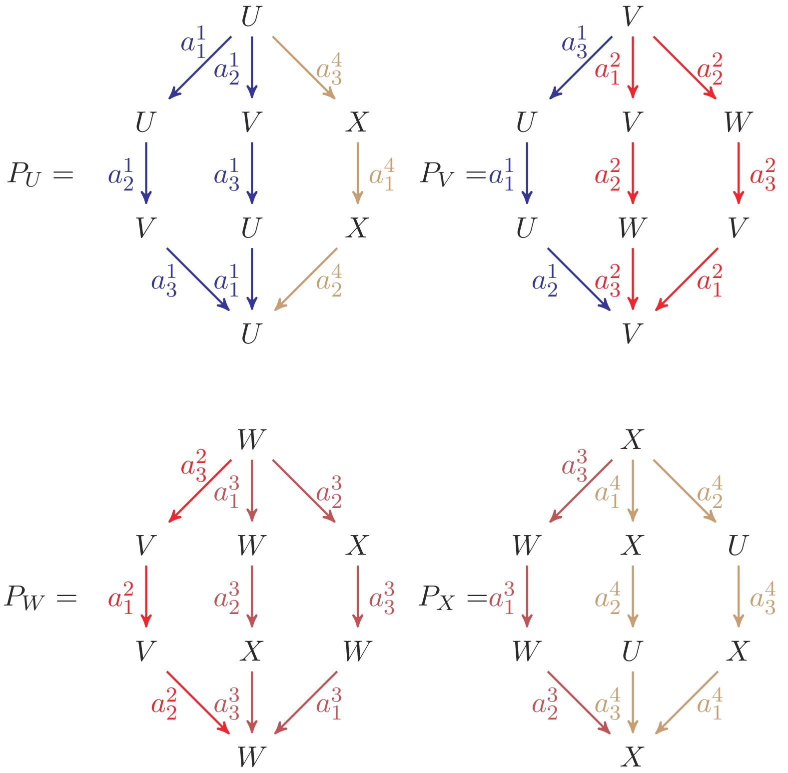


FIGURE 5. Indecomposable projective modules P_U, P_V, P_W, and P_X. Note that, the corresponding radicals are multiserial with series given by special cycles.

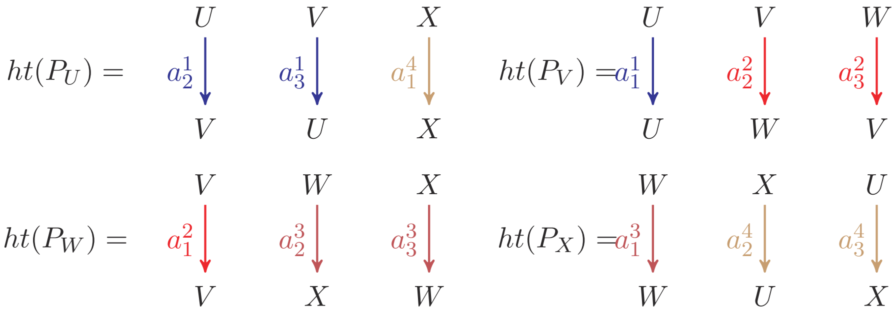


FIGURE 6. Hearts of the indecomposable projective modules P_U, P_V, P_W, and P_X.

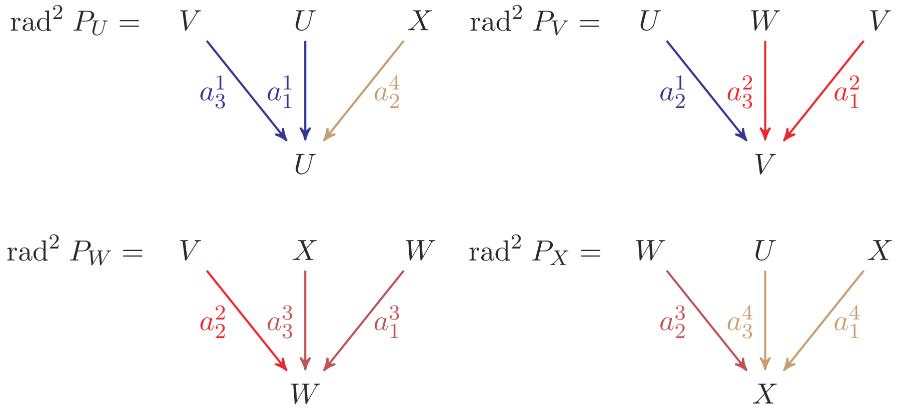


Figure 7. Radical square of the projective modules P_U, P_V, P_W, and P_X.

The following are the numeric data associated to the algebra $\Lambda_\Gamma = \mathbb{F}Q_\Gamma/I$ with Q_Γ as shown in Figure 4 and special cycles given in (2), ($|r(Q_\Gamma)|$ is the number of indecomposable projective modules, r_U, r_V, r_W and r_X denote the number of summands in the heart of the indecomposable projective modules P_U, P_V, P_W and P_X. Note that, $|C_i| = \mathrm{val}(i)$):

$$\begin{aligned}
|r(Q_\Gamma)| &= 4,\\
r_U = 3, \quad r_V &= 3, \quad r_W = 3, \quad r_X = 3,\\
|C_1| = 3, \quad |C_2| &= 3, \quad |C_3| = 3, \quad |C_4| = 3,\\
\sum_{\alpha\in\Gamma_0}\sum_{X\in\Gamma_1} \mathrm{occ}(\alpha, X) &= 12, \quad \text{the number of special cycles},\\
\dim_{\mathbb{F}} \Lambda &= 2(4) + 3(2) + 3(2) + 3(2) + 3(2) = 32,\\
\dim_{\mathbb{F}} Z(\Lambda) &= 1 + 4 + (4 - 4) + 4 - 0 = 9.
\end{aligned}$$

Remark 12. Note that according to Proposition 9, the Brauer configuration algebra Λ_Γ with quiver Q_Γ shown in Figure 4 has a length grading induced by the path algebra $\mathbb{F}Q_\Gamma$.

1.3. Labeled Brauer configuration

The notion of labeled Brauer configurations is helpful to define suitable specializations of some Brauer configuration algebras.

Let $\Gamma = \{\Gamma_0, \Gamma_1, \mu, \mathcal{O}\}$ be a Brauer configuration and let $U \in \Gamma_1$ be a polygon such that $U = \{\alpha_1^{f_1}, \alpha_2^{f_2}, \ldots, \alpha_n^{f_n}\}$, where $f_i = \mathrm{occ}(\alpha_i, U)$. The term

$$w(U) = \alpha_1^{f_1}\alpha_2^{f_2}\dots\alpha_n^{f_n}$$

is said to be the *word associated to* U. The formal sum (or word sum)

$$M(\Gamma) = \sum_{U\in\Gamma_1} w(U) \tag{3}$$

is said to be the *message of the Brauer configuration* Γ.

An *integer specialization* of a Brauer configuration $\Gamma = \{\Gamma_0, \Gamma_1, \mu, \mathcal{O}\}$ is a Brauer configuration $\Gamma^e = (\Gamma_0^e, \Gamma_1^e, \mu^e, \mathcal{O}^e)$ endowed with a preserving orientation map $e : \Gamma_0 \to \mathbb{N}$, such that

$$\begin{gathered}\Gamma_0^e = \operatorname{Img} e \subset \mathbb{N}, \qquad \Gamma_1^e = e(\Gamma_1),\\ \text{if } H \in \Gamma_1 \text{ then } e(H) = \{e(\alpha_i) \mid \alpha_i \in H\} \in e(\Gamma_1),\\ \mu^e(e(\alpha)) = \mu(\alpha) \text{ for any } \alpha \in \Gamma_0.\end{gathered} \tag{4}$$

Orientation $\mathcal{O}^e$ is given by defining a linear order $\lhd$ such that $e(U) \lhd e(V)$ in Γ_1^e provided that $U < V$ in Γ_1.

We let $w^e(U) = (e(\alpha_1))^{f_1}(e(\alpha_2))^{f_2}\dots(e(\alpha_n))^{f_n}$ denote the specialization under e of a word $w(U)$. In such a case, $M(\Gamma^e) = \sum_{U\in\Gamma_1^e} w^e(U)$ is the *specialized message* of the Brauer configuration Γ with the usual integer sum and product (in general with the sum and product associated to $\operatorname{Img} e$).

A Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ is said to be *labeled* if each polygon is labeled by an element of $\mathbb{N}^s$ for some $s \geqslant 1$. In such a case we often write

$$\Gamma_1 = \{(U_1, n_1), (U_2, n_2), \dots, (U_k, n_k)\}, \quad n_j \in \mathbb{N}^s,$$

with $(U_i, n_i) \prec (U_{i+1}, n_{i+1})$, for $1 \leqslant i \leqslant k-1$ if $U_i < U_{i+1}$ in Γ_1, i.e., the labeling do not alter the orientation $\mathcal{O}$.

Example 13. As an example, we define the labeled Brauer configuration $\mathcal{K} = (\mathcal{K}_0, \mathcal{K}_1, \mu, \mathcal{O})$, where:

$$\begin{aligned}&\mathcal{K}_0 = \{\alpha_w^i \mid 1 \leqslant i \leqslant k, w \in \{0,1\}^{k-1}, k \geqslant 2 \text{ fixed}\},\\ &\mathcal{K}_1 = \{(U_w, n) \mid \alpha_w^i \in (U_w, n), n = (n_1, n_2, \dots, n_k), \text{ fixed}, n_j \geqslant 2\}.\end{aligned} \tag{5}$$

Vertices $\alpha_w^i \in (U_w, n) \in \mathcal{K}_1$ are given by the following formula bearing in mind that w is of the form $w = (w_1, w_2, \dots, w_{k-1})$.

$$\alpha_w^i = n_i - g(w_{i-1}, i) - g(w_i, i) + 2, \tag{6}$$

where g is a map $g := \{0,1\} \times \mathbb{Z}^+ \to \{1,2\}$ defined by

$$g(0,i) = \begin{cases} 2, & \text{if } i \text{ is even;} \\ 1, & \text{if } i \text{ is odd;} \end{cases} \quad \text{and} \quad g(1,i) = \begin{cases} 1, & \text{if } i \text{ is even;} \\ 2, & \text{if } i \text{ is odd;} \end{cases}.$$

In particular, $g(w_0, 1) = g(w_k, k) = 0$. The definition of g can be reformulated by the rule $g(x, n) = 2 - (x + n \pmod 2)$.

In this case, $\mu(\alpha) = 2$, for any vertex $\alpha \in \mathscr{K}_0$ and the orientation $\mathcal{O}$ is given by the relation $\prec$. Actually, henceforth $\mu(\alpha)$ will be chosen in such a way that each polygon will contain at least one non-truncated vertex.

2. On the number of perfect matchings of snake graphs via Brauer configuration algebras

In this section, we give formulas for the number of perfect matchings of snake graphs with the shape $\mathscr{G}_f(n_1, n_2, \ldots, n_k)$ (see Remark 5). Firstly, we note that;

$$\text{Match}(\mathscr{G}_f(n)) = F_{n+2}, \tag{7}$$

where F_n denotes the n-th Fibonacci number. In the next result, we assume that $\mathscr{G}_f(1)$ is a snake graph with only one tile.

Theorem 14. $\text{Match}(\mathscr{G}_f(n)) = F_{n+2}$, *for all* $n \geqslant 1$.

Proof. For any perfect matching of $\mathscr{G}_f((n))$ there are two options: either the vertical right edge of the last square is contained in the matching or the horizontal edges of the last square are contained in the matching, see Figure 8.

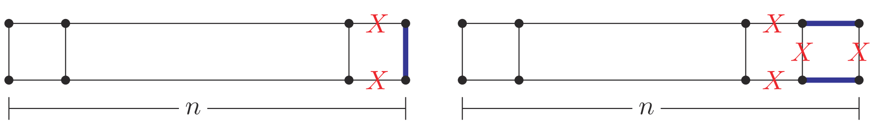


Figure 8. A perfect matching of $\mathscr{G}_f(n)$.

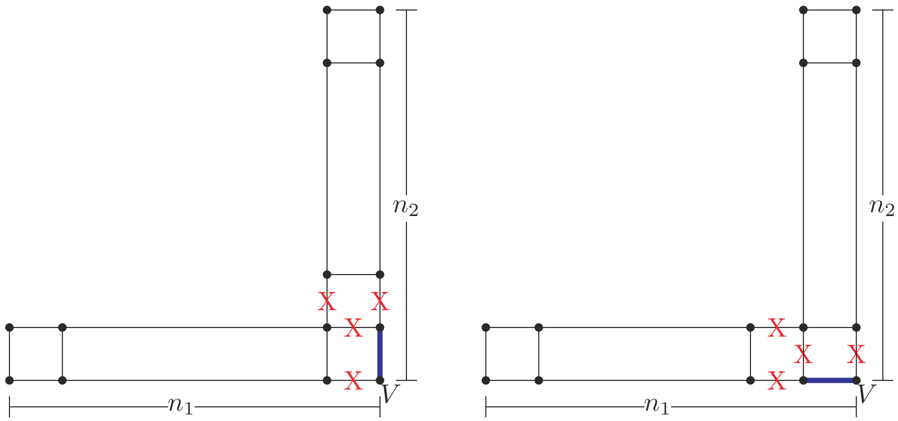
From the definition of perfect matching it is clear that the edges labeled with red X cannot be used. Therefore we have the recurrence relation

$$\text{Match}(\mathscr{G}_f(n)) = \text{Match}(\mathscr{G}_f(n-1)) + \text{Match}(\mathscr{G}_f(n-2)).$$

Since $\text{Match}(\mathscr{G}_f(1)) = 2$ and $\text{Match}(\mathscr{G}_f(2)) = 3$, we conclude that $\text{Match}(\mathscr{G}_f(n)) = F_{n+2}$ for all $n \geqslant 1$. □

Corollary 15. $\operatorname{Match}(\mathscr{G}_f(n_1,n_2))=F_{n_1+1}F_{n_2}+F_{n_1}F_{n_2+1}$ *for all* $n_1,n_2\geqslant 2$.

Proof. Let V be the vertex on the lower right corner of $\mathscr{G}_f(n_1,n_2)$. We consider the adjacent edges to the vertex V. So, we have the following possible configurations:



Therefore, it holds that

$$\operatorname{Match}(\mathscr{G}_f(n_1,n_2))=M_1+M_2.$$

Where

$$M_1=\operatorname{Match}(\mathscr{G}_f(n_1-1))\operatorname{Match}(\mathscr{G}_f(n_2-2))$$
$$M_2=\operatorname{Match}(\mathscr{G}_f(n_1-2))\operatorname{Match}(\mathscr{G}(n_2-1)).$$

Theorem 14 allows to obtain the desired result. We are done. □

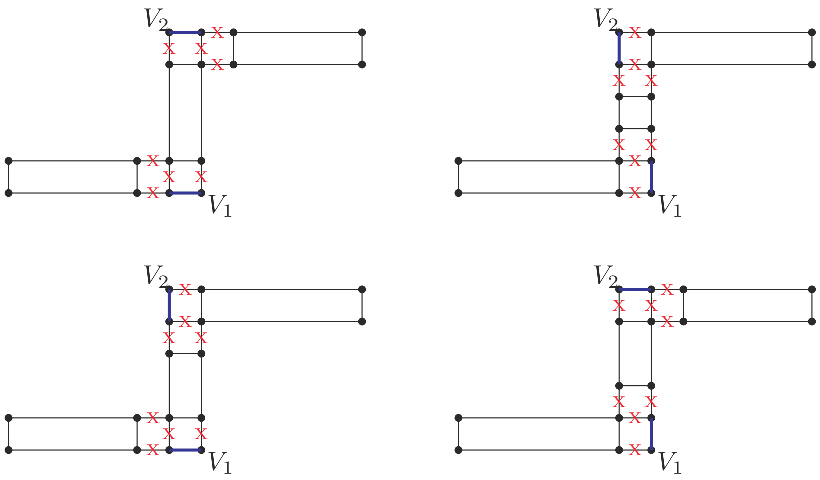
The following result corresponds to the case of a snake graph with three straight subsnake graphs.

Corollary 16.

$$\operatorname{Match}(\mathscr{G}_f(n_1,n_2,n_3))=F_{n_1}F_{n_2}F_{n_3}+F_{n_1+1}F_{n_2-2}F_{n_3+1} + F_{n_1}F_{n_2-1}F_{n_3+1}+F_{n_1+1}F_{n_2-1}F_{n_3}$$

for all $n_1,n_2,n_3\geqslant 2$.

Proof. Firstly, let us suppose that $n_2\geqslant 4$. For the cases $n_2=2,3$, we can use a similar argument. Let V_1 and V_2 be the vertices in the lower right corner and the upper left corner, respectively. By considering the adjacent edges with the vertices V_1 and V_2, we obtain the following four options:



From the above decomposition, we obtain that if

$$
\begin{aligned}
M_1 &= \mathrm{Match}(\mathscr{G}_f(n_1-2))\mathrm{Match}(\mathscr{G}_f(n_2-2))\mathrm{Match}(\mathscr{G}_f(n_3-2)),\\
M_2 &= \mathrm{Match}(\mathscr{G}_f(n_1-1))\mathrm{Match}(\mathscr{G}_f(n_2-4))\mathrm{Match}(\mathscr{G}_f(n_3-1)),\\
M_3 &= \mathrm{Match}(\mathscr{G}_f(n_1-2))\mathrm{Match}(\mathscr{G}_f(n_2-3))\mathrm{Match}(\mathscr{G}_f(n_3-1),\\
M_4 &= \mathrm{Match}(\mathscr{G}_f(n_1-1))\mathrm{Match}(\mathscr{G}_f(n_2-3))\mathrm{Match}(\mathscr{G}_f(n_3-2)),
\end{aligned}
$$

then

$$\mathrm{Match}(\mathscr{G}_f(n_1,n_2,n_3)) = \sum_{i=1}^{4} M_i.$$

And Theorem 14 allows us to conclude the desired result. We are done. □

The following result gives the number of perfect matchings of a snake graph of type $\mathscr{G}_f(n_1,n_2,\ldots,n_k)$ as a specialized message of the Brauer configuration defined in Example 13. In this case, words concatenation arising from the configuration is specialized by the usual product of natural numbers.

Theorem 17. *For all $n_1,n_2\ldots,n_k \geqslant 2$, we have*

$$\mathrm{Match}(\mathscr{G}_f(n_1,n_2,\ldots,n_k)) = M(\mathscr{K}^e),$$

where $\mathscr{K}$ is the Brauer configuration given in Example 13, $M(\mathscr{K})$ defined as in (3). And e is an integer specialization of $\mathscr{K}$ with associated map e of the form $e:\mathscr{K}_0 \to \mathbb{N}$ such that $e(\alpha_w^i) = F_{\alpha_w^i}$ with F_j being the j-th Fibonacci number.

Proof. The definition of the Brauer configuration $\mathscr{K}$ and the corresponding specialization e allow us to infer that it suffices to see that

$$\text{Match}(\mathscr{G}_f(n_1, n_2, \ldots, n_k)) = \sum_{w \in \{0,1\}^{k-1}} \prod_{\ell=1}^{k} F_{n_\ell - g(w_{\ell-1},\ell) - g(w_\ell,\ell)+2}.$$

Note that, for l fixed a product of the form $\prod_{w_l} F_{n_\ell - g(w_{\ell-1},\ell) - g(w_\ell,\ell)+2}$ is a specialized message $w^e((U_l, n_l))$ of the labeled polygon (U_l, n_l). Now, we proceed to prove the proposed identity.

Let $V_1, V_2, \ldots, V_{k-1}$ be the vertices on the $k-1$ corners of the snake graph $\mathscr{G}_f(n_1, n_2, \ldots, n_k)$, see Figure 9. There are 2^{k-1} ways to choose the adjacent edges with the vertices $V_1, V_2, \ldots, V_{k-1}$.

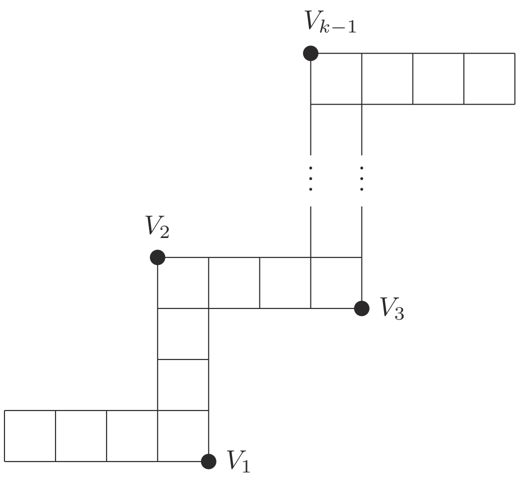


Figure 9. Snake graph $\mathscr{G}_f(n_1, n_2, \ldots, n_k)$.

Let e_i be one of the incident edge with V_i, for $i = 1, 2, \ldots, k-1$. For each V_i, there are two options for e_i: either e_i is vertical or horizontal. If e_i is vertical and i is odd, we have to consider the number of perfect matchings for the snake graphs $\mathscr{G}_f(n_i - 1)$ and $\mathscr{G}_f(n_{i+1} - 2)$. Note that for the first case we do not consider the last tile of the row that contains the vertex V_i, and for the second case we do not consider the first two tiles of the column that contains the vertex V_i. Analogously, if e_i is horizontal and i is odd, we have to consider the perfect matching for the snake graphs $\mathscr{G}_f(n_i - 2)$ and $\mathscr{G}_f(n_{i+1} - 1)$. Similarly, for the case when i is even.

Finally, we can encode this situation with binary words. We use `0` for vertical edges and `1` for horizontal edges. So, it is clear that the function

$g(x,n)$ encodes the subtraction of the tiles that we must apply to each vertex V_i.

Identity (7), the multiplication principle and the definition of the message $M(\mathscr{K}^e)$ of the Brauer configuration $\mathscr{K}^e$ allow us to conclude that

$$\begin{aligned}\text{Match}(\mathscr{G}_f(n_1,n_2,\dots,n_k))\\ &= \sum_{w\in\{0,1\}^{k-1}}\prod_{\ell=1}^{k}\text{Match}(\mathscr{G}_f(n_\ell - g(w_{\ell-1},\ell)-g(w_\ell,\ell))\\ &= \sum_{w\in\{0,1\}^{k-1}}\prod_{\ell=1}^{k}F_{n_\ell-g(w_{\ell-1},\ell)-g(w_\ell,\ell)+2} = M(\mathscr{K}^e). \qquad \square\end{aligned}$$

Example 18. As an example, we define a Brauer configuration algebra induced by the Brauer configuration $\mathscr{K}$ for $k=3$ (see, (4), (5) and (6)). The relations defined here can be adapted for all the distinct values of $\mathbb{F}$ in order to define the corresponding Brauer configuration algebras, in this case $w\in\{0,1\}^2=\{(0,0),(1,0),(0,1),(1,1)\}$, $n=(n_1,n_2,n_3)$ and

$$\begin{aligned}\mathscr{K}_0 &= \{\alpha^1_{(0,0)},\alpha^2_{(0,0)},\alpha^3_{(0,0)},\alpha^1_{(1,0)},\alpha^2_{(1,0)},\alpha^3_{(1,0)},\alpha^1_{(0,1)},\alpha^2_{(0,1)},\alpha^3_{(0,1)},\\ &\qquad \alpha^1_{(1,1)},\alpha^2_{(1,1)},\alpha^3_{(1,1)}\},\\ \mathscr{K}_1 &= \{(U_{(0,0)},n),(U_{(1,0)},n),(U_{(0,1)},n),(U_{(1,1)},n),\\ &\qquad \text{with } n=(n_1,n_2,n_3), n_j\geqslant 2\}.\end{aligned}$$

In Table 1 we compute all the vertices and polygons of $\mathscr{K}$ by using the values of i and w.

i/w		$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
1	$n_1-g(w_0,1)-g(w_1,1)+2$	n_1+1	n_1	n_1+1	n_1
2	$n_2-g(w_1,2)-g(w_2,2)+2$	n_2-2	n_2-1	n_2-1	n_2
3	$n_3-g(w_2,3)-g(w_3,3)+2$	n_3+1	n_3+1	n_3	n_3

Table 1. In this table entries correspond to the vertices and columns correspond to polygons of the Brauer configuration $\mathscr{K}$.

Explicitly, $(U_{(0,0)},n)=\{n_1+1,n_2-2,n_3+1\}$, $(U_{(1,0)},n)=\{n_1,n_2-1,n_3+1\}$, $(U_{(0,1)},n)=\{n_1+1,n_2-1,n_3\}$, $(U_{(1,1)},n)=\{n_1,n_2,n_3\}$ and

$$\begin{aligned}w((U_{(0,0)},n)) &= n_1+1\cdot n_2-2\cdot n_3+1,\\ w((U_{(1,0)},n)) &= n_1\cdot n_2-1\cdot n_3+1,\\ w((U_{(0,1)},n)) &= n_1+1\cdot n_2-1\cdot n_3,\\ w((U_{(1,1)},n)) &= n_1\cdot n_2\cdot n_3.\end{aligned}$$

The following identities are obtained by using the specialization $e(\alpha_w^i) = F_{\alpha_w^i}$ defined in Theorem 17 with F_j being the j-th Fibonacci number:

$$\begin{aligned}
w^e((U_{(0,0)}, n)) &= F_{n_1+1}F_{n_2-2}F_{n_3+1},\\
w^e((U_{(1,0)}, n)) &= F_{n_1}F_{n_2-1}F_{n_3+1},\\
w^e((U_{(0,1)}, n)) &= F_{n_1+1}F_{n_2-1}F_{n_3},\\
w^e((U_{(1,1)}, n)) &= F_{n_1}F_{n_2}F_{n_3}.
\end{aligned}$$

The specialized message $M(\mathscr{K}^e) = \sum_{U\in\Gamma_1^e} w^e(U)$ of the Brauer configuration $\mathscr{K}$ has the following form:

$$\begin{aligned}
M(\mathscr{K}^e) &= F_{n_1+1}F_{n_2-2}F_{n_3+1} + F_{n_1}F_{n_2-1}F_{n_3+1}\\
&\quad + F_{n_1+1}F_{n_2-1}F_{n_3} + F_{n_1}F_{n_2}F_{n_3}\\
&= \mathrm{Match}(\mathscr{G}_f(n_1, n_2, n_3))
\end{aligned}$$

For $k = 3$, the Brauer configuration algebra associated to $\mathscr{K}$ is defined as follows:

1) $\mathscr{K}_0 = \{n_1 + 1, n_2 - 2, n_3 + 1, n_2 - 1, n_1, n_2, n_3\}$,
2) $\mathscr{K}_1 = \{(U_{(0,0)}, n), (U_{(1,0)}, n), (U_{(0,1)}, n), (U_{(1,1)}, n)$, with $n = (n_1, n_2, n_3)\}$,
3) At vertex n_1+1, it holds that $(U_{(0,0)}, n) < (U_{(0,1)}, n)$, $\mathrm{val}(n_1+1) = 2$,
4) At vertex $n_2 - 2$, it holds that $(U_{(0,0)}, n)$, $\mathrm{val}(n_2 - 2) = 1$,
5) At vertex n_3+1, it holds that $(U_{(0,0)}, n) < (U_{(1,0)}, n)$, $\mathrm{val}(n_3+1) = 2$,
6) At vertex n_2-1, it holds that $(U_{(1,0)}, n) < (U_{(0,1)}, n)$, $\mathrm{val}(n_2-1) = 2$,
7) At vertex n_1, it holds that $(U_{(1,0)}, n) < (U_{(1,1)}, n)$, $\mathrm{val}(n_1) = 2$,
8) At vertex n_2, it holds that $(U_{(1,1)}, n)$, $\mathrm{val}(n_2) = 1$,
9) At vertex n_3, it holds that $(U_{(0,1)}, n) < (U_{(1,1)}, n)$, $\mathrm{val}(n_3) = 2$,
10) $\mu(\alpha) = 2$ for any vertex α.

The ideal I of the corresponding Brauer configuration algebra $\Lambda_{\mathscr{K}}$ is generated by the following relations (see Figure 10), for which it is assumed the following notation for the special cycles:

$$\begin{gathered}
C_{n_1+1}^{U_{(0,0)},1} = a_1^{n_1+1}a_2^{n_1+1}, \quad C_{n_1+1}^{U_{(0,1)},1} = a_2^{n_1+1}a_1^{n_1+1}, \quad C_{n_2-2}^{U_{(0,0)},1} = a_1^{n_2-2},\\
C_{n_3+1}^{U_{(0,0)},1} = a_1^{n_3+1}a_2^{n_3+1}, \quad C_{n_3+1}^{U_{(1,0)},1} = a_2^{n_3+1}a_1^{n_3+1}, \quad C_{n_2-1}^{U_{(1,0)},1} = a_1^{n_2-1}a_2^{n_2-1},\\
C_{n_2-1}^{U_{(0,1)},1} = a_2^{n_2-1}a_1^{n_2-1}, \quad C_{n_1}^{U_{(1,0)},1} = a_1^{n_1}a_2^{n_1}, \quad C_{n_1}^{U_{(1,1)},1} = a_2^{n_1}a_1^{n_1},\\
C_{n_2}^{U_{(1,1)},1} = a_1^{n_2}, \quad C_{n_3}^{U_{(0,1)},1} = a_1^{n_3}a_2^{n_3}, \quad C_{n_3}^{U_{(1,1)},1} = a_2^{n_3}a_1^{n_3}.
\end{gathered} \tag{8}$$

1) $a_i^h a_r^s$, if $h \neq s$, for all possible values of i and r,
2) $\big(C_j^{U_{(0,0)},i}\big)^2 - \big(C_l^{U_{(0,0)},k}\big)^2$, for all possible values of i, j, k and l,
3) $\big(C_j^{U_{(0,1)},i}\big)^2 - \big(C_l^{U_{(0,1)},k}\big)^2$, for all possible values of i, j, k and l,
4) $\big(C_j^{U_{(1,0)},i}\big)^2 - \big(C_l^{U_{(1,0)},k}\big)^2$, for all possible values of i, j, k and l,
5) $\big(C_j^{U_{(1,1)},i}\big)^2 - \big(C_l^{U_{(1,1)},k}\big)^2$, for all possible values of i, j, k and l,
6) $\big(C_i^{U_{(0,0)},j}a\big)^2$ $\big(\big(C_i^{U_{(0,1)},j}a'\big)^2\big)$, with a (a') being the first arrow of $C_i^{U_{(0,0)},j}$ $(C_i^{U_{0,1},j})$ for all i, j,
7) $\big(C_i^{U_{(1,0)},j}a\big)^2$ $\big(\big(C_i^{U_{(1,1)},j}a'\big)^2\big)$, with a (a') being the first arrow of $C_i^{U_{(1,0)},j}$ $(C_i^{U_{(1,1)},j})$ for all i, j.

Figure 10 shows the quiver $Q_{\mathscr{K}}$ associated to this configuration.

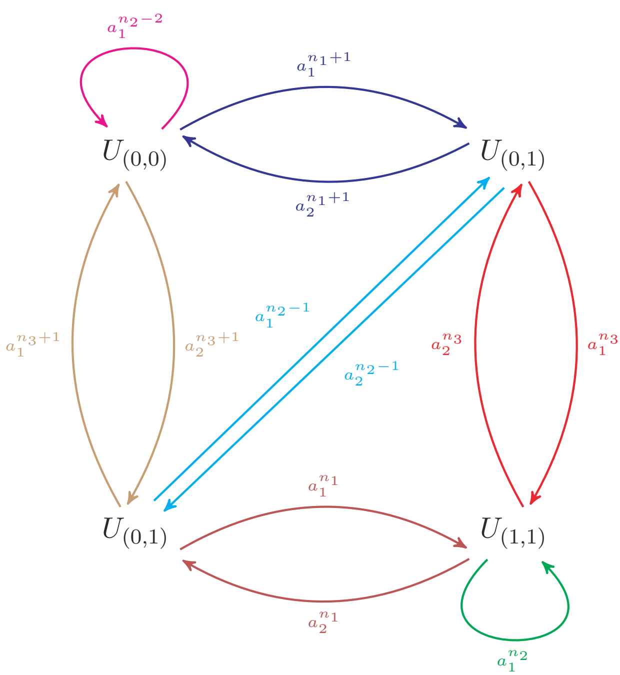
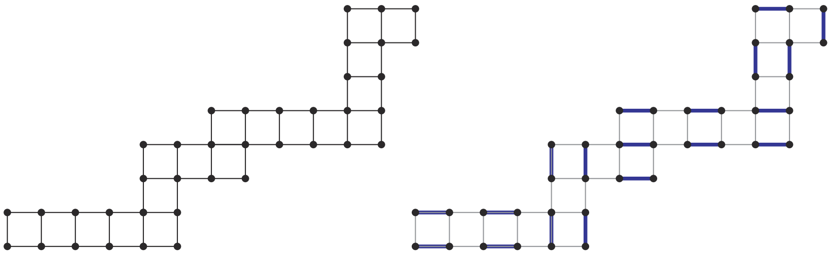


FIGURE 10. Quiver $Q_{\mathscr{K}}$ associated to configuration $\mathscr{K}$. Colors are assigned as in Figure 4.

The following is the numeric data associated to the algebra $\Lambda_{\mathscr{K}} = \mathbb{F}Q_{\mathscr{K}}/I$ with $Q_{\mathscr{K}}$ as shown in Figure 10 and special cycles given in (8), ($|r(Q_{\mathscr{K}})|$ is the number of indecomposable projective modules, $r_{U_{(0,0)}}$, $r_{U_{(0,1)}}$, $r_{U_{(1,0)}}$ and $r_{U_{(1,1)}}$ denote the number of summands in the heart of the indecomposable projective modules $P_{U_{(0,0)}}$, $P_{U_{(0,1)}}$, $P_{U_{(1,0)}}$ and $P_{U_{(1,1)}}$. Note that, $|C_i| = \mathrm{val}(i)$):

$$|r(Q_{\mathscr{K}})| = 4,$$
$$r_{U_{(0,0)}} = 3, \quad r_{U_{(0,1)}} = 3, \quad r_{U_{(1,0)}} = 3, \quad r_{U_{(1,1)}} = 3,$$
$$|C_{n_1+1}| = 2, \quad |C_{n_2-2}| = 1, \quad |C_{n_3+1}| = 2, \quad |C_{n_2-1}| = 2,$$
$$|C_{n_1}| = 2, \quad |C_{n_2}| = 1, \quad |C_{n_3}| = 2,$$
$$\sum_{\alpha\in\mathscr{K}_0}\sum_{X\in\mathscr{K}_1} \mathrm{occ}(\alpha, X) = 12, \quad \text{the number of special cycles,}$$
$$\dim_{\mathbb{F}} \Lambda_{\mathscr{K}} = 8 + 2(3) + 1(1) + 2(3) + 2(3) + 2(3) + 1(1) + 2(3) = 40,$$
$$\dim_{\mathbb{F}} Z(\Lambda_{\mathscr{K}}) = 1 + 14 + 4 - 7 + 2 - 2 = 12.$$

Example 19. As another example of Theorem 17 consider the following snake graph of type $\mathscr{G}_f(5, 3, 3, 2, 5, 4, 2)$:



In this case,

$$\begin{aligned}\mathrm{Match}(\mathscr{G}_f(5, 3, 3, 2, 5, 4, 2)) &= 3221\\ &= 4F_3F_4F_5F_2^4 + 12F_1F_3F_4F_6F_2^3 + 16F_1F_3^2F_4F_5F_2^2\\ &\quad + 12F_1^2F_3^2F_4F_6F_2 + 4F_1^2F_3^3F_4F_5.\end{aligned}$$

Note that, sequences (Fibonacci words) $F_3F_4F_5F_2^4$, $F_1F_3F_4F_6F_2^3, \ldots$ are specialized polygons of the Brauer configuration (5).

2.1. Determinants and Path Problems Via Brauer Configurations

In this section, we describe the way that specializations of suitable Brauer configurations (or Brauer configuration algebras) can be used to define determinants thus solutions of some very well known problems, as the paths problem solved by Lindström, Gessel and Viennot can be interpreted as a specialization of a Brauer configuration and as a consequence of such interpretation the message described in Theorem 17 can be viewed as a product of specialized Brauer configurations as well.

Let us consider a labeled Brauer configuration $\mathscr{D}(k) = \{\mathscr{D}_0(k), \mathscr{D}_1(k), \nu, \mathcal{O}\}$ obtained from the labeled Brauer configuration $\mathscr{K}$ defined by identities (5) and (6) by redefining vertices labels and polygons as follows:

$$\begin{gathered}\mathscr{D}_0(k) = \{\alpha_\pi^i = \alpha_{(i,\pi(i))} \in G \mid 1 \leqslant i \leqslant k, \pi \in S_k, k > 2 \text{ fixed}\},\\ \mathscr{D}_1(k) = \{(U_\pi, \pi) \mid \pi \in S_k\}, \quad (U_\pi, \pi) = \{\alpha_{(i,\pi(i))} \mid \pi \in S_k \text{ fixed}\}, \\ \nu(\alpha_{(i,\pi(i))}) = 1 \quad \text{for any vertex } \alpha_{(i,\pi(i))} \in \mathscr{D}_0(k),\end{gathered} \tag{9}$$

where π is an element of the group $(S_k, \prec)$ of permutations of k elements endowed with a linear order $\prec$, the labels in this case have the form $(\pi(1), \dots, \pi(k))$, ν is a multiplicity function. And the orientation $\mathcal{O}$ is defined in such a way that labeled polygons (U_{π_j}, π_j) and $(U_{\pi_{j+1}}, \pi_{j+1})$ are consecutive in $\mathscr{D}_1(k)$ provided that π_j and π_{j+1} are consecutive in $(S_k, \prec)$.

For the sake of accuracy in this case, to each word $w(U_\pi, \pi)$ associated to the polygon (U_π, π) it is defined $\operatorname{sign}(w(U_\pi, \pi)) = \operatorname{sign}(\pi)$ and the message $M(\mathscr{D})$ of the Brauer configuration $\mathscr{D}(k)$ is given by the identity:

$$M(\mathscr{D}(k)) = \sum_{(U_\pi, \pi) \in \mathscr{D}_1} \operatorname{sign}(w) w(U_\pi, \pi). \tag{10}$$

The following result follows immediately from the definitions (9) and (10).

Theorem 20. $M(\mathscr{D}(k)) = \det(\alpha_{(i,j)})$ *where* $\det(\alpha_{(i,j)})$ *is the determinant with entries* $\alpha_{(i,j)} \in \mathscr{D}_0(k)$.

Now several specializations can be defined for the message (10).

Henceforth, we let $M(\mathscr{D}^{e^k_{\mathscr{F}}}(k))$ denote the specialization of the message (10) with an associated function of the form $e^k_{\mathscr{F}} : \mathscr{D}_0(k) \to \mathbb{C}$ such that

$$e^k_{\mathscr{F}}(\alpha_{(r,s)}) = \begin{cases} \mathbf{i} = \sqrt{-1}, & \text{if } s = r+1, \ 1 \leqslant r \leqslant k-1, \\ \mathbf{i}, & \text{if } s = r-1, \ 2 \leqslant r \leqslant k, \\ 1, \ \text{if } s = r, \ 1 \leqslant r \leqslant k, \\ 0, \ \text{elsewhere.} \end{cases}$$

Then the following result holds (see [2] for the calculus of this family of determinants).

Corollary 21. *$M(\mathscr{D}^{e^k_{\mathscr{F}}}(k)) = F_{k+1}$ where F_j is the jth Fibonacci number.*

Proof. $M(\mathscr{D}^{e^k_{\mathscr{F}}}(k))$ is a $k \times k$-determinant whose entries are given by identities $e^k_{\mathscr{F}}(\alpha_{(r,s)})$ then column transformations of the form $C'_{j+1} \leftrightarrow -\frac{F_j}{F_{j+1}}C_j i + C_{j+1}$, for $1 \leqslant j \leqslant k-1$ reduce $\det(e^k_{\mathscr{F}}(\alpha_{(r,s)}))$ to a determinant with entries of the form:

$$T(e^k_{\mathscr{F}}(\alpha_{(r,s)})) = \begin{cases} \frac{F_{j+1}}{F_j}, & \text{if } s = r, \; 1 \leqslant r \leqslant k, \\ \mathbf{i}, & \text{if } s = r-1, \; 2 \leqslant r \leqslant k, \\ 0, \; \text{elsewhere.} \end{cases}$$

Thus, $T(e^k_{\mathscr{F}}(\alpha_{(r,s)}))$ is a diagonal determinant such that

$$\det(T(e^k_{\mathscr{F}}(\alpha_{(r,s)}))) = \prod_{j=1}^{k} \frac{F_{j+1}}{F_j} = F_{k+1}. \qquad \square$$

The following result is a direct consequence of Theorem 17 and Corollary 21.

Corollary 22. *For all $n_1, n_2 \dots, n_k \geqslant 2$, we have that* $\mathrm{Match}(\mathscr{G}_f(n_1, n_2, \dots, n_k)) = \sum_{w \in \{0,1\}^{k-1}} \prod_{\ell=1}^{k} M(\mathscr{D}^{e^{h_l}_{\mathscr{F}}}(k))$ *where $h_l = n_\ell - g(w_{\ell-1}, \ell) - g(w_\ell, \ell) + 1$.*

The Lindström's theorem. Specializations of the Brauer configuration $\mathscr{D}(k)$ allow us to interpret the Lindström's theorem as a message $M(\mathscr{D}(k))$. To do that, let us first recall the description of such a result as Gessel and Viennot described in [10].

If Q is an acyclic digraph with finitely many paths between any two vertices. Let k be a fixed positive integer. A k-vertex is a k-tuple of vertices of Q, if $u = (u_1, u_2, \dots, u_k)$ and $v = (v_1, v_2, \dots, v_k)$ are k-vertices of Q then a k-path from u to v is a k-tuple $A = (A_1, A_2, \dots, A_k)$ such that A_i is a path from u_i to v_i. The k-path A is disjoint if the paths A_i are vertex disjoint. Let S_k be the set of permutations of $\{1, 2, \dots, k\}$ then for $\pi \in S_k$, by $\pi(v)$ we mean the k-vertex $(v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(k)})$.

Let us assign a weight to every edge of Q we define the weight of a path to be the product of the weights of its edges and the weight of a k-path to be the product of the weights of its components.

Let $\mathscr{P}(u_i, v_j)$ be the set of paths from u_i to v_j and $P(u_i, v_j)$ be the sum of their weights. Define $\mathscr{P}(u, v)$ and $P(u, v)$ analogously for k paths from u to v.

Let $\mathscr{N}(u,v)$ be the subset of $\mathscr{P}(u,v)$ of disjoint paths and let $N(u,v)$ be the sum of their weights then it is clear that for any permutation $\pi \in \{1,2,\dots,k\}$, it holds that $P(u,\pi(v)) = \prod_{i=1}^{k} P(u_i, v_{\pi(i)})$. Thus the specialization with associated function of the form $h : \mathscr{D}_0(k) \to \mathbb{N}$ such that $h(\alpha_{(i,\pi(i))}) = P(u_i, v_{\pi(i)})$ and words defined by the specialized polygons $h(U_\pi,\pi) = \{P(u_i, v_{\pi(i)}) \mid 1 \leqslant i \leqslant k, \pi \in S_k \text{ fixed}\}$ of the form $w(h(U_\pi,\pi)) = \operatorname{sign}(\pi)P(u,\pi(v))$ build the following Brauer configuration version of the theorem of Lindström [10].

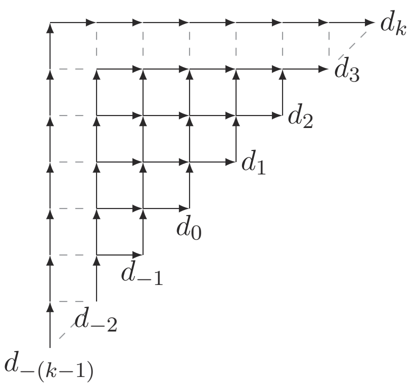
Theorem 23. $M(\mathscr{D}^h(k)) = \sum_{\pi\in S_k} \operatorname{sign}(\pi)N(u,\pi(v))$.

Proof. By definition $M(\mathscr{D}^h(k)) = \det(P(u_i,v_j))_{1\leqslant i,j\leqslant k}$. □

The following results are well known consequences of Theorem 23 giving values of $n\times n$, t-Catalan determinants. For which, we define specialized messages $M(\mathscr{D}^{h_t}(n))$ with $P(u_{1+h}, v_{j-h}) = P(u_1, v_j) = C_{t-1+j}$, $t \geqslant 1$ fixed, $0 \leqslant h \leqslant j-1$, $1 \leqslant j \leqslant n$, $j-h > 0$, and $P(u_{k+l}, v_{n-l}) = P(u_k, v_n) = C_{t+n+k-2}$, for $2 \leqslant k \leqslant n$ and $0 \leqslant l \leqslant n-k$, C_s denotes the sth Catalan number.

Corollary 24. $M(\mathscr{D}^{h_1}(k)) = 1$.

Proof. Consider the infinite directed graph G with $\mathbb{Z}\times\mathbb{Z}$ as the set of vertices and directed edges from (i,j) to $(i+1,j)$ and to $(i,j+1)$ for every $i,j\in\mathbb{Z}$. Let d_i denote the vertex (i,i) in G, $i\in\mathbb{Z}$. Note that the number of directed paths in G from d_i to d_j, with $j \geqslant i$ is equal to the Catalan number C_{j-i}. Let Q_k^1 be the family consisting of all k pairwise vertex disjoint directed paths $(A_0, A_1, \dots, A_{k-1})$ in G such that A_i joins d_{-i} with d_{i+1}, $i = 0,1,\dots,k-1$ then $M(\mathscr{D}^{h_1}(k)) = |Q_k^1| = 1$, where $|Q_k^1|$ is the number of vertices of the graph Q_k^1, see the diagram below.



□

The following is a more general result obtained via specializations $M(\mathscr{D}^{h_t}(k))$ and digraphs Q_k^t (as described in the proof of Corollary 24) where the system of k-paths $(A_0, A_1, \ldots, A_{k-1})$ and A_i joins vertices d_{-i} and d_{t+i} [14].

Corollary 25. $M(\mathscr{D}^{h_t}(k)) = |Q_k^t|$.

For example $M(\mathscr{D}^{h_2}(k)) = k+1$ and $M(\mathscr{D}^{h_3}(k)) = \frac{(k+1)(k+2)(2k+3)}{6}$.

On the Brauer configuration algebra $\Lambda_{\mathscr{D}(k)}$ induced by the Brauer configuration $\mathscr{D}(k)$. Note that each vertex $\alpha_{(i,j)} \in \mathscr{D}_0(k)$ has associated a successor sequence of the form

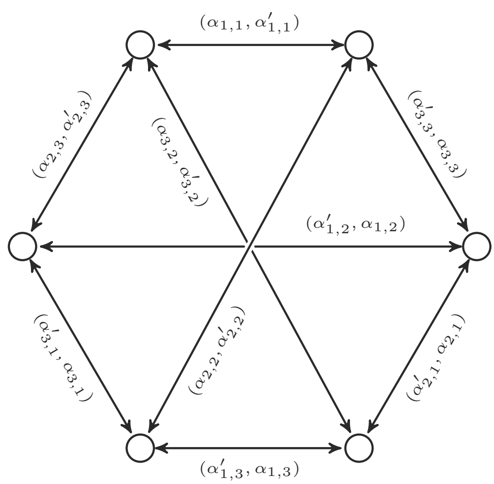
$$S_{(i,j)} = (U_{\pi_{i_1}}, \pi_{i_1}) \prec (U_{\pi_{i_2}}, \pi_{i_2}) \prec \cdots \prec (U_{\pi_{i_k}}, \pi_{i_k}),$$

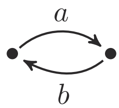
$(i,j) \in \pi_{i_j}$ with π_{i_j} being a k-set permutation and for any j, it holds that $1 \leqslant i \leqslant j$. Successor sequences $S_{(i,j)}$ define the corresponding special cycles $C_{(i,j)}$. Then the following are relations generating the admissible ideal I of $\Lambda_{\mathscr{D}(k)}$.

1) If $x_i = (i, \pi(i))$ and $x_j = (j, \pi(j))$ are elements of $\pi \in S_k$ then a relation of the form $C_{(i,\pi(i))} - C_{(j,\pi(j))}$ takes place,
2) If a is the first arrow of a special cycle $C_{(i,j)}$ then a relation $C_{(i,j)}a$ takes also place,
3) If γ is an arrow of a given special cycle $C_{(i,j)}$ and β is arrow of a special cycle $C'_{(i',j')}$ with the final vertex $e(\alpha)$ being the initial vertex $s(\beta)$ and $C_{(i,j)} \neq C'_{(i,j)}$ then a relation of the form $\alpha\beta$ holds in I,
4) The Brauer quiver $Q_{\mathscr{D}(k)}$ has no loops.

The following is the Brauer quiver $Q_{\mathscr{D}(3)}$:

$Q_{\mathscr{D}(3)} =$



For the sake of clarity any cycle of the form  is written as

$\overset{(b,a)}{\longleftrightarrow}$.

Note that in this case relations of the following form take place $\alpha'_{(i,j)}\alpha_{(j+n) \bmod 3,r}$ and $\alpha_{(i,j)}\alpha'_{(j+n) \bmod 3,s}$, besides $\alpha_{(1,1)}\alpha_{(2,3)}$, $\alpha_{(2,3)}\alpha_{(3,1)}$, $\alpha_{(3,1)}\alpha_{(1,3)}$, $\alpha_{(1,3)}\alpha_{(2,1)}$, $\alpha_{(1,3)}\alpha_{(3,2)}$, $\alpha_{(3,2)}\alpha_{(2,3)}$, $\alpha_{(2,3)}\alpha_{(1,2)}$, $\alpha_{(2,2)}\alpha_{(1,3)}$, $\alpha'_{(2,3)}\alpha'_{(1,1)}$, $\alpha'_{(2,1)}\alpha'_{(1,3)}$, $\alpha'_{(1,3)}\alpha'_{(2,2)}$, $\alpha'_{(1,3)}\alpha'_{(3,1)}$, $\alpha'_{(3,1)}\alpha'_{(2,3)}$, $\alpha'_{(3,2)}\alpha'_{(1,3)}$, $\alpha'_{(1,2)}\alpha'_{(2,3)}$ for all possible values of r and s. Thus in general the following result holds.

Theorem 26. *For the Brauer configuration $\Lambda_{\mathscr{D}(k)}$ induced by the Brauer configuration $\mathscr{D}(k)$ the following statements hold:*

1) *$\Lambda_{\mathscr{D}(k)}$ has $k!$ indecomposable projective modules.*
2) *If $\alpha_{(i,j)} \in \mathscr{D}_0(k)$ then $\mathrm{val}(\alpha_{(i,j)}) = (k-1)!$.*
3) *The number of summands in the heart of an indecomposable projective module given by a polygon of the form (U_π, π) is k.*
4) *$\dim_{\mathbb{F}} \Lambda_{\mathscr{D}(k)} = 2(k!+k^2 t_{((k-1)!-1)})$ where t_s denotes the sth triangular number.*
5) *$\dim_{\mathbb{F}} Z(\Lambda_{\mathscr{D}(k)}) = 1 + k!$.*

Proof. 1) The assertion follows from Theorem 6 (item 1) and the fact that $|\mathscr{D}_1(k)| = |S_k| = k!$.

2) By definition of a $k \times k$-determinant it holds that each entry-vertex $\alpha_{(i,\pi(i))}$ occurs in $(k-1)!$ summands-polygons of the form

$$\alpha_{(1,\pi(1))}\alpha_{(2,\pi(2))}\cdots\alpha_{(k,\pi(k))}.$$

3) We note that if P is an indecomposable projective $\Lambda_{\mathscr{D}(k)}$-module corresponding to a polygon (U_π, π) then $\mathrm{rad}^2 P \neq 0$ and the result follows bearing in mind that any polygon (U_π, π) has k vertices each of them occurring in $(k-1)!$ polygons (i.e., all vertices in a given polygon are non-truncated).

4) Proposition 8 allows to conclude that

$$\dim_{\mathbb{F}} \Lambda_{\mathscr{D}(k)} = 2k! + \sum_{\alpha_{(i,j)} \in \mathscr{D}_0} |C_{\alpha_{(i,j)}}|(|C_{\alpha_{(i,j)}}| - 1)$$

where for each $\alpha_{(i,j)}$, $|C_{\alpha_{(i,j)}}| = \mathrm{val}(\alpha_{(i,j)}) = (k-1)!$. Thus, the statement holds taking into account that for any $j \geqslant 2$, $j(j-1) = 2t_{j-1}$.

5) Since $\operatorname{rad}^2 \Lambda_{\mathscr{D}(k)} \neq 0$, the statement is a consequence of Theorem 10 with $\nu(\alpha_{(i,j)}) = 1$, for all $\alpha_{(i,j)} \in \mathscr{D}_0(k)$, $|\mathscr{D}_0(k)| = k^2$, $|\mathscr{D}_1(k)| = k!$, $\#(\text{Loops}\, Q_{\mathscr{D}(k)}) = 0$ and $|\mathscr{C}_{\mathscr{D}(k)}| = 0$. □

Corollary 27. *For $n > 2$ the algebra $\Lambda_{\mathscr{D}(n)}$ associated to the Brauer configuration $\mathscr{D}(n)$ has a length grading induced from the path algebra $\mathbb{F}Q_{\mathscr{D}(n)}$.*

Proof. Since by definition $\mathscr{D}(n)$ is connected, then the corollary holds as a consequence of Proposition 9, taking into account that for any $\alpha_{(i,j)} \in \mathscr{D}_0(n)$, $\nu(\alpha_{(i,j)}) = 1$ and $\operatorname{val}(\alpha_{(i,j)}) = (n-1)!$. □

2.2. Some particular sequences

In this section, we study some sequences obtained from the number of perfect matchings of some particular snake graphs. Let us denote by $a(n,k)$ the sequence whose terms are given by the number of perfect matchings of the snake graph $\mathscr{G}_f(\underbrace{n, n, \ldots, n}_{k \text{ times}})$, that is $a(n,k) := \operatorname{Match}(\mathscr{G}_f(\underbrace{n, n, \ldots, n}_{k \text{ times}}))$. For example, $a(2,3) = 5$, the relevant perfect matchings for the graph $\mathscr{G}_f(2,2,2)$ are shown in Figure 11.

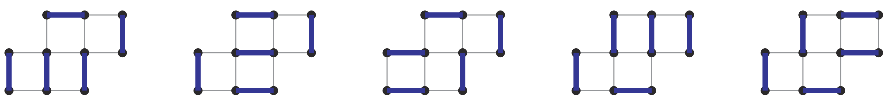


FIGURE 11. Perfect matchings of the snake graph $\mathscr{G}_f(2,2,2)$.

In Table 2, we give several terms of sequences $a(n,k)$ for different values of k and their ordinary generating functions. Notice that each term in the sequence $a(n,2)$ (sequence A079472 in the OEIS [17]) corresponds to the number of perfect matchings of a symmetric L-shaped graph [13].

k	Sequence $a(n,k)$	Generating Funct.	A-Sequence
2	4, 12, 30, 80, 208, 546, 1428, 3740, 9790, …	$\frac{2(2+2x-x^2)}{x^3-2x^2-2x+1}$	A079472
3	5, 29, 112, 493, 2059, 8770, 37073, 157169, 665576, …	$\frac{-2x^3-5x^2+14x+5}{x^4+3x^3-6x^2-3x+1}$	—
4	6, 70, 418, 3038, 20382, 140866, 962470, 6604838, …	$-\frac{2(x^4-6x^3-11x^2+20x+3)}{x^5-5x^4-15x^3+15x^2+5x-1}$	—
5	165, 1153, 7811, 53745, 367797, 2522395, 17284853, …	$-\frac{2(x^4-6x^3-11x^2+20x+3)}{x^5-5x^4-15x^3+15x^2+5x-1}$	—

TABLE 2. Particular cases of the sequence $a(n,k)$.

We note that the coefficients of the polynomial in the denominators of the generating functions $\sum_{n\geqslant 2} a(n,k)x^n$ coincide with the Fibonomial triangle (array A010048 in the OEIS), except for the signs. Let $\begin{bmatrix} n \\ k \end{bmatrix}_F$ be the (n, k)-th entry of the Fibonomial triangle, which is defined by (cf. [11])

$$\begin{bmatrix} n \\ k \end{bmatrix}_F := \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}.$$

The first few rows of the Fibonomial matrix are

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}_F\right]_{n,k\geqslant 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 6 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 15 & 15 & 5 & 1 & 0 & 0 & 0 \\ 1 & 8 & 40 & 60 & 40 & 8 & 1 & 0 & 0 \\ 1 & 13 & 104 & 260 & 260 & 104 & 13 & 1 & 0 \\ 1 & 21 & 273 & 1092 & 1820 & 1092 & 273 & 21 & 1 \end{pmatrix}$$

In fact, we conjecture that

$$\sum_{n\geqslant 2} a(n,k)x^n = \frac{p_k(x)}{\sum_{k=0}^{n}(-1)^{\binom{k+1}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_F x^k},$$

where $p_k(x)$ is a polynomial of degree k.

Let $b(n, k)$ denote the number of perfect matchings of the snake graph $\mathscr{G}_f(n, n+1, \ldots, n+k-1)$. In Table 3, we give terms of sequences of this type for different values of k and their ordinary generating functions. The sequence $b(n, 2) = F_{n+1}^2 + F_n F_{n+2}$ (sequence A061646 in the OEIS) was studied by Tauraso [18]. This sequence counts the number of domino tilings of a L-grid obtained by removing the upper-right $(n-1)\times(n-2)$ rectangle from a $(n+1)\times n$ rectangle. Moreover, this sequence is a particular example of the family of nonlinear recurrences studied by Alperin in [1].

k	Sequence $b(n, k)$	Generating Funct.	A-Sequence
2	7, 19, 49, 129, 337, 883, 2311, 6051, 15841, …	$\frac{-3x^2+5x+7}{x^3-2x^2-2x+1}$	A061646
3	27, 116, 487, 2069, 8754, 37099, 157127, 665644, 2819643, …	$\frac{-7x^3-23x^2+35x+27}{x^4+3x^3-6x^2-3x+1}$	—
4	165, 1153, 7811, 53745, 367797, 2522395, 17284853, …	$\frac{-27x^4+130x^3+429x^2-328x-165}{x^5-5x^4-15x^3+15x^2+5x-1}$	—
5	1640, 18493, 202901, 2258082, 25006855, 277477625, 3076643824, …	$\frac{-165x^5-1339x^4+6446x^3+10643x^2-5373x-1640}{x^6+8x^5-40x^4-60x^3+40x^2+8x-1}$	—

TABLE 3. Particular cases of the sequence $b(n, k)$.

3. Acknowledgements

The authors are deeply indebted to the referee for his helpful suggestions and comments.

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Received by the editors: 10.07.2020
and in final form 17.01.2021.