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Capable groups of order p^3q

O. Kalteh and S. Hadi Jafari*

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Abstract. In this paper, we study on the capability of groups of order p^3q, where p and q are distinct prime numbers and $p > 2$.

1. Introduction and motivation

A group G is called capable if there exists a group E such that $G \cong E/Z(E)$. The study of capable groups goes back to Baer [1], who determined all finitely generated abelian capable groups. P. Hall remarked in [7] that characterizations of capable groups are important in classifying groups of prime power order. In 1979, Beyl et al. [2] studied capable groups by focusing on a characteristic subgroup $Z^*(G)$, called the epicenter of G, which is the smallest central subgroup of G such that $G/Z^*(G)$ is capable. Therefore the triviality of the epicenter of a group is a criterion for capability of the group.

Graham Ellis [6] characterized the epicenter in terms of the nonabelian exterior square as defined below. Once the nonabelian exterior square of a group is known, it is not too hard to determine its epicenter. For a group G, the nonabelian tensor square $G \otimes G$ is the group generated by the symbols $g \otimes h$ subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \qquad \text{and} \qquad g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h'),$$

*Corresponding author.
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for all $g, g', h, h' \in G$, where G acts on itself by conjugation, i.e. ${}^{g}g' = gg'g^{-1}$. The nonabelian tensor square is a special case of the nonabelian tensor product which introduced by Brown and Loday in [4]. The nonabelian exterior square $G \wedge G$ is obtained by imposing the additional relations $g \otimes g = 1$ on $G \otimes G$, for all $g \in G$. The nonabelian exterior square of a group G, defines the central subgroup $Z^{\wedge}(G)$ of G called the exterior center, which is defined as follows:

$$Z^{\wedge}(G) = \{g \in G \mid g \wedge x = 1; \forall\, x \in G\}.$$

G. Ellis [6] established that $Z^{\wedge}(G) = Z^{*}(G)$. So the following criterion follows immediately:

A group G is capable if and only if $Z^{\wedge}(G) = 1$.

Regarding to the P. Hall's remark mentioned above, many authors have been interested in characterizing the capable groups among the specific classes of groups, for example, see [3, 9, 13, 16]. In particular, the capability of groups of order $8q$, q is an odd prime, was studied in [15]. In this paper, our goal is to complete the latter work by studying on the capability of groups of order p^3q, where p and q are distinct prime numbers and $p > 2$. We first determine the epicenter for those groups and then identify the capable ones among them. In 1899, Western [17] classified the groups of order p^3q. He proved that there are 25 types of nonabelian groups of order p^3q, where p is an odd prime.

Theorem 1. *[17, pp. 258-261]. Let G be a nonabelian group of order p^3q, where p and q are distinct prime numbers and $p > 2$. Then G is one of the following types:*

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1,\ b^{-1}ab = a^{p+1},\ ad = da, bd = db\rangle, \tag{1}$$

$$\begin{aligned}&\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1,\ ab = ba,\ ac = ca,\ c^{-1}bc = ab,\\ &\quad ad = da,\ bd = db,\ cd = dc\rangle\end{aligned} \tag{2}$$

If $q \equiv 1 \pmod{p}$, there are the following:

$$\langle a, d \mid a^{p^3} = d^q = 1,\ a^{-1}da = d^{\alpha}\rangle \tag{3}$$

where α (here and in the next five groups) is any primitive root of $\alpha^p \equiv 1 \pmod q$.

$$\langle a,b,d \mid a^{p^2}=b^p=d^q=1,\ ab=ba,\ ad=da,\ b^{-1}db=d^{\alpha}\rangle \tag{4}$$

$$\langle a,b,d \mid a^{p^2}=b^p=d^q=1,\ ab=ba,\ a^{-1}da=d^{\alpha},\ bd=db\rangle \tag{5}$$

$$\begin{aligned}\langle a,b,c,d \mid a^p=b^p=c^p=d^q=1,\ ab=ba,\ ac=ca,\ bc=cb,\\ ad=da,\ bd=db,\ c^{-1}dc=d^{\alpha}\rangle\end{aligned} \tag{6}$$

$$\langle a,b,d \mid a^{p^2}=b^p=d^q=1,\ b^{-1}ab=a^{p+1},\ ad=da,\ b^{-1}db=d^{\beta}\rangle \tag{7}$$

where $\beta=\alpha$, or $\alpha^2,\ldots$, or α^{p-1}.

$$\begin{aligned}\langle a,b,c,d \mid a^p=b^p=c^p=d^q=1,\ ab=ba,\ ac=ca,\ ad=da,\\ bd=db,\ c^{-1}bc=ab,\ c^{-1}dc=d^{\alpha}\rangle\end{aligned} \tag{8}$$

$$\langle a,d \mid a^{p^3}=d^q=1,\ a^{-1}da=d^{\alpha}\rangle \tag{9}$$

where α (here and in the next group) is any primitive root of $\alpha^{p^2} \equiv 1 \pmod q$.

$$\langle a,b,d \mid a^{p^2}=b^p=d^q=1,\ ab=ba,\ a^{-1}da=d^{\alpha},\ bd=db\rangle \tag{10}$$

$$\langle a,d \mid a^{p^3}=d^q=1,\ a^{-1}da=d^{\alpha}\rangle \tag{11}$$

where α is any primitive root of $\alpha^{p^3} \equiv 1 \pmod q$.

When $p \equiv 1 \pmod q$, there are the following types (where α, α_2 and α_3 are the primitive qth root of unity modulo p, p^2 and p^3 respectively):

$$\langle a,d \mid a^{p^3}=d^q=1,\ d^{-1}ad=a^{\alpha_3}\rangle \tag{12}$$

$$\langle a,b,d \mid a^{p^2}=b^p=d^q=1,\ ab=ba,\ ad=da,\ d^{-1}bd=b^{\alpha}\rangle \tag{13}$$

$$\langle a,b,d \mid a^{p^2}=b^p=d^q=1,\ ab=ba,\ d^{-1}ad=a^{\alpha_2},\ db=bd\rangle \tag{14}$$

$$\langle a,b,d \mid a^{p^2}=b^p=d^q=1,\ ab=ba,\ d^{-1}ad=a^{\alpha_2},\ d^{-1}bd=b^{\alpha_2^i}\rangle \tag{15}$$

where $1 \leqslant i \leqslant q-1$.

$$\begin{aligned}\langle a,b,c,d \mid a^p=b^p=c^p=d^q=1,\ ab=ba,\ ac=ca,\ bc=cb,\\ ad=da,\ bd=db,\ d^{-1}cd=c^{\alpha}\rangle\end{aligned} \tag{16}$$

$$\begin{aligned}q=2\cdot\langle a,b,c,d \mid a^p=b^p=c^p=d^2=1,\ ab=ba,\ ac=ca,\\ bc=cb,\ ad=da,\ dbd=b^{-1},\ dcd=c^{-1}\rangle\end{aligned} \tag{17}$$

$$\begin{aligned}q>2\cdot\langle a,b,c,d \mid a^p=b^p=c^p=d^q=1,\ ab=ba,\ ac=ca,\\ bc=cb,\ ad=da,\ d^{-1}bd=b^{\alpha},\ d^{-1}cd=c^{\alpha^{\lambda}}\rangle\end{aligned}$$

where λ represents the different solutions of $xy \equiv 1 \pmod q$, in which $b \equiv a^x \pmod p$, and a and b are the primitive roots of $q \pmod p$.

$$\langle a,b,c,d \mid a^p = b^p = c^p = d^q = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ d^{-1}ad = a^{\alpha}, d^{-1}bd = b^{\alpha^x},\ d^{-1}cd = c^{\alpha^y}\rangle \tag{18}$$

where $q \equiv 0$ or $\pm 1 \pmod 3$, and x and y may have any of the values $1, 2, \ldots, q-1$.

$$\langle a,b,d \mid a^{p^2} = b^p = d^q = 1,\ b^{-1}ab = a^{p+1},\ bd = db,\ d^{-1}ad = a^{\alpha_2}\rangle \tag{19}$$

$$\langle a,b,c,d \mid a^p = b^p = c^p = d^q = 1,\ ab = ba,\ ac = ca,\ ad = da,\ c^{-1}bc = ab,\ d^{-1}bd = b^{\alpha},\ d^{-1}cd = c^{\alpha^{q-1}}\rangle \tag{20}$$

$$\langle a,b,c,d \mid a^p = b^p = c^p = d^q = 1,\ ab = ba,\ ac = ca,\ c^{-1}bc = ab,\ d^{-1}ad = a^{\alpha},\ db = bd,\ d^{-1}cd = c^{\alpha}\rangle \tag{21}$$

$$\langle a,b,c,d \mid a^p = b^p = c^p = d^q = 1,\ ab = ba,\ ac = ca,\ c^{-1}bc = ab,\ d^{-1}ad = a^{\alpha},\ d^{-1}bd = b^{\alpha^x},\ d^{-1}cd = c^{\alpha^{q+1-x}}\rangle \tag{22}$$

where $x = 2$ or $3,\ldots,$or $\frac{q+1}{2}$ and $q > 2$.

When $p \equiv -1 \pmod q$, and $q > 2$, there are the following two types:

$$\langle a,b,c,d \mid a^p = b^p = c^p = d^q = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ ad = da,\ d^{-1}bd = c,\ d^{-1}cd = b^{-1}c^{t^p+t}\rangle \tag{23}$$

where t (here and in the next group) is any primitive Galoisian root of $t^q \equiv 1 \pmod p$.

$$\langle a,b,c,d \mid a^p = b^p = c^p = d^q = 1,\ ab = ba,\ ac = ca,\ c^{-1}bc = ab,\ ad = da,\ d^{-1}bd = c,\ d^{-1}cd = b^{-1}c^{t^p+t}\rangle. \tag{24}$$

And, lastly, when $p^2+p+1 \equiv 0 \pmod q$, and $q > 3$, there is the following type:

$$\langle a,b,c,d \mid a^p = b^p = c^p = d^q = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ d^{-1}ad = b,\ d^{-1}bd = c,\ d^{-1}cd = ab^{-\lambda^{-1}-\lambda^{-p}-\lambda^{-p^2}}c^{\lambda+\lambda^p+\lambda^{p^2}}\rangle \tag{25}$$

where λ is a Galois imaginary of the third order, which is a primitive root of $\lambda^q \equiv 1 \pmod p$.

In what follows, we use the following notations frequently:

- $\mathcal{M}(G)$ is the Schur multiplier of G;
- G^{ab} is the abelianization of G;
- C_p^k is the direct product of k copies of the cyclic group of order p;
- $E^1_{p^3}$ is the extraspecial p-group of order p^3 and exponent p;
- Φ_i is the isoclinic family of groups of order p^n, where $n \leqslant 6$ and $p \neq 2$ given in [10].

2. Capable and unicentral groups

For a group G, the commutator map induces the homomorphisms $k : G \otimes G \to G$ and $k' : G \wedge G \to G$ such that $k(g \otimes h) = k'(g \wedge h) = [g, h]$ for all $g, h \in G$. The kernel of k and k' denoted by $J_2(G)$ and $\mathcal{M}(G)$, respectively. In order to study the capability of groups of order p^3q, $p > 2$, we first compute their nonabelian exterior squares. Also it will be useful to know the nonabelian tensor squares and Schur multipliers of such groups which are described in [11]. The following result (part (ii)) will be used in proof of Proposition 3. It is an immediate consequence of part (i).

Proposition 1. *Let G be a finite polycyclic group with a polycyclic generating sequence $g_1, \dots, g_k$. Then*

(i) $G \otimes G = \langle g_i \otimes g_i, g_i \otimes g_j, (g_i \otimes g_j)(g_j \otimes g_i)\rangle$ ([3, Proposition 20]).

(ii) $G \wedge G = \langle g_i \wedge g_j\rangle$ *where* $1 \leqslant j < i \leqslant k$.

Proposition 2. *Let G be a nonabelian group of order p^3q with $2 < p < q$. Then*

$$G \wedge G \cong \begin{cases} C_p, & \textit{if } G \textit{ is of type (1)} \\ C_q, & \textit{if } G \textit{ is any group of types (3), (9) or (11)} \\ C_{pq}, & \textit{if } G \textit{ is any group of types (4), (5), (7) or (10)} \\ C_p^3, & \textit{if } G \textit{ is of type (2)} \\ C_p^3 \times C_q, & \textit{if } G \textit{ is any group of types (6) or (8)} \end{cases}$$

Proof. When $\mathcal{M}(G) \cong 1$ we have $G \wedge G \cong G'$. So for groups of types (1), (3), (7), (9) or (11) the result follows by [11, Proposition 3.1]. If G is of type (4), then $\mathcal{M}(G) \cong C_p$ by [11, Proposition 3.1]. Moreover, we have $G' \cong C_q$. Since G' is isomorphic with the central factor group $(G \wedge G)/\mathcal{M}(G)$, then $G \wedge G$ is an abelian group of order pq. Utilizing the same method we can show that $G \wedge G \cong C_{pq}$ for any group of types (5) or (10).

If G is of type (2), then [11, Proposition 3.1] shows that $\mathcal{M}(G) \cong C_p^2$. Also we have $G' \cong C_p$. Hence $|G \wedge G| = p^3$. Now it is easily seen that $G/N \cong E^1_{p^3}$, where $N = \langle d\rangle$. Since $E^1_{p^3} \wedge E^1_{p^3} \cong C_p^3$ (see [13, Proposition

34]), we have $G \wedge G \cong E^1_{p^3} \wedge E^1_{p^3}$ which is generated by $a \wedge b, a \wedge c$ and $b \wedge c$.

If G is of type (6), then [11, Proposition 3.1] shows that $\mathcal{M}(G) \cong C_p^3$. Also we have $G' \cong C_q$. Hence $|G \wedge G| = p^3 q$. Set $N = \langle d \rangle \cong C_q$, then $G/N \cong C_p^3$ and $G/N \wedge G/N \cong C_p^3$. Therefore $a \wedge b, a \wedge c$ and $b \wedge c$ are non-trivial independent generators of $G \wedge G$. On the other hand, the epimorphism $G \wedge G \longrightarrow G'$ together with the equation $d^q \wedge c = (d \wedge c)^q = 1$ imply that $d \wedge c$ is a non-trivial generator of $G \wedge G$ whose order is q. Moreover, $d \wedge c$ is independent of the above three generators. Thus $G \wedge G$ is an abelian group isomorphic with $C_p^3 \times C_q$. For group (8), the desired result follows similarly by considering the factor group $G/\langle d \rangle \cong E_1$. □

Proposition 3. *Let G be a nonabelian group of order $p^3 q$ with $p > q$ and $p > 2$. Then*

$$G \wedge G \cong \begin{cases} C_p, & \text{if } G \text{ is of type (13)} \\ C_p^2, & \text{if } G \text{ is of type (16)} \\ C_p^2 \text{ or } E^1_{p^3}, & \text{if } G \text{ is any group of types (17), (21) or (23)} \\ C_{p^2}, & \text{if } G \text{ is any group of types (14) or (19)} \\ C_{p^2} \times C_p, & \text{if } G \text{ is of type (15) and } 1 \leqslant i \leqslant q-2 \\ \Phi_2(211)c, & \text{if } G \text{ is of type (15) and } i = q-1 \\ C_{p^3}, & \text{if } G \text{ is of type (12)} \\ C_p^3,\ \Phi_2(1^4),\ \Phi_4(1^5) \text{ or } \Phi_{11}(1^6), & \text{if } G \text{ is any group of types (18) or (25)} \\ E^1_{p^3}, & \text{if } G \text{ is any group of types (20) or (24)} \\ E^1_{p^3},\ \Phi_3(1^4) \text{ or } \Phi_3(1^5), & \text{if } G \text{ is of type (22)} \end{cases}$$

Proof. If G is any group of types (12), (13), (14), (19), (20) or (24), then $\mathcal{M}(G) \cong 1$ by [11, Proposition 3.2], whence $G \wedge G \cong G'$. If G is of type (16), then [11, Proposition 3.2] shows that $\mathcal{M}(G) \cong C_p$. Also we have $G' \cong C_p$. Hence $|G \wedge G| = p^2$. Now set $N = \langle c, d \rangle$. The natural homomorphism $G \wedge G \longrightarrow G/N \wedge G/N$ implies that $a \wedge b$ is a non-trivial generator of

$G \wedge G$. Since $a^p \wedge b = (a \wedge b)^p = 1$, then $|a \wedge b| = p$. In addition, it is clear that $c \wedge d$ is a non-trivial generator of $G \wedge G$ which is independent of $a \wedge b$, for $[c, d] \neq 1$. As $c^p \wedge d = (c \wedge d)^p = 1$, it follows that $|c \wedge d| = p$. Therefore we deduce that $G \wedge G = \langle a \wedge b, c \wedge d \rangle \cong C_p^2$.

Assume G is any group of types (17), (21) or (23). Then $\mathcal{M}(G) \cong 1$ or C_p by [11, Proposition 3.2]. If $\mathcal{M}(G) \cong 1$, then $G \wedge G \cong C_p^2$. Suppose G is of type (17) and $\mathcal{M}(G) \cong C_p$. Then $|G \wedge G| = p^3$. Set $N = \langle a \rangle$. Obviously $(G/N)' \cong C_p^2$ and $(G/N)^{ab} \cong C_q$. By the [11, proof of Proposition 3.2] we know that in this case $\mathcal{M}(G/N) \cong C_p$. Hence it follows from [8, Theorem C] that $G/N \otimes G/N \cong C_q \times H$ where H is an extraspecial p-group of order p^3. Now, [5, Proposition 8] implies that $G/N \wedge G/N \cong H$. Therefore $G \wedge G \cong G/N \wedge G/N$. As $b \wedge c, b \wedge d$ and $c \wedge d$ are non-trivial independent generators of orders p, then $G \wedge G = \langle b \wedge c, b \wedge d, c \wedge d \rangle \cong E^1_{p^3}$.

Assume G is of type (21) and $\mathcal{M}(G) \cong C_p$. We know that $|G \wedge G| = p^3$. The polycyclic presentation of G is as follows:

$$\begin{aligned}\langle a_1, a_2, a_3, a_4 \mid a_1^q = a_2^p = a_3^p = a_4^p = 1, a_2^{a_1} = a_2^{\alpha}, a_3^{a_1} = a_3,\\ a_4^{a_1} = a_4^{\alpha}, a_3^{a_2} = a_3a_4, a_4^{a_2} = a_4, a_4^{a_3} = a_4\rangle.\end{aligned}$$

The above generating set form polycyclic generating sequence so that Proposition 1 provides a generating set $\{a_2 \wedge a_1, a_3 \wedge a_1, a_4 \wedge a_1, a_3 \wedge a_2, a_4 \wedge a_2, a_4 \wedge a_3\}$ for $G \wedge G$. Obviously $a_4 \wedge a_3 = a_3 \wedge a_1 = 1$. We claim that $a_3 \wedge a_2$ can be generated by $a_4 \wedge a_2$ and $a_4 \wedge a_1$. First by induction observe that for any integer n, $a_3 \wedge a_2^n = (a_3 \wedge a_2)^n (a_4 \wedge a_2)^{\binom{n}{2}}$. Furthermore $a_3^{-1} \wedge a_2^n = (a_4 \wedge a_2)^{-\binom{n}{2}} (a_3 \wedge a_2)^{-n}$, which implies that $a_3^{-1} \wedge a_2^{\alpha-1} = (a_4 \wedge a_2)^{-\binom{\alpha-1}{2}} (a_3 \wedge a_2)^{1-\alpha}$. On the other hand $a_3^{-1} \wedge a_2^{\alpha-1} = a_3^{-1} \wedge [a_2, a_1] = (a_2 \wedge a_4)^{\alpha-1}(a_4 \wedge a_1)$. So the claim holds by equating the last two equalities. Therefore $G \wedge G = \langle a_4 \wedge a_2, a_4 \wedge a_1, a_2 \wedge a_1 \rangle$. The epimorphism $G \wedge G \longrightarrow G'$ implies that $a_4 \wedge a_1$ and $a_2 \wedge a_1$ are non-trivial independent generators whose orders are divided by p. On the other hand $a_4^p \wedge a_1 = (a_4 \wedge a_1)^p = 1$ and $a_2^p \wedge a_1 = (a_2 \wedge a_1)^p = 1$. Hence $|a_4 \wedge a_1| = |a_2 \wedge a_1| = p$. As $[a_4 \wedge a_1, a_2 \wedge a_1] = \langle a_4 \wedge a_2 \rangle = \mathcal{M}(G)$, then $G \wedge G \cong E^1_{p^3}$.

Assume G is of type (23) and $\mathcal{M}(G) \cong C_p$. So $|G \wedge G| = p^3$. Put $Z = \langle a \rangle \leqslant Z(G)$. Since $(G/Z)'$ has the cyclic complement $(G/Z)^{ab} \cong C_q$, then $G/Z \otimes G/Z \cong (G/Z \wedge G/Z) \times C_q$ by [5, Proposition 8]. On the other hand it follows from [12, Theorem 2.5.5] that $\mathcal{M}(G/Z) \cong C_p$. Hence [8, Theorem C] yields that $G/Z \wedge G/Z \cong E^1_{p^3}$ so that $G/Z \wedge G/Z \cong G \wedge G \cong E^1_{p^3}$.

Assume G is of type (15). Then by [11, Proposition 3.2], either $\mathcal{M}(G) \cong 1$ when $1 \leqslant i \leqslant q-2$ or $\mathcal{M}(G) \cong C_p$ when $i = q-1$. If $\mathcal{M}(G) \cong 1$, then $G \wedge G \cong C_{p^2} \times C_p$. In the case $\mathcal{M}(G) \cong C_p$, first observe by [11, Corollary 3.4] that $|G \otimes G| = p^4 q$. Also, [11, Proposition 2.4] and exact sequence $1 \to J_2(G) \to G \otimes G \to G' \to 1$ show that $J_2(G) \cong C_p \times C_q$. Since G' is abelian, we also have $(G \otimes G)' \subseteq J_2(G)$. From the presentation of G, we find that $a \otimes b$ and $d \otimes d$ belong to $J_2(G)$, which imply that $J_2(G) = \langle a \otimes b\rangle \times \langle d \otimes d\rangle$, as $|J_2(G)| = pq$. Thus $|a \otimes b| = p$. On the other hand, from the natural epimorphism $\pi : G \otimes G \to G^{ab} \otimes G^{ab}$, we have $|ker\pi| = p^4$ and $(G \otimes G)' \subseteq ker\pi$. Hence $|(G \otimes G)'| \mid p^4$. Since $(G \otimes G)' \subseteq J_2(G)$, we also get $|(G \otimes G)'| \mid pq$, from which it follows that $|(G \otimes G)'| \mid p$. Consequently we have $(G \otimes G)' = \langle a \otimes b\rangle \cong C_p$. As every element g in G may be presented by $g = a^r b^s d^t$ for some integers r, s, t, then one can easily show that $G \wedge G = \langle a \wedge b, b \wedge d, a \wedge d\rangle$. Now the epimorphism $k' : G \wedge G \longrightarrow G'$ implies that $p^2 \mid |a \wedge d|$. Also it follows by induction on any integer n that $a^n \wedge d = (a \wedge d)^n$. So $|a \wedge d| \mid p^2$ whence $|a \wedge d| = p^2$. Similarly $|b \wedge d| = p$. Now from the fact that $G \wedge G$ is a nonabelian group of order p^4, it follows that $G \wedge G \cong \Phi_2(211)c$ (see [10]).

Assume G is of type (18). Then by [11, Proposition 3.2], $\mathcal{M}(G) \cong 1, C_p, C_p^2$ or C_p^3. If $\mathcal{M}(G) = 1$, then $G \wedge G \cong G'$. Suppose that $\mathcal{M}(G) \cong C_p$. It is readily seen that $|G \wedge G| = p^4$. The polycyclic presentation of G is as follows:

$$\langle a_1, a_2, a_3, a_4 \mid a_1^q = a_2^p = a_3^p = a_4^p = 1,\ a_2^{a_1} = a_2^{\alpha}, a_3^{a_1} = a_3^{c_3}, \\ a_4^{a_1} = a_4^{c_4}, a_3^{a_2} = a_3, a_4^{a_2} = a_4, a_4^{a_3} = a_4\rangle,$$

in which $c_3 = \alpha^x \bmod p$ and $c_4 = \alpha^y \bmod p$. By Proposition 1 we get

$$G \wedge G = \langle a_2 \wedge a_1, a_3 \wedge a_1, a_4 \wedge a_1, a_3 \wedge a_2, a_4 \wedge a_2, a_4 \wedge a_3\rangle,$$

and $\mathcal{M}(G) = \langle a_3 \wedge a_2, a_4 \wedge a_2, a_4 \wedge a_3\rangle$. It follows from the epimorphism $G \wedge G \longrightarrow G'$ that $a_2 \wedge a_1$, $a_3 \wedge a_1$ and $a_4 \wedge a_1$ are non-trivial independent generators such that p divides their orders. On the other hand $a_2^p \wedge a_1 = (a_2 \wedge a_1)^p = 1$, $a_3^p \wedge a_1 = (a_3 \wedge a_1)^p = 1$, and $a_4^p \wedge a_1 = (a_4 \wedge a_1)^p = 1$. Hence $|a_2 \wedge a_1| = |a_3 \wedge a_1| = |a_4 \wedge a_1| = p$. Moreover it is shown in [11, proof of Theorem B] that

$$(G \wedge G)' = \langle (a_2 \wedge a_3)^{(\alpha-1)(c_3-1)}, (a_2 \wedge a_4)^{(\alpha-1)(c_4-1)}, \\ (a_3 \wedge a_4)^{(c_3-1)(c_4-1)} (a_2 \wedge a_3)^{(c_4-1)\binom{c_3-1}{2}}\rangle,$$

and consequently $(G \wedge G)' = \mathcal{M}(G)$. Therefore by applying [10] we conclude that $G \wedge G \cong \Phi_2(1^4)$. For the case $\mathcal{M}(G) \cong C_p^2$, as $(G \wedge$

$G)/Z(G\wedge G)$ is abelian group of order p^3 and by using [10], likewise above we deduce that $G\wedge G\cong\Phi_4(1^5)$. Finally, if $\mathcal{M}(G)\cong C_p^3$, the result follows by a same method.

Assume G is of type (25). Then from [11, Proposition 3.2] we have $\mathcal{M}(G)\cong 1$, C_p, C_p^2 or C_p^3. By a same argument as for the group (18) we can show that $G\wedge G\cong C_p^3$, $\Phi_2(1^4)$, $\Phi_4(1^5)$ or $\Phi_{11}(1^6)$, respectively (for more details see the proof of Theorem B in [11]). Note that the polycyclic presentation of G is as follows:

$$\langle a_1,a_2,a_3,a_4 \mid a_1^q=a_2^p=a_3^p=a_4^p=1, a_2^{a_1}=a_3, a_3^{a_1}=a_4,\\ a_4^{a_1}=a_2a_3^sa_4^t, a_3^{a_2}=a_3, a_4^{a_2}=a_4, a_4^{a_3}=a_4\rangle,$$

where $s=-\lambda^{-1}-\lambda^{-p}-\lambda^{-p^2} \bmod p$ and $t=\lambda+\lambda^p+\lambda^{p^2} \bmod p$.

For the group (22) the result follows similarly. Here the polycyclic presentation of G is as follows:

$$\langle a_1,a_2,a_3,a_4 \mid a_1^q=a_2^p=a_3^p=a_4^p=1, a_2^{a_1}=a_2^{c_3}, a_3^{a_1}=a_3^{c_2},\\ a_4^{a_1}=a_4^{\alpha}, a_3^{a_2}=a_3a_4, a_4^{a_2}=a_4, a_4^{a_3}=a_4\rangle,$$

in which $c_3=\alpha^{q+1-x} \bmod p$ and $c_2=\alpha^x \bmod p$. □

Now we are ready to compute the exterior centers of all the groups in Theorem 1, and then determine those that are capable.

Theorem 2. *Let G be a nonabelian group of order p^3q, where p and q are distinct prime numbers and $p>2$. Then*

$$Z^{\wedge}(G)\cong\begin{cases} C_{pq}, & \text{if } G \text{ is of type (1)}\\ C_q, & \text{if } G \text{ is of type (2)}\\ C_p, & \text{if } G \text{ is any group of types (4), (5), (7), (9), (14), (17), (20), (23) or (24)}\\ C_{p^2}, & \text{if } G \text{ is of type (3) or (13)}\end{cases}$$

In other cases, $Z^{\wedge}(G)=1$.

Proof. Note that if $\mathcal{M}(G)=1$, then $Z^{\wedge}(G)=Z(G)$. So for groups of types (1), (3), (7), (9), (11), (12), (13), (14), (19), (20) or (24) the result follows easily. Assume G is of type (2). Then $Z(G)=\langle a,d\rangle$. By Proposition 2, $G\wedge G=\langle a\wedge b, a\wedge c, b\wedge c\rangle$, which implies that $a\notin Z^{\wedge}(G)$, whence $Z^{\wedge}(G)=\langle d\rangle\cong C_q$. If G is of type (4), then $Z(G)=\langle a\rangle\cong C_{p^2}$. We claim that $G\wedge G=\langle a\wedge b, b\wedge d\rangle$. First, it can be shown that $1=a\wedge b^p=(a\wedge b)^p$ and $1=b\wedge d^q=(d^{\alpha-1}\wedge d)^q(b\wedge d)^q=(b\wedge d)^q$. Note that $d^i\wedge d^j=1$ modulo $G\wedge G$ for any integers i,j. On the other hand, the epimorphisms $G\wedge G\longrightarrow G^{ab}\wedge G^{ab}$ and $G\wedge G\longrightarrow G'$ imply that $a\wedge b$ and $b\wedge d$ are non-trivial generators, respectively. Now, since p and q are coprime, it follows that these generators are independent, as desired. It shows that $a\notin Z^\wedge(G)$. Moreover, by induction on any integer n, we can prove that $a^n\wedge b=a\wedge b^n$. Hence $a^p\wedge b=1$. Also $a^p\wedge d=1$, because $a^p\wedge d$ has order dividing p and q, which implies that $Z^\wedge(G)=\langle a^p\rangle\cong C_p$. For group (5), one can readily show that $G\wedge G=\langle a\wedge b, a\wedge d\rangle$ and $Z^\wedge(G)=\langle a^p\rangle\cong C_p$ as same as above.

If G is of type (6), then $Z(G)=\langle a,b\rangle$. Also by Proposition 2 we have

$$G\wedge G=\langle a\wedge b, a\wedge c, b\wedge c, c\wedge d\rangle,$$

which implies that $a,b\notin Z^\wedge(G)$. Thus $Z^\wedge(G)=1$. For the type (16), the result follows by a same method. The exterior square of group (10) is generated by $a\wedge b$ and $a\wedge d$. So for any group of types (8) or (10), as $a\wedge b\neq 1$ then $Z^\wedge(G)=1$. If G is of type (17), then $Z(G)=\langle a\rangle\cong C_p$. When $\mathcal{M}(G)=1$, we have $Z^\wedge(G)=Z(G)$. If $\mathcal{M}(G)\cong C_p$, it follows from the generators set of $G\wedge G$ described in proof of Proposition 3 that $Z^\wedge(G)=\langle a\rangle$. For the group (23) the result follows similarly.

For any group of types (15), (18), (21), (22) and (25), it is clear that $Z(G)=1$. Hence $Z^\wedge(G)=1$. □

Corollary 1. *Let G be a nonabelian group of order p^3q, where p and q are distinct prime numbers and $p>2$. Then G is capable if and only if G is any group of types (6), (8), (10), (11), (12), (15), (16), (18), (19), (21), (22) or (25).*

We now consider the case of unicentral groups. Recall that the unicentral groups lie at the opposite extreme as the capable groups: a group G is unicentral if and only if the smallest subgroup N of $Z(G)$ such that G/N is capable is $N=Z(G)$ [2]; that is, if and only if $Z^\wedge(G)=Z(G)$.

Corollary 2. *Let G be a nonabelian group of order p^3q, where p and q are distinct prime numbers and $p>2$. Then G is unicentral if and only if G is any group of types (1), (3), (7), (9), (11), (12), (13), (14), (15), (17), (18), (19), (20), (21), (22), (23), (24) or (25).*

Remark 1. There are three abelian groups of order p^3q. It follows from [2, Proposition 7.3] (see also [14, Proposition 2.6]) that none of them are capable. Also the cyclic group C_{p^3q} is the only abelian unicentral group of order p^3q.

Remark 2. The final summary table of groups determined by Western [17] has a group missing in the case that $q \equiv 1 \pmod p$. The following missing group appears in Western's analysis in Section 13:

$$G = \langle a, b, d \mid a^{p^2} = b^p = d^q = 1,\ b^{-1}ab = a^{p+1},\ a^{-1}da = d^{\alpha},\ bd = db \rangle \tag{26}$$

where α is any primitive root of $\alpha^p \equiv 1 \pmod q$. It follows from [11] that $G \wedge G \cong G' \cong C_{pq}$. One could readily seen that $Z^{\wedge}(G) = Z(G)$ which implies that G is unicentral.

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Contact information

O. Kalteh,
S. Hadi Jafari

Department of Mathematics, Mashhad Branch,
Islamic Azad University, Mashhad, Iran.
E-Mail(s): kaltehstudent@yahoo.com,
s.hadi_jafari@yahoo.com

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