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Categorical properties of some algorithms of differentiation for equipped posets

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Abstract. In this paper it is proved that the algorithms of differentiation VIII-X (introduced by A.G. Zavadskij to classify equipped posets of tame representation type) induce categorical equivalences between some quotient categories, in particular, an algorithm A_z is introduced to build equipped posets with a pair of points (a, b) suitable for differentiation VII such that the subset of strong points related with the weak point a is not empty.

Introduction

The theory of representation of partially ordered sets or posets was introduced and developed by Nazarova, Roiter and their students in the 1970s in Kiev. According to Simson such theory allowed to Nazarova and Roiter to give a solution to the second Brauer-Thrall conjecture [14, 20]. We recall that one of the main goals of the theory of representation of posets consists of giving a complete description of the indecomposable objects and irreducible morphisms of the category of representations rep $\mathscr{P}$ over a field k of a given poset $\mathscr{P}$.

Perhaps the most useful tool to classify posets are the algorithms of differentiation [13, 20]. For instance, Nazarova and Roiter introduced an algorithm known as the algorithm of differentiation with respect to a maximal point which allowed to Kleiner in 1972 to obtain a classification

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of posets of finite representation type [12]. The categorical properties of such an algorithm were given by Gabriel in 1973 [11]. Soon afterwards between 1974 and 1977, Zavadskij defined the more general algorithm I (also named DI) with respect to a suitable pair of points, this algorithm was used in 1981 by Nazarova and Zavadskij in order to give a criterion for the classification of posets of finite growth representation type [16, 17, 22]. Actually, several years later Zavadskij himself described the structure of the Auslander-Reiten quiver of this kind of posets, to do that, it was established that such an algorithm I together with a completion algorithm are in fact categorical equivalences between some quotient categories [23].

The theory of representation of posets with additional structures was developed in the 1980s and 1990s, for instance, posets endowed with an equivalence relation in particular with an involution were introduced and classified by Nazarova and Roiter in [15], and Bondarenko and Zavadskij in [1] whereas the theory of representation of equipped posets was introduced by Zabarilo and Zavadskij in [30] and [31]. Posets with involution were classified by using DI and some algorithms of differentiation named DII-DV together with some additional (more simple) algorithms, such collection of algorithms is currently called the apparatus of differentiation DI-DV [9].

A tameness criterion for equipped posets with and without involution was given by Zavadskij. It was obtained by using both the apparatus of differentiation DI-DV and some additional differentiations VII-XVII [24–26]. In particular, algorithms of differentiation I, VII VIII and IX allowed to classify equipped posets of finite growth representation type. In fact, according to Zavadskij [25] the use of algorithms of differentiation makes of the classification problems for posets a fairly easy task based only on combinatorial methods.

Since algorithms of differentiation are additive functors it is necessary to establish the behavior of the objects and morphisms involved in the process, in this direction Gabriel proved that the algorithm of differentiation with respect to a maximal point induces a categorical equivalence and the same was proved by Zavadskij, Cañadas et al for the algorithms of differentiation I-V, and VII, actually advances on the subject have been proposed for algorithms of differentiation VIII and IX [2–4, 6, 7, 9, 11, 23].

We recall that according to Zavadskij the *main problem* regarding the theory of the algorithms of differentiation consists of *proving that they induce categorical equivalences between appropriated quotient categories* [5]. In this paper, we address this problem by proving that algorithms of differentiation A_z (introduced in this paper by the authors), VIII, IX and

X satisfy this property. Actually, we will establish the following theorem 1 bearing in mind that when Zavadskij introduced algorithms VII-XVII for equipped posets he was focused on the behavior of the objects under the action of functors of type D_S^{J}, in fact, he proved the denseness property of such algorithms without pay attention to its faithfulness and fullness properties [25, 28].

Theorem 1. *Let $(\mathscr{P}, \Phi)$ be an equipped poset endowed with an involution $*$ and with a set of points S, J-suitable. Then if J is one of the symbols A_z, VIII, IX, X the corresponding differentiation functor $' = D_S^{\mathrm{J}}\colon \operatorname{rep}\mathscr{P} \longrightarrow \operatorname{rep}\mathscr{P}'_S$ defined by one of the formulas (19), (20), (21), (27), (31) induces an equivalence between quotient categories:*

$$\operatorname{rep}\mathscr{P}/\mathscr{I} \xrightarrow{\sim} \operatorname{rep}\mathscr{P}'_S/\mathscr{I}'$$

in particular the functor D_S^{J} induces mutually inverse bijections between indecomposable representations of the form

$$\operatorname{Ind}\operatorname{rep}\mathscr{P} \setminus [\mathscr{I}(I)] \rightleftarrows \operatorname{Ind}\operatorname{rep}\overline{\mathscr{P}}'_S = \operatorname{Ind}\operatorname{rep}\mathscr{P}'_S \setminus [\mathscr{I}'(I)].$$

In Theorem 1 we let $[\mathscr{I}]$ $([\mathscr{I}'(I)])$ denote a suitable ideal (collection of isomorphic classes of indecomposable representations) defined by the action of the corresponding functor. Generally such ideal consists of morphisms that pass through sums of some suitable indecomposable representations in $[\mathscr{I}(I)]$ and $[\mathscr{I}'(I)]$. Moreover, for two representations or representatives $U, V \in [\mathscr{I}(I)]$ it holds that $U' = V' \in [\mathscr{I}'(I)]$. Besides, it is considered that the involution $*$ is trivial (i.e., $x^* = x$ for all $x \in \mathscr{P}$) for each of the differentiations A_z, VIII and IX.

The following lemma proved by Zavadskij for differentiations VII-XVII in [25, 28] establishes that each of these functors is dense. In this case, Y denotes a suitable representation of the category of representations of an equipped poset with a set of points S suitable for differentiation J, $\mathscr{P}'_S$ is a corresponding derived poset and $\overline{\mathscr{P}}'_S$ stands for the derivative of a completed poset with an additional strong relation.

Lemma 1. *For each representation $W \in \operatorname{rep}\overline{\mathscr{P}}'_S$, there exists a representation $W^{\uparrow} \in \operatorname{rep}\mathscr{P}$ such that $(W^{\uparrow})' \simeq W \oplus Y^m$, for some $m \geqslant 0$.*

This paper is organized as follows; in section 1 basic notation and definitions regarding the category of representations of posets with additional structures are included. In section 2, we recall some categorical properties of the algorithms of differentiation, I, completion, and VII. We prove the main result by describing in section 3 the algorithms of differentiation A_z, VIII-X.

1. Preliminaries

In this section, for the sake of better understanding, we introduce main notation and definitions regarding equipped posets and its category of representations [2–4, 6, 7, 25, 26, 30, 31].

1.1. Category of representations of posets with additional structures

In this section, we recall the definition of equipped posets and posets with involution and their corresponding category of representations as Zavadskij et al have described in [3, 4, 6, 7, 25, 26]. Worth noting that although equipped posets were introduced and classified in [25, 26, 30, 31] over the pair of fields $(\mathbb{R}, \mathbb{C})$, in this paper, we consider notation and definitions adopted by Zavadskij and Rodriguez in [19] where representations of equipped posets are defined over a pair of fields (F, G) with $G = F(\boldsymbol{\xi})$ a quadratic extension of F associated with a minimal polynomial of the form $t^2 + \alpha t + \beta$, $\alpha, \beta \in F$, $\beta \neq 0$ and $\boldsymbol{\xi} \in G$ such that

$$\boldsymbol{\xi}^2 + \alpha\boldsymbol{\xi} + \beta = 0. \tag{1}$$

Equipped posets. A poset $(\mathscr{P}, \leqslant)$ is called *equipped* if all the order relations between its points $x \leqslant y$ are separated into strong (denoted $x \trianglelefteq y$) and weak (denoted $x \preceq y$) in such a way that

$$x \leqslant y \trianglelefteq z \quad \text{or} \quad x \trianglelefteq y \leqslant z \quad \text{implies} \quad x \trianglelefteq z, \tag{2}$$

i.e., a composition of a strong relation with any other relation is strong.

In general relations $\trianglelefteq$ and $\preceq$ are not order relations. These relations are antisymmetric but not reflexive. In particular $\preceq$ is not reflexive (meanwhile $\trianglelefteq$ is transitive) [19].

We let $x \leqslant y$ denote an arbitrary relation in an equipped poset $(\mathscr{P}, \leqslant)$. The order $\leqslant$ on an equipped poset $\mathscr{P}$ gives rise to the relations $\prec$ and $\lhd$ of *strict inequality*: $x \prec y$ (respectively, $x \lhd y$) in $\mathscr{P}$ if and only if $x \preceq y$ (respectively, $x \trianglelefteq y$) and $x \neq y$.

A point $x \in \mathscr{P}$ is called *strong* (*weak*) if $x \trianglelefteq x$ (respectively, $x \preceq x$). These points are denoted $\circ$ (respectively, $\otimes$) in diagrams. We also denote $\mathscr{P}^{\circ} \subseteq \mathscr{P}$ (respectively, $\mathscr{P}^{\otimes} \subseteq \mathscr{P}$) the subset of strong points (respectively, weak points) of $\mathscr{P}$. If $\mathscr{P}^{\otimes} = \varnothing$ then the equipment is *trivial* and the poset $\mathscr{P}$ is ordinary.

Remark 1. Note that if $x \preceq y$ in an equipped poset $(\mathscr{P}, \leqslant)$ and there exists $t \in \mathscr{P}$ such that $x \leqslant t \leqslant y$ then $x, y \in \mathscr{P}^{\otimes}$, $x \preceq t$ and $t \preceq y$.

Otherwise, if $x \trianglelefteq t$ or $t \trianglelefteq y$ then by definition it is obtained the contradiction $x \trianglelefteq y$.

If $\mathscr{P}$ is an equipped poset and $a \in \mathscr{P}$ then the subsets of $\mathscr{P}$ denoted $a^{\vee}$, $a_{\wedge}$, $a^{\triangledown}$, $a_{\vartriangle}$, $a^{\blacktriangledown}$, $a_{\blacktriangle}$, $a^{\curlyvee}$ and $a_{\curlywedge}$ are defined in such a way that:

$$\begin{aligned}
a^{\vee} &= \{x \in \mathscr{P} \mid a \leqslant x\}, \quad a_{\wedge} = \{x \in \mathscr{P} \mid x \leqslant a\},\\
a^{\triangledown} &= \{x \in \mathscr{P} \mid a \trianglelefteq x\}, \quad a_{\vartriangle} = \{x \in \mathscr{P} \mid x \trianglelefteq a\},\\
a^{\blacktriangledown} &= a^{\vee} \setminus a, \quad a_{\blacktriangle} = a_{\wedge} \setminus a,\\
a^{\curlyvee} &= \{x \in \mathscr{P} \mid a \preceq x\}, \quad a_{\curlywedge} = \{x \in \mathscr{P} \mid x \preceq a\}.
\end{aligned}$$

Subset $a^{\vee}$ ($a_{\wedge}$) is called the *ordinary upper (lower) cone*, associated with the point $a \in \mathscr{P}$ and subset $a^{\triangledown}$ ($a_{\vartriangle}$) is called the *strong upper (lower) cone* associated with the point $a \in \mathscr{P}$. Whereas subsets $a^{\blacktriangledown}$ and $a_{\blacktriangle}$ are called *truncated cones* (upper and lower) associated with the point $a \in \mathscr{P}$.

In general, subsets $a^{\curlyvee}$ and $a_{\curlywedge}$ are not cones. Note that, if $x \in \mathscr{P}^{\circ}$ then $x^{\curlyvee} = x_{\curlywedge} = \varnothing$.

For an equipped poset $(\mathscr{P}, \leqslant)$ and $A \subset \mathscr{P}$, we define the subsets, $A^{\triangledown}$, $A^{\curlyvee}$ and $A^{\vee}$ in such a way that

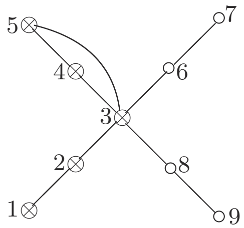
$$A^{\triangledown} = \bigcup_{a \in A} a^{\triangledown}, \qquad A^{\curlyvee} = \bigcup_{a \in A} a^{\curlyvee}, \qquad A^{\vee} = \bigcup_{a \in A} a^{\vee}$$

Subsets $A_{\vartriangle}$, $A_{\curlywedge}$ and $A_{\wedge}$ are defined in the same way.

If $\mathscr{P}$ is an equipped poset then a *chain* $C = \{c_i \in \mathscr{P} \mid 1 \leqslant i \leqslant n, c_{i-1} < c_i \text{ if } i \geqslant 2\} \subseteq \mathscr{P}$ is a *weak chain* if and only if $c_{i-1} \prec c_i$ for each $i \geqslant 2$. If $c_1 \prec c_n$ then we say that C is a *completely weak chain*. Moreover, a subset $X \subset \mathscr{P}$ is *completely weak* if $X = X^{\otimes}$ and weak relations are the only relations between points of X. Often, we let $\{c_1 \prec c_2 \prec \cdots \prec c_n\}$ denote a weak chain which consists of points $c_1, c_2, \ldots, c_n$. An ordinary chain C is denoted in the same way (by using the corresponding symbol $<$).

The diagram of an equipped poset $(\mathscr{P}, \leqslant)$ may be obtained via its Hasse diagram (with strong ($\circ$) and weak points ($\otimes$)). In this case, a new line is added to the line connecting two points $x, y \in \mathscr{P}$ with $x \lhd y$ if and only if such relation cannot be deduced of any other relations in $\mathscr{P}$. Figure 1 shows an example of this kind of diagrams.

In this case if $A = \{4, 6\}$, then $A^{\triangledown} = \{6, 7\}$, $A^{\curlyvee} = \{4, 5\}$, $A^{\vee} = \{4, 5, 6, 7\}$, $A_{\vartriangle} = \{1, 2, 3, 6, 8, 9\}$, $A_{\wedge} = \{1, 2, 3, 4, 6, 8, 9\}$ and $A_{\curlywedge} = \{1, 2, 3, 4\}$. Note that $A \neq A^{\otimes}$, subsets $C_1 = \{9 < 8 < 3 < 4 < 5\}$



$1^{\curlyvee} = \{1,2,3,4\}$	$1^{\triangledown} = \{5,6,7\}$
$2^{\curlyvee} = \{2,3,4\}$	$2^{\triangledown} = \{5,6,7\}$
$3^{\curlyvee} = \{3,4\}$	$3^{\triangledown} = \{5,6,7\}$
$4^{\curlyvee} = \{4,5\}$	$4^{\triangledown} = \varnothing$
$5^{\curlyvee} = \{5\}$	$5^{\triangledown} = \varnothing$
$6^{\curlyvee} = \varnothing$	$6^{\triangledown} = \{6,7\}$
$7^{\curlyvee} = \varnothing$	$7^{\triangledown} = \{7\}$
$8^{\curlyvee} = \varnothing$	$8^{\triangledown} = \{3,4,5,6,7,8\}$
$9^{\curlyvee} = \varnothing$	$9^{\triangledown} = \{3,4,5,6,7,8,9\}$

Figure 1. The diagram of an equipped poset and some of its subsets.

and $C_2 = \{1 \prec 2 \prec 3 \prec 4\}$ constitute a chain and a completely weak chain, respectively.

For an equipped poset $\mathscr{P}$ and $A, B \subset \mathscr{P}$ we write $A < B$ if $a < b$ for each $a \in A$ and $b \in B$. Notations $A \prec B$ and $A \lhd B$ are assumed in the same way.

Equipped posets endowed with an involution. An *equipped poset with involution* is an equipped poset $(\mathscr{P}, \leqslant, \preceq, \unlhd)$ with an involution $*$ satisfying the following two additional conditions:

(i) on the set of all points $\mathscr{P}$, it is given an involution $* : \mathscr{P} \longrightarrow \mathscr{P}$ which preserves strong and weak points and independent of the relation $\leqslant$. Hence, strong points are divided into *small* ($x = x^*$) and *big* ($x \neq x^*$), and weak points are partitioned into *weak* ($x = x^*$) and *biweak* ($x \neq x^*$);

(ii) to each biweak point x it is assigned the number $g(x) = g(x^*) \in \{\pm 1\}$ called its *genus* (or *genus of the pairs* x, x^*).

In the case $x \neq x^*$, we called the points x and x^* *equivalents* and write $x \sim x^*$. The involution $*$ is said to be *primitive* if it leaves fixed all weak points (i.e. there are no biweak points).

In diagrams of equipped posets with involution, symbols $\circ$, $\bullet$, $\otimes$, $\odot$ depict small, big, weak and biweak points, respectively. All order relations with a participation of at least one strong point, as well as all weak relations between weak points, are pictured by a single line. But all strong relations between weak points, which are not consequences of some other relations, are pictured by a double line (or by an additional line) [25].

If some group of points is encircled by a contour connected by some (single or double) line with some other points, it means that all points located inside the contour have the same order relations with the mentioned other points (determined by the type of the line).

Note that in Figure 2, $a \sim a^*$, $c \sim c^*$; $q = q^*$; $b = b^*$; $c^* \unlhd b$, $a \unlhd a^* \unlhd q$, $a \unlhd c \preceq b$, $a \unlhd A$, $B \unlhd b$; $b, c, c^*, q \in \mathscr{P}^{\otimes}$.

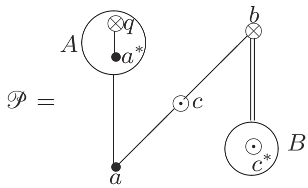


FIGURE 2. The diagram of an equipped poset with involution.

1.2. Complexification

In this section, we give definitions of complexification and reellification of a vector space and its respective extension to complexification of linear transformations [2, 3, 21]. Some particular subspaces whose properties are useful in the theory of representation of equipped posets are described as well [25].

Let $F \subset G$ be an arbitrary quadratic field extension with $G = F(\boldsymbol{\xi})$ for some fixed element $\boldsymbol{\xi} \in G$. Then each element $x \in G$ can be written uniquely in the form $\alpha + \boldsymbol{\xi}\beta$ with $\alpha, \beta \in F$ in this case (analogously to the case $(F, G) = (\mathbb{R}, \mathbb{C})$) α is called the *real* part of x and β is the corresponding *imaginary* part of x.

Complexification of $\boldsymbol{F}$-spaces. The *complexification* of a real vector space U_0 is the complex vector space $\widetilde{U_0} = U_0 \times U_0 = U_0^2$ in which the addition $+ : \widetilde{U_0} \times \widetilde{U_0} \longrightarrow \widetilde{U_0}$ and the scalar multiplication $\cdot : \mathbb{C} \times \widetilde{U_0} \longrightarrow \widetilde{U_0}$ are defined by

$$\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} v + v' \\ w + w' \end{pmatrix} \quad \text{and} \quad (a + \boldsymbol{i}b) \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} av - bw \\ bv + aw \end{pmatrix}. \tag{3}$$

If we identify the space U_0 with the real subspace $U_0 \times \{0\}$ of $\widetilde{U_0}$ and write simply v instead of $(v, 0)^t$ then an arbitrary element $z \in \widetilde{U_0}$, may be written in the following form

$$z = \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} + \boldsymbol{i} \begin{pmatrix} w \\ 0 \end{pmatrix} = v + \boldsymbol{i}w, \quad v, w \in U_0.$$

Therefore the complexification of a real vector space U_0 has the form $\widetilde{U_0} = U_0 + \boldsymbol{i}U_0$. Thus, if $W \subset \widetilde{U_0}$ is a $\mathbb{R}$-subspace of $\widetilde{U_0}$ then the real part of W denoted $\operatorname{Re} W$ and its corresponding imaginary part denoted $\operatorname{Im} W$ are defined in such a way that if $W = \mathbb{R}\{x_t + \boldsymbol{i}y_t \mid x_t, y_t \in U_0, t \in A\} \subset \widetilde{U_0}$ for a fixed basis then

$$\operatorname{Re} W = \operatorname{span}\{x_t \mid t \in A\} \subset U_0, \qquad \operatorname{Im} W = \operatorname{span}\{y_t \mid t \in A\} \subset U_0.$$

In this case, if k is a field and $T = \{e_1, e_2, \ldots, e_n\}$ is a set of generators of a k-vector space V then $k\{e_1, e_2, \ldots, e_n\}$ denotes the subspace generated by T.

In [21] it is proved that every basis in a real vector space V is also a basis (over $\mathbb{C}$) of the complex vector space $\widetilde{V}$ consequently $\dim_{\mathbb{C}} \widetilde{V} = \dim_{\mathbb{R}} V$.

If W is a complex vectorial space then the *reellification* $W_{\mathbb{R}}$ of W is the real vector space which is obtained from W by restricting the scalar multiplication to $\mathbb{R} \times W$, (Sloppily, this is just W considered as a real vector space). Thus, if $\{w_t \mid t \in A\}$ is a basis of $W_{\mathbb{R}}$ over $\mathbb{C}$ then

$$\{w_t \mid t \in A\} \cup \{\boldsymbol{i} w_t \mid t \in A\}$$

is a basis of $W_{\mathbb{R}}$ over $\mathbb{R}$ and $\dim_{\mathbb{R}} W_{\mathbb{R}} = 2 \dim_{\mathbb{C}} W$ [21].

A real subspace V of $W_{\mathbb{R}}$ is called a *real form* of W if $W = \widetilde{V} = V + \boldsymbol{i}V$, therefore $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} W$. In [21] it is also proved that if V is a $\mathbb{R}$-space then $(\widetilde{V})_{\mathbb{R}} \simeq V \oplus V$. Thus, if W is a $\mathbb{C}$-subspace of $\widetilde{U_0}$ with $W = \mathbb{C}\{x_t + \boldsymbol{i} y_t \mid x_t, y_t \in U_0, t \in A\} \subset \widetilde{U_0}$ then

$W_{\mathbb{R}} = \mathbb{R}\{x_t + \boldsymbol{i} y_t, -y_t + \boldsymbol{i} x_t \mid x_t, y_t \in U_0, t \in A\}$, therefore $\operatorname{Re} W_{\mathbb{R}} = \operatorname{Im} W_{\mathbb{R}} = \operatorname{span}\{x_t \mid t \in A\} + \operatorname{span}\{y_t \mid t \in A\} = \operatorname{span}\{x_t, y_t \mid t \in A\}$.

The complexification of a real vector space may be generalized to the case (F, G) where $G = F(\boldsymbol{\xi})$ is a quadratic extension of F. In this case, we assume that $\boldsymbol{\xi}$ is a root of the minimal polynomial $t^2 + \alpha t + \beta$, $\beta \neq 0$, $(\alpha, \beta \in F)$. In particular if U_0 is a F-space then the corresponding complexification is the G-vector space also denoted $U_0^2 = \widetilde{U_0}$ with a scalar product of the form (see identity (1)):

$$(a + \boldsymbol{\xi} b) \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} av - \beta bw \\ bv + (a - \alpha b)w \end{pmatrix}, \quad v, w \in U_0. \tag{4}$$

As in the case $(\mathbb{R}, \mathbb{C})$, we write $U_0^2 = U_0 + \boldsymbol{\xi} U_0 = \widetilde{U_0}$.

To each G-subspace W of $\widetilde{U_0}$ it is possible to associate the following F-subspaces of U_0, $W^+ = \operatorname{Re} W_F = \operatorname{Im} W_F$ and $W^- = \operatorname{span}\{x \in U_0 \mid (x, 0)^t \in W\} \subset W^+$.

$$\widetilde{W^+} = F(W) \text{ is called the } F\text{-hull of } W \text{ such that } W \subset F(W). \tag{5}$$

If Y is a F-subspace of U_0 and $X = \widetilde{Y}$ then $X^+ = X^- = Y$. Therefore, Y is a F-form of X. For example, if we consider $F = \mathbb{R}$, $G = \mathbb{C}$ and $U_0 = \mathbb{R}^2 = \mathbb{R}\{e_1, e_2\}$ then $\widetilde{U_0} = \mathbb{C}^2$, in this case, we can assume $\boldsymbol{\xi} = \boldsymbol{i}$. Thus, if W is a $\mathbb{C}$-subspace of $\mathbb{C}^2$ such that $W = \mathbb{C}\{e_1 + \boldsymbol{i} e_2\}$ then

$$W^+ = \mathbb{R}^2 \quad \text{and} \quad W^- = 0.$$

If $\mathbb{R}^3 = \mathbb{R}\{e_1, e_2, e_3\}$, and $W = \mathbb{C}\{e_1, e_2 + \boldsymbol{i}e_3\} \subset \mathbb{C}^3 = \widetilde{\mathbb{R}^3}$ then

$$W^+ = \mathbb{R}^3, \qquad F(W) = \widetilde{W^+} = \mathbb{C}^3 \quad \text{and} \quad W^- = \mathbb{R}\{e_1\}.$$

Remark 2. Any G-subspace W of $\widetilde{U_0}$ can be written as a direct sum of G-subspaces, $W = \widetilde{W^-} \oplus H$ where H is a complement of $\widetilde{W^-}$ in W. Therefore, $H^+ \simeq W^+/W^-$. If $X \subset \widetilde{U_0}$ is a G-subspace with a F-hull such that $F(X) = X$ then we say that X is a *strong* space. Therefore any G-subspace $X \subset \widetilde{U_0}$ always has a strong direct summand of the form $\widetilde{X^-}$.

1.3. Representation of equipped posets

In this section, we recall the definition given by Zavadskij et al. of the category of representations of equipped posets with and without an involution defined on its set of points. It should be noted that Zavadskij gave a generalization of equipped posets over a pair of fields (F, G), where G is a Galois extension of the ground field F [29].

A *representation* of an equipped poset over the pair (F, G) is a system of subspaces of the form

$$U = (U_0; U_x \mid x \in \mathscr{P}), \tag{6}$$

where U_0 is a finite dimensional F-space; and for each $x \in \mathscr{P}$, U_x is a G-subspace of $\widetilde{U_0}$, such that, if $x \preceq y$ then $U_x \subset U_y$, and if $x \trianglelefteq y$ then $F(U_x) \subset U_y$ (see (5)).

We let rep $\mathscr{P}$ denote the category whose objects are the representations of an equipped poset $\mathscr{P}$ over a pair of fields (F, G). In this case, a morphism $\varphi : (U_0; U_x \mid x \in \mathscr{P}) \longrightarrow (V_0; V_x \mid x \in \mathscr{P})$, between two representations U and V is a F-linear map $\varphi : U_0 \longrightarrow V_0$ such that $\widetilde{\varphi}(U_x) \subset V_x$, for each $x \in \mathscr{P}$, where $\widetilde{\varphi} : \widetilde{U_0} \longrightarrow \widetilde{V_0}$ is the complexification of φ ($\widetilde{\varphi} = \varphi + \boldsymbol{\xi}\varphi$). The composition between morphisms of rep $\mathscr{P}$ is defined in a natural way.

Two representations $U, V \in$ rep $\mathscr{P}$ are said to be *isomorphic* if and only if there exists an F-isomorphism $\varphi : U_0 \longrightarrow V_0$ such that $\widetilde{\varphi}(U_x) = V_x$, for each $x \in \mathscr{P}$

The sum $U \oplus V \in$ rep $\mathscr{P}$ is defined as in the classical way, that is, the sum $U \oplus V$ of two representations of a given equipped poset $\mathscr{P}$ is defined in such a way that $U \oplus V = (U_0 \oplus V_0; U_x \oplus V_x \mid x \in \mathscr{P})$. Therefore, rep $\mathscr{P}$ is a Krull-Schmidt category. A representation $U \in$ rep $\mathscr{P}$ is *indecomposable* if $U \neq 0$ and there is not a direct sum decomposition of U into two non-zero representations. Often, we let Ind $\mathscr{P}$ denote a set of representatives of the isomorphism classes of all the indecomposable objects of a category rep $\mathscr{P}$.

Let $\mathscr{P}$ be an equipped poset and $U, V \in \operatorname{rep}\mathscr{P}$. Then U is a *subrepresentation* of V if and only if the spaces U_0, V_0, U_x and V_x satisfy the inclusions $U_0 \subset V_0$ and $U_x \subset V_x$, for each $x \in \mathscr{P}$.

For each $x \in \mathscr{P}$, we let $\underline{U_x}$ denote the *radical subspace* of U_x, that is, $\underline{U_x} = \sum_{z \lhd x} F(U_z) + \sum_{z \prec x} U_z$.

Let $\mathscr{P}$ be an equipped poset. The *dimension* of a representation $U \in \operatorname{rep}\mathscr{P}$ is the vector $d = \underline{\dim}\, U = (d_0; d_x \mid x \in \mathscr{P})$, where $d_0 = \dim_F U_0$ and $d_x = \dim_G U_x/\underline{U_x}$. A representation $U \in \operatorname{rep}\mathscr{P}$ is *sincere* if $d_0 \neq 0$ and $d_x \neq 0$, for each $x \in \mathscr{P}$. In other words, the vector d of a sincere representation U has not null coordinates.

Let $X \subset \mathscr{P}$ and $U \in \operatorname{rep}\mathscr{P}$. The subspaces of U_0, denoted respectively by U_X, U_X^+, $\overset{\frown}{U}_X$ and $(\overset{\frown}{U}_X)^-$, are defined as follows:

$$U_X = \sum_{x \in X} U_x, \quad U_X^+ = \sum_{x \in X} U_x^+, \quad \overset{\frown}{U}_X = \bigcap_{x \in X} U_x, \quad (\overset{\frown}{U}_X)^- = \bigcap_{x \in X} U_x^-.$$

Note that $U_\varnothing^+ = 0$, $\overset{\frown}{U}_\varnothing = U_0$, and if $x, y \in \mathscr{P}$ with $x \lhd y$ then $U_x^+ \subset U_y^-$.

Let $\mathscr{P}$ be an equipped poset with involution $*$ which naturally induces an equivalence relation on the points of $\mathscr{P}$, let Φ be the set of all equivalence classes on $\mathscr{P}$ respect to such an involution. Then classes $\kappa \in \Phi$ consist either of one or two points, in the second case it holds that $x \neq x^*$ and $\kappa = (x, x^*)$.

Now, we recall the definition of a representation of an equipped poset with involution as given by Zavadskij in [25]. In this case, we let $(\mathscr{P}, \Phi)$ denote an equipped poset with an involution inducing a set of classes Φ over $\mathscr{P}$, if there is not doubt with the order $\leqslant$ and the corresponding equipment, we will write simply $\mathscr{P}$ to denote an equipped poset with involution.

Let $(\mathscr{P}, \Phi)$ be an equipped poset with involution. A *representation* U of $(\mathscr{P}, \Phi)$ is a system of vector spaces of the form

$$U = (U_0; U_\kappa \mid \kappa \in \Phi), \tag{7}$$

where U_0 is a finite dimensional F-vector space and $\widetilde{U_0}$ is its corresponding complexification, which is a G-vector space, such that,

if x is a small point	$\Longrightarrow$	$U_x \subset U_0$;
if x is a weak point	$\Longrightarrow$	$U_x \subset \widetilde{U_0}$;
if x is a big point	$\Longrightarrow$	$U_{(x,x^*)} \subset U_0 \oplus U_0$;
if x is a biweak point	$\Longrightarrow$	$U_{(x,x^*)} \subset \widetilde{U_0} \oplus \widetilde{U_0}$;
if $x < y$	$\Longrightarrow$	$U_x^+ \subset U_y^-$.

A morphism $\varphi : (U_0; U_\kappa \mid \kappa \in \Phi) \longrightarrow (V_0; V_\kappa \mid \kappa \in \Phi)$ between two representations U and V, is an F-linear map $\varphi : U_0 \longrightarrow V_0$ such that: $\varphi^\kappa(U_\kappa) \subset V_\kappa$, for each $\kappa \in \Phi$. In the natural sense, if $z = (z_1, z_2) \in U_\kappa$, then $\varphi^\kappa(z) = (\varphi(z_1), \varphi(z_2))$.

1.4. Examples of some indecomposable objects

In this section, we give some examples of indecomposable objects in the category rep$\mathscr{P}$, where $\mathscr{P}$ is an equipped poset. The matrix problem of these kind of posets and the matrix presentations of the indecomposable objects were defined by Zavadskij in [25].

Later on a subset $X \subset \mathscr{P}$ will be called *small* (*big, weak,...*) if all its points are small (big, weak,...). A subset consisting of two (three, four) mutually incomparable points is called a *dyad* (*triad, tetrad*).

we often write $a \parallel b$ to denote that points a, b in a poset $\mathscr{P}$ are incomparable and if there is not confusion hereinafter $\mathscr{P}$ denotes an equipped poset unless otherwise stated.

If $\mathscr{P}$ is an equipped poset and $A \subset \mathscr{P}$ then we denote by $P(A)$ an indecomposable representation of the equipped poset $\mathscr{P}$ such that $P(A) = P(\min A) = (P_0; P_x \mid x \in \mathscr{P})$, where $P_0 = F$ and $P_x = G$ if $x \in A^\vee$, $P_x = 0$ otherwise. In particular, $P(\varnothing) = (F; 0, \ldots, 0)$.

If $a, b \in \mathscr{P}$ with $a \parallel b$ then $P(a, b)$ denotes an indecomposable object such that $P(a, b) = (P_0; P_x \mid x \in \mathscr{P})$ with $P_0 = F$ and $P_x = G$ if $x \in a^\vee \cup b^\vee$, $P_x = 0$ otherwise.

If $a, b, p \in \mathscr{P}^\otimes$, $c \in \mathscr{P}^\circ$, with $a \prec b$, $a \parallel p$, $a \parallel c$ then $T(a)$, $T(a, b)$, $T(a, p)$ $G_1(a, c)$ and $G_2(a, c)$ denote indecomposable objects with matrix presentation of the following form ($T_0 = G_0 = F^2$ in each case):

$$\underset{T(a)}{\begin{array}{c} a \\ \begin{array}{|c|}\hline 1 \\ \boldsymbol{\xi} \\ \hline\end{array}\end{array}} \qquad \underset{T(a,b)}{\begin{array}{cc} a & b \\ \end{array} \atop \begin{array}{|c|c|}\hline 1 & 0 \\ \boldsymbol{\xi} & 1 \\ \hline\end{array}} \qquad \underset{T(a,p)}{\begin{array}{cc} a & p \end{array} \atop \begin{array}{|c|c|}\hline 1 & 1 \\ \boldsymbol{\xi} & \boldsymbol{\xi} \\ \hline\end{array}} \qquad \underset{G_1(a,c)}{\begin{array}{cc} a & c \end{array} \atop \begin{array}{|c|c|}\hline 1 & 0 \\ \boldsymbol{\xi} & 1 \\ \hline\end{array}} \qquad \underset{G_2(a,c)}{\begin{array}{cc} a & c \end{array} \atop \begin{array}{|c|cc|}\hline 1 & 0 & 1 \\ \boldsymbol{\xi} & 1 & 0 \\ \hline\end{array}}$$

If $\mathscr{P}$ is an equipped poset with a primitive involution $*$, and $a \in \mathscr{P}^\bullet$, $b \in \mathscr{P}^\otimes$, with $a \parallel b$, then $G_1(b, a)$ and $G_2(b, a)$ denote indecomposable representations with the matrix presentations described below ($G_0 = F^2$ in each case):

$$G_1(b,a) = \begin{array}{c|c|cc|} \multicolumn{1}{c}{b} & \multicolumn{1}{c}{a} & \multicolumn{2}{c}{a^*} \\ \hline 1 & 0 & 0 & 0 \\ \boldsymbol{\xi} & 1 & 0 & 0 \\ \hline \end{array} \qquad G_2(b,a) = \begin{array}{|c|cc|cc|} \multicolumn{1}{c}{b} & \multicolumn{2}{c}{a} & \multicolumn{2}{c}{a^*} \\ \hline 1 & 1 & 0 & 0 & 0 \\ \boldsymbol{\xi} & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

Remark 3. Zavadskij proved in [25] that $P(\varnothing)$, $P(c_i)$, $T(c_i)$ and $T(c_i, c_j)$ for $1 \leqslant i < j \leqslant n$, are the only indecomposable representations (up to isomorphisms) over the pair $(\mathbb{R}, \mathbb{C})$ of a completely weak chain $C = \{c_1 \prec \cdots \prec c_n\}$. In fact, if $U = (U_0; U_{c_i} \mid 1 \leqslant i \leqslant n)$ is a representation of C over $(\mathbb{R}, \mathbb{C})$, then in the corresponding matrix representation to each block U_{c_i}, $1 \leqslant i \leqslant n$, can be reduced via admissible transformations to the following standard form:

$$U_{c_i} = \begin{array}{|c|c|} \hline I & \\ \hline & I \\ \hline & \boldsymbol{i}I \\ \hline & \\ \hline \end{array},$$

where the columns consist of generators of U_{c_i} modulo its radical subspace $\underline{U_{c_i}} = U_{c_{i-1}}$ with respect to a fixed basis of U_0 (in this case, empty cells indicate null coordinates). This result can be generalized in a natural way to the case (F, G) by using a suitable scalar $\boldsymbol{\xi} \in G$ instead of the constant $\boldsymbol{i} \in \mathbb{C}$ in the matrix presentation of U_{c_i} shown above.

1.5. (A,B)-cleaving and the Zavadskij symbol

In this section, we recall the notion of a cleaving pair of subspaces in the sense of Zavadskij [25] and the definition of the Zavadskij symbol as Cañadas and Cifuentes described in [9].

Henceforth, the disjoint union of subsets X, $Y \in \mathscr{P}$ will be called a *sum* and it will be denoted by $X + Y$. A sum $X + Y$ is called *cardinal* (*ordinal*) if there is no order relations between points $x \in X$ and $y \in Y$ (if $x < y$ for all $x \in X$ and $y \in Y$, or conversely). By $(\widetilde{p_1}, \ldots, \widetilde{p_k}, q_1, \ldots, q_l)$ we denote an analogous cardinal sum in which l chains are ordinary with $q_1, \ldots, q_l$ points, and k chains are completely weak with $p_1, \ldots, p_k$ points, respectively.

The following lattice allows defining a cleaving pair of subspaces as Zavadskij described in [27].

The order relation in this poset is given by the natural inclusion of subspaces, E_0 is a complementary subspace of $A \cap B$ in A, and W_0 is a complementary subspace of $A + B$ in U_0. Let U_0 be an F-vector space and E_0, W_0, A, $B \subset U_0$. The pair of subspaces (E_0, W_0) is an (A, B)-*cleaving* of U_0 if the poset of subspaces described in Figure 3 is a lattice (with the obvious meets $\wedge = \cap$ and sums $\vee = +$). In other words, (E_0, W_0) is an (A, B)-cleaving pair of U_0 if and only if

$$U_0 = E_0 \oplus W_0, \quad A = E_0 + (A \cap B) \quad \textit{and} \quad B = W_0 \cap (A + B). \tag{8}$$

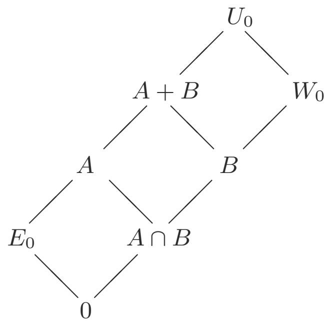


FIGURE 3. The diagram of an (A, B)-cleaving of U_0.

Set U_0, V_0 be two arbitrary finite-dimensional F-vector spaces. For any subspaces $X \subset U_0$ and $Y \subset V_0$, the *Zavadskij symbol* $[X, Y]$ associated with X and Y is a subspace of $\mathrm{Hom}_F\,(U_0, V_0)$ such that $\varphi \in [X, Y]$ if

$$X \;\subset\; \mathrm{Ker}\,\varphi \quad \text{and} \quad \mathrm{Im}\,\varphi \subset Y.$$

Note that, if $X' \subset X$ and $Y \subset Y'$ then $[X, Y] \subset [X', Y']$ [27].

If $U = X \oplus Y$ is a vector space decomposition then we let e_X denote the idempotent $i\pi$ in End (U), where $\pi : U \to X$ and $i : X \to U$ are the natural projection and injection, respectively.

For a category $\mathscr{R}$, we let $\langle U_i \mid i \in I\rangle_{\mathscr{R}}$ denote the ideal consisting of all morphisms passing through finite direct sums of the objects U_i. That is, if $\varphi : U \to V \in \langle U_i \mid i \in I\rangle_{\mathscr{R}}$, then there exist morphisms $f, g \in \mathscr{R}$ such that $\varphi = U \xrightarrow{f} \bigoplus\limits_i U_i^{m_i} \xrightarrow{g} V$ with $m_i = 0$ for almost all i.

1.6. Auslander-Reiten quiver

The *Gabriel's quiver* $\Delta(\mathscr{K})$ of a Krull-Schmidt category $\mathscr{K}$ is a directed graph whose vertices are the isomorphism classes $[U]$ of the indecomposable objects U in $\mathscr{K}$ and there is an arrow $[U] \to [V]$ if $\mathrm{Irr}(U, V) \neq 0$ with $\mathrm{Irr}(U, V) = \mathrm{Rad}(U, V)/\mathrm{Rad}^2(U, V)$. A *component* of $\mathscr{K}$ is the class objects generated by the indecomposable objects belonging to a connected component of $\Delta(\mathscr{K})$ [18].

The *Auslander-Reiten quiver* $\Gamma(\mathscr{K})$ of a Krull-Schmidt category $\mathscr{K}$ is the Gabriel's quiver of $\mathscr{K}$ in which it is defined a particular translation denominated the Auslander-Reiten translation (τ).

2. Some preliminary algorithms

In this section, for the sake of clarity, we recall some categorical properties of the algorithms of differentiation, I (section 2.1), completion (section 2.2) and VII (section 2.3).

2.1. Algorithm of differentiation I

The following is the definition of the *algorithm of differentiation* I (DI) with respect to a suitable pair of points [26].

A pair of incomparable points (a, b), of a poset $\mathscr{P}$ is called I- *suitable* or suitable for differentiation I, if $\mathscr{P} = a^{\nabla} + b_{\vartriangle} + C$

where $C = \{c_1 < \cdots < c_n\}$ is an ordinary chain incomparable with points a, b. The *derived poset* of the set $\mathscr{P}$ with respect to the pair (a, b) is a poset $\mathscr{P}' = \mathscr{P}'_{(a,b)} = (\mathscr{P} \setminus C) + C^+ + C^-$, where $C^- = \{c_1^- < \cdots < c_n^-\}$ and $C^+ = \{c_1^+ < \cdots < c_n^+\}$ are new ordinary chains, replacing the chain C, with the relations $c_i^- < c_i^+$; $a < c_i^+$ and $c_i^- < b$ for all $1 \leqslant i \leqslant n$.

The differentiation functor $D^{\mathrm{I}}_{(a,b)} : \operatorname{rep}\mathscr{P} \longrightarrow \operatorname{rep}\mathscr{P}'$ assigns to each representation $U = (U_0; U_x \mid x \in \mathscr{P})$ of $\mathscr{P}$ the derivative representation $U' = (U'_0; U'_x \mid x \in \mathscr{P}')$ accordingly to the formulae:

$$
\begin{aligned}
U'_0 &= U_0,\\
U'_{c_i^+} &= U_a + U_{c_i}, \quad \text{for } 1 \leqslant i \leqslant n,\\
U'_{c_i^-} &= U_b \cap U_{c_i}, \quad \text{for } 1 \leqslant i \leqslant n,\\
U'_x &= U_x \quad \text{for the remaining points } x \in \mathscr{P}'_{(a,b)},\\
\varphi' &= \varphi \quad \text{for all F linear map-morphism}, \varphi : U_0 \to V_0.
\end{aligned} \tag{9}
$$

$\mathscr{P}'_{(a,b)}$ can be considered as a subposet of the free lattice generated by $\mathscr{P}$. Figure 4 shows the Hasse diagram for this differentiation.

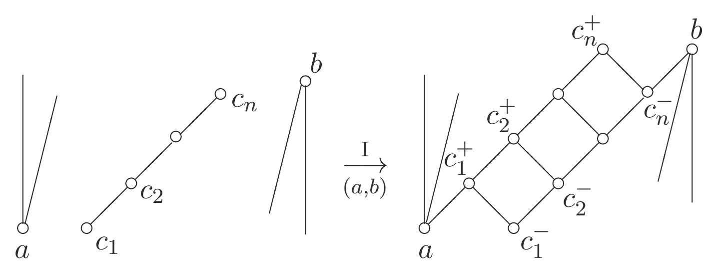


Figure 4. Hasse diagrams of an equipped poset $\mathscr{P}$ and its corresponding derived poset $\mathscr{P}'_{(a,b)}$.

Since usually the derived representation U' is decomposable and contains trivial summands $P(a)$, it is convenient to consider (besides U') the *reduced* derived representation $U^{\downarrow}$ such that $U' \simeq U^{\downarrow} \oplus P^m(a)$, where $m \geqslant 0$ and $U^{\downarrow}$ is free of direct summands $P(a)$. There exist an alternative definition of $U^{\downarrow}$, namely, $U^{\downarrow} = W = (W_0; W_x \mid x \in \mathscr{P})$, where W_0 is any subspace in U_0 satisfying the conditions $U_a + W_0 = U_0$, $(U_a + U_b) \cap W_0 = U_b$ and $W_x = {U'}_x \cap W_0^{m(x)}$ for all $x \in \mathscr{P}$ (here $m(x) = l_x$ is the multiplicity of a point x). The representation $U^{\downarrow}$ does not depend (up to isomorphism), on the choice of W_0.

The inverse (in some sense) operation $\uparrow$, called integration, assigns to each representation W of the set $\mathscr{P}'$ the primitive representation $W^{\uparrow}$ of the initial set $\mathscr{P}$ such that $(W^{\uparrow})^{\downarrow} \simeq W$ as soon as W contains no direct summands $P(a)$.

Zavadskij proved the following result in [22], [23] and [27].

Theorem 2. *Let $\mathscr{P}$ be a poset with a pair of points (a,b) I-suitable. Then:*

(a) *The functor $D^{\mathrm{I}}_{(a,b)} : \operatorname{rep}\mathscr{P} \longrightarrow \operatorname{rep}{\mathscr{P}'}_{(a,b)}$, defined by formulas (9) induces an equivalence of the quotient categories*

$$\operatorname{rep}\mathscr{P}/\langle P(a), P(a,c_1), \ldots, P(a,c_n)\rangle \xrightarrow{\sim} \operatorname{rep}{\mathscr{P}'}_{(a,b)}/\langle P(a)\rangle.$$

(b) *The operations $\downarrow$ and $\uparrow$ induce mutually inverse bijections*

$$\operatorname{Ind}\mathscr{P} \setminus [P(a), P(a,c_1), \ldots, P(a,c_n)] \rightleftarrows \operatorname{Ind}\mathscr{P}'_{(a,b)} \setminus [P(a)].$$

Remark 4. It should be noted that Zavadskij proved numerals (a) and (b) of Lemma 2 in [22, 23] for the algorithm of differentiation I and completion, whereas for algorithms A_z, VII-X he only proved numeral (b) [25].

2.2. Completion algorithm

In this section, we present the algorithm of completion as Zavadskij defined in [23, 25, 26].

A pair of weak points a, b weakly comparable $a \prec b$ of an equipped poset $\mathscr{P}$ will be called *special* if $\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + \Sigma$, where Σ is the interior of the interval $[a,b]$.

The following is the definition of the completion algorithm which is a differentiation with respect to a special pair of points (a,b) of an equipped poset.

The *completion* of $\mathscr{P}$ with respect to such special pair (a,b) is a transition from $\mathscr{P}$ to a slightly different equipped poset $\overline{\mathscr{P}} = \overline{\mathscr{P}}_{(a,b)}$ obtained from $\mathscr{P}$ by *strengthening* the relation between the points a and b for which we have the following two situations:

(a) $\mathscr{P} = a^{\triangledown} + b_{\vartriangle}$, where a, b are incomparable strong points,
(b) $\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + \Sigma$, where a, b are weak points, $a \prec b$ and Σ is the interior of the segment $[a, b]$.

In both cases the completed equipped poset $\overline{\mathscr{P}}$ is obtained from $\mathscr{P}$ by adding the only one strong relation $a \lhd b$. In the case (a) this is in fact the classical completion of an ordinary poset (see, [23]). In the case (b) the completion $a \lhd b$ of $\mathscr{P}$ conforms to a pair of mutually symmetric completions of the evolvent $\widehat{\mathscr{P}}$ (i.e., the ordinary poset associated to $\mathscr{P}$) with respect to ordinary special pairs (a', b'') and (a'', b').

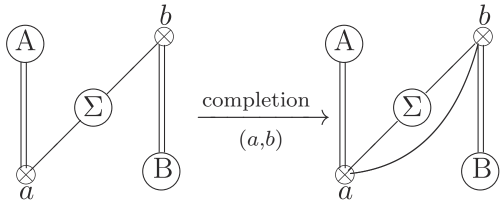


Figure 5. The diagrams of an equipped poset $\mathscr{P}$ and its corresponding completed poset $\overline{\mathscr{P}}_{(a,b)}$.

Let $\overline{D}_{(a,b)} : \operatorname{rep}\mathscr{P} \longrightarrow \operatorname{rep}\overline{\mathscr{P}}_{(a,b)}$ be the functor induced by the algorithm of completion. This functor is defined as follows: for $U = (U_0; U_x \mid x \in \mathscr{P}) \in \operatorname{rep}\mathscr{P}$,

$$\overline{D}_{(a,b)}\,(U) := \overline{U} = (\overline{U}_0; \overline{U}_x \mid x \in \overline{\mathscr{P}}) \in \operatorname{rep}\overline{\mathscr{P}}_{(a,b)},$$

where

$$\begin{aligned} \overline{U}_0 &= U_0, \\ \overline{U}_b &= U_b + F(U_a), \\ \overline{U}_x &= U_x, \ \text{ for the remaining points } \ x \in \overline{\mathscr{P}}_{(a,b)}, \\ \overline{\varphi} &= \varphi, \ \text{ for all F linear map-morphism } \varphi : U_0 \longrightarrow V_0. \end{aligned} \tag{10}$$

It is clear that $\operatorname{rep}\overline{\mathscr{P}}$ is a full subcategory of the category $\operatorname{rep}\mathscr{P}$. Moreover, the following statement holds, see [23, 25].

Lemma 2. *The category* $\operatorname{rep}\overline{\mathscr{P}}$ *coincides with the full subcategory of the category* $\operatorname{rep}\mathscr{P}$ *formed by the objects without direct summands of type* $P(a)$ *in the case* (a)*, and of type* $T(a)$ *in the case* (b)*. Therefore*

$$\operatorname{Ind}\overline{\mathscr{P}}_{(a,b)} = \begin{cases} \operatorname{Ind}\mathscr{P} \setminus \{P(a)\} & \textit{in the case } (a), \\ \operatorname{Ind}\mathscr{P} \setminus \{T(a)\} & \textit{in the case } (b). \end{cases}$$

Regarding the completion functor Cañadas and Zavadskij proved the following results in [2] and [27] respectively.

Lemma 3. *The completion functor* $\overline{D}_{(a,b)}$ *induces the following categorical equivalence of quotient categories.*

$$\operatorname{rep}\mathscr{P}/\langle T(a), T(a,b)\rangle \xrightarrow{\sim} \operatorname{rep}\overline{\mathscr{P}}/\langle T(a)\rangle.$$

As a consequence of Lemmas 2 and 3, the following corollary is obtained giving an isomorphism ($\simeq$) between Gabriel quivers of the corresponding categories.

Corollary 1. *Let* $\Gamma(\mathscr{R})$ *and* $\Gamma(\overline{\mathscr{R}})$ *be, respectively, the Gabriel's quivers of the categories* $\mathscr{R} = \operatorname{rep}\mathscr{P}$ *and* $\overline{\mathscr{R}} = \operatorname{rep}\overline{\mathscr{P}}$*, then*

$$\Gamma(\mathscr{R}) \setminus [T(a), T(a,b)] \simeq \Gamma(\overline{\mathscr{R}}) \setminus [T(a)].$$

2.3. Categorical properties of the algorithm of differentiation VII for equipped posets

The differentiation VII is one of the seventeen differentiations developed by Zavadskij to classify (in particular) equipped posets of tame and of finite growth representation type [25, 26].

Let $\mathscr{P}$ be an equipped poset then a pair of points (a,b) of the poset $\mathscr{P}$ is said to be VII-*suitable* or suitable for differentiation VII, if $a \in \mathscr{P}^{\otimes}$, $b \in \mathscr{P}^{\circ}$, $a \parallel b$ and $\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + C$, where $\{c_1 \prec \cdots \prec c_n\}$ is a

completely weak chain (possibly empty) incomparable with the point b, and $a \prec c_1$ (note that automatically $a \prec c_n$).

The *derived poset* $\mathscr{P}'_{(a,b)}$ of an equipped poset $\mathscr{P}$ with respect to a pair (a,b) of points VII-suitable is an equipped poset defined in such a way that

$$\mathscr{P}'_{(a,b)} = (\mathscr{P} \setminus \{a + C\}) + \left\{a^- < a^+\right\} + C^- + C^+,$$

where $a^- \in (\mathscr{P}'_{(a,b)})^{\otimes}$, $a^+ \in (\mathscr{P}'_{(a,b)})^{\circ}$, $C^- = \left\{c_1^- \prec \cdots \prec c_n^-\right\}$ and $C^+ = \left\{c_1^+ \prec \cdots \prec c_n^+\right\}$ are completely weak chains, $c_i^- \prec c_i^+$ for all i; $a^- \prec c_1^-$; $a^+ < c_1^+$; $c_n^- < b$, and the following conditions hold:

1. (1) each of the points a^-, a^+, (c_i^-, c_i^+) inherits all the previous order relations of the point a (c_i) with the points of the subset $\mathscr{P} \setminus \{a + C\}$;
2. (2) the order relations in $\mathscr{P}'_{(a,b)}$ are induced by the relations in its subset $\mathscr{P} \setminus \{a + C\}$, and by the relations described above (note that, in particular, $a^- \prec c_n^-$).

The following functor $D^{\mathrm{VII}}_{(a,b)}$ was given by Zavadskij in [25], soon afterwards, it was updated by Rodriguez and Zavadskij in [19] by using some short versions of this algorithm via representations of posets with additional lattice relations.

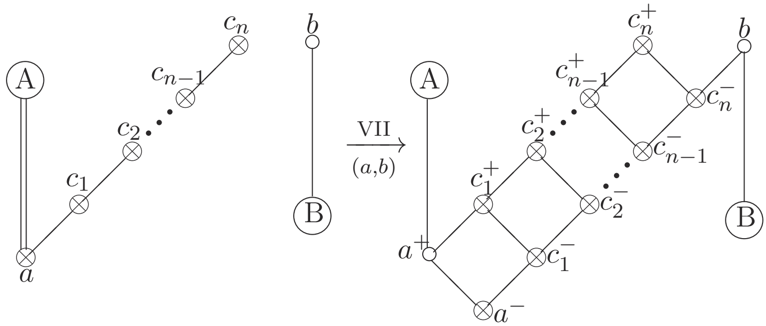


FIGURE 6. Diagrams of an equipped poset $\mathscr{P}$ and its derivative poset $\mathscr{P}'_{(a,b)}$.

Let $\mathscr{P}$ be an equipped poset with a pair of points (a,b), VII-suitable, the following formulas define the *differentiation functor* $D^{\mathrm{VII}}_{(a,b)} : \operatorname{rep}\mathscr{P} \longrightarrow \operatorname{rep}\mathscr{P}'_{(a,b)}$, induced by the algorithm of differentiation VII. Thus for a given representation $U = (U_0; U_x \mid x \in \mathscr{P}) \in \operatorname{rep}\mathscr{P}$, we get the *derived representation* $U' = (U'_0; U'_x \mid x \in \mathscr{P}'_{(a,b)})$, if $1 \leqslant i \leqslant n$, where:

$$\begin{gathered}
U'_0 = U_0, \qquad U'_{a^-} = U_a \cap U_b, \qquad U'_{a^+} = F(U_a), \\
U'_{c_i^-} = U_{c_i} \cap U_b, \qquad U'_{c_i^+} = U_{c_i} + F(U_a), \\
U'_x = U_x \text{ for the remaining points } x \in \mathscr{P}'_{(a,b)}, \\
\varphi' = \varphi, \text{ for all F linear map-morphism } \varphi : U_0 \longrightarrow V_0.
\end{gathered} \tag{11}$$

Note that, $P'(a) = P(a^+)$ and $T'(a) = T'(a, c_i) = P^2(a^+)$. A representation of $\mathscr{P}$, containing no direct summands of the form $P(a)$, $T(a)$ and $T(a, c_i)$, will be called *reduced*. Obviously, $P^{\downarrow}(a) = T^{\downarrow}(a) = T^{\downarrow}(a, c_i) = 0$, for all $1 \leqslant i \leqslant n$. By construction of the *reduced* derivative representation.

The following results were proved by Cañadas, Zavadskij and Zavadskij et al in [2], [19] and [25].

Lemma 4. *For each object* $W \in \operatorname{rep}\overline{\mathscr{P}}'_{(a,b)}$ *there exists an object* $U = W^{\uparrow} \in \operatorname{rep}\mathscr{P}$ *such that* $U' \simeq W \oplus P^m(a^+)$*, for some* $m \geqslant 0$.

Zavadskij proved that $(W^{\uparrow})^{\downarrow} \simeq W$, $(U^{\downarrow})^{\uparrow} \simeq U$ for each reduced representation U of $\mathscr{P}$ and each representation W of $\overline{\mathscr{P}'}$ where $W \simeq U^{\downarrow}$ [25].

Lemma 5. *Let* $\mathscr{P}$ *be an equipped poset with a pair of points* (a,b) VII-*suitable. Then:*

(a) *The functor* $D^{\mathrm{VII}}_{(a,b)} : \operatorname{rep}\mathscr{P} \longrightarrow \operatorname{rep}\mathscr{P}'_{(a,b)}$*, defined by formulas* (11) *induces an equivalence of the quotient categories*

$$\operatorname{rep}\mathscr{P}/\langle T(a), T(a,c_i), P(a) \mid 1 \leqslant i \leqslant n\rangle \xrightarrow{\sim} \operatorname{rep}\mathscr{P}'_{(a,b)}/\langle P(a^+)\rangle.$$

(b) *The operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\operatorname{Ind}\mathscr{P} \setminus [T(a), T(a,c_i), P(a) \mid 1 \leqslant i \leqslant n] \rightleftarrows \operatorname{Ind}\overline{\mathscr{P}}'_{(a,b)} = \operatorname{Ind}\mathscr{P}'_{(a,b)} \setminus [P(a^+)].$$

The following result holds as a consequence of Lemma 5.

Corollary 2. *If* $\Gamma(\mathscr{R})$ *and* $\Gamma(\mathscr{R}')$ *are the Gabriel's quivers of the categories* $\mathscr{R} = \operatorname{rep}\mathscr{P}$ *and* $\mathscr{R}' = \operatorname{rep}\mathscr{P}'$*, then*

$$\Gamma(\mathscr{R}) \setminus [T(a), T(a,c_i), P(a) \mid 1 \leqslant i \leqslant n] \simeq \Gamma(\mathscr{R}') \setminus [P(a^+)].$$

Remark 5. Henceforth, if X is an F-subspace of a vector space U_0 then, we let λ_X denote a linear combination of the form $\lambda_{i_1}x_1+\lambda_{i_2}x_2+\cdots+\lambda_{i_k}x_k$ for a fixed basis $\{x_i\} \subset X$ with $\lambda_{i_j} \in F$.

3. Proof of Theorem 1

In this section, we prove that algorithms A_z, VIII-X induce categorical equivalences between quotient categories of equipped posets.

3.1. Some remarks regarding the algorithm of differentiation VII for equipped posets

In this section, it is defined an algorithm A_z which in some sense can be considered as a generalization of the algorithm of differentiation VII defined by the structure of a chain of (F,G) subspaces of a given F-vector space U_0. Actually, algorithm A_z is a way to obtain equipped posets with a pair of points (a,b), VII-suitable for which the set $(a^{\triangledown})^{\circ} \neq \varnothing$.

Let us consider the following chain of G-subspaces of a vector space $\widetilde{U_0}$.

$$U_{c_0} \subseteq U_{c_1} \subseteq U_{c_2} \subseteq \cdots \subseteq U_{c_{n-1}} \subseteq U_{c_n},\ n \geqslant 1,$$

which are incomparable with a G-subspace U_b such that:

$$F(U_b) = U_b.$$

We also consider that, for any i, $1 \leqslant i \leqslant n$,

$$U_{c_i} \subsetneq F(U_{c_i}).$$

Moreover, each G-subspace U_{c_i} can be seen as a sum of subspaces of the form

$$\begin{aligned} U_{c_i} &= U_{c_{i-1}} \oplus H_i, & H_i &= \widetilde{H_i^-} \oplus S_i, \\ \widetilde{H_i^-} &= \widetilde{H_i^-} \cap U_b \oplus Y_i^-, & S_i &= S_i \cap U_b \oplus \overline{S_i}, \end{aligned} \tag{12}$$

where S_i is a complementary subspace of $\widetilde{H_i^-}$ in H_i, as well as, Y_i^- and $\overline{S_i}$ are complementary subspaces of $\widetilde{H_i^-} \cap U_b$ and $S_i \cap U_b$ in $\widetilde{H_i^-}$ and S_i, respectively.

Another finest way to express U_{c_i} as a sum of subspaces goes as follows:

$$U_{c_i} = U_{c_{n-1}} \oplus \widetilde{H_i^-} \cap U_b \oplus S_i \cap U_b \oplus Y_i^- \oplus \sum_{i<j\leqslant n} T_{c_i}^{c_j} \oplus N_i. \tag{13}$$

In (13) the spaces $\sum_{i<j\leqslant n} T_{c_i}^{c_j}$ and N_i are subspaces of $\overline{S_i}$. In particular, for j fixed

$$\begin{aligned} T_{c_i}^{c_j} &\simeq T^{k_1}(c_i, c_j), \quad \text{for some } k_1 \geqslant 0, \\ N_i &\simeq T^{k_2}(c_i), \quad \text{for some } k_2 \geqslant 0. \end{aligned} \tag{14}$$

The corresponding subspace $T(c_i)$ associated with an indecomposable representation $T(c_i, c_j)$ will be denoted $T(i^j) = (T_1(i^j), T_2(i^j))$, thus, $T_1(i^j)$, $T_2(i^j) \subseteq \widetilde{U_{c_j}^-}$.

We assume that

$$\begin{aligned} T(i^j) &= (T_1(i^j) \cap U_b, \overline{T_2}(i^j)) + (X_1(i^j), X_2(i^j)), \\ T_1(i^j) &= T_1(i^j) \cap U_b \oplus \overline{T_1}(i^j) \oplus X_1(i^j), \\ T_2(i^j) &= T_2(i^j) \cap U_b \oplus \overline{T_2}(i^j) \oplus X_2(i^j), \end{aligned} \tag{15}$$

with $T_2(i^j) \cap U_b = 0 = \overline{T_1}(i^j) = X_k(i^j) \cap U_b$, $k \in \{1, 2\}$.

$$\begin{aligned} N_i = (N_{i(1)}, N_{i(2)}) &= (P_{i(1)b}, P_{i(2)}) + (Q_{i(1)}, Q_{i(2)}), \\ P_{i(1)b} \subseteq U_b, \qquad Q_{i(1)} \cap U_b &= P_{i(2)} \cap U_b = Q_{i(2)} \cap U_b = 0. \end{aligned} \tag{16}$$

The algorithm A_z (adding a subspace $F(U_z)$). In this subsection, it is described the way that a subspace U_{c_i} changes when adding a subspace $F(U_z)$, $z \geqslant 0$.

Firstly, we note that for $0 \leqslant i \leqslant n-1$ and z fixed $0 \leqslant z \leqslant i$,

$$\begin{aligned} U_{c_i} + F(U_{c_z}) &\subseteq U_{c_{i+1}} + F(U_{c_z}), \\ U_{c_i} \cap U_b + F(U_{c_z}) &\subseteq U_{c_{i+1}} \cap U_b + F(U_{c_z}). \end{aligned} \tag{17}$$

and that under these circumstances, the subspaces $T_1(k^i),\ T_2(k^i)\ X_h(k^i) \subseteq \widetilde{U_{c_i}^-}$ (see identities (15)), $h \in \{1, 2\}$.

The following lattice arises for each i, $0 \leqslant i \leqslant n-1$.

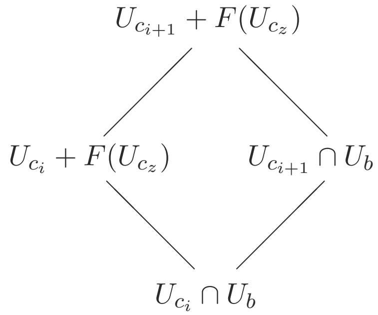


FIGURE 7. The diagram of subspaces associated with U_{c_z} and U_b.

Then, the subspaces $U_{c_i},\ U_{c_i} \cap U_b,\ U_b$ and $U_{c_i} + F(U_{c_z})$ build (F, G)-representations of the following equipped posets:

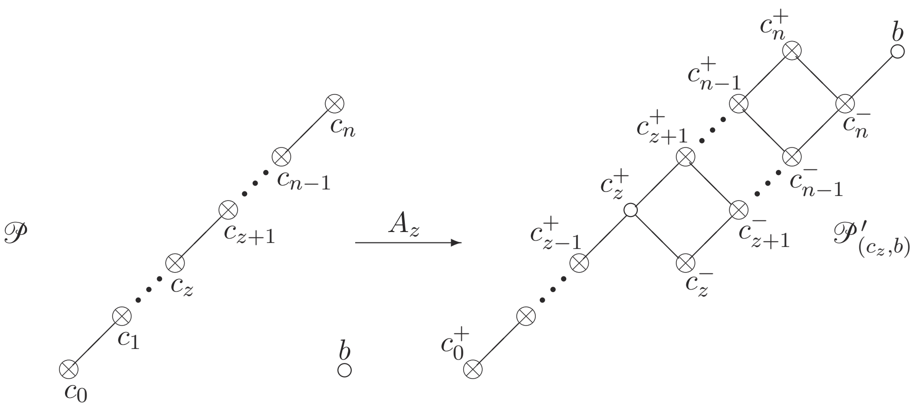


FIGURE 8. The diagram of the algorithm A_z.

We let $\dot{\mathscr{P}}_{(c_z,b)}$ denote the equipped poset obtained from the derived poset $\mathscr{P}'_{(c_z,b)} = (c_0^+)^{\triangledown} + b_{\vartriangle} + (c_{z-1}^+)_{\curlywedge}$ by adding a lattice relation of the form $(c_0^+ + c_1^+ + c_2^+ + \cdots + c_{z-1}^+)b = c_{z-1}^+ b \subset c_z^-$, sometimes it is written as $\dot{\mathscr{P}}_{(c_z,b)} = (\mathscr{P}'_{(c_z,b)} \mid \sum_{(c_z,b)})$ where $\sum_{(c_z,b)}$ consists only of the lattice relation $c_{z-1}^+ b \subset c_z^-$, and as in [2, 19] it means that $\operatorname{rep}\dot{\mathscr{P}}_{(c_z,b)}$ is the full subcategory of $\operatorname{rep}\mathscr{P}'_{(c_z,b)}$ whose objects W satisfy the condition

$$W_{(c_{z-1}^+)_{\curlywedge}} \cap W_b \subseteq W_{c_z^-}. \tag{18}$$

Actually, any representation of $\dot{\mathscr{P}}_{(c_z,b)}$ is obtained via the following assignments of subspaces of $\widetilde{U_0}$ to the points of $\mathscr{P}$ and $\dot{\mathscr{P}}_{(c_z,b)}$:

$$\begin{aligned} U_b &\text{ to the point } b, \\ U_{c_i} &\text{ to each point } c_i,\ 0 \leqslant i \leqslant n, \\ U_{c_i} \cap U_b &\text{ to each point } c_i^-,\ z \leqslant i \leqslant n, \\ U_{c_i} + F(U_{c_z}) &\text{ to each point } c_i^+,\ z \leqslant i \leqslant n, \\ U_{c_{z-1}^+} \cap U_b \subseteq U_{c_z^-}&. \end{aligned} \tag{19}$$

Keeping without changes the other subspaces or points in the chain, i.e., $U_{c_h} = U_{c_h^+}$, $0 \leqslant h \leqslant c_{z-1}$.

Note that by definition $P'(c_z) = P(c_z^+)$, $T'(c_z, c_i) = T'(c_z) = P^2(c_z^+)$ in $\operatorname{rep} \dot{\mathscr{P}}_{(c_z,b)}$, $i > z$. Therefore the algorithm A_z transforms the poset $\mathscr{P}$ in the new poset $\dot{\mathscr{P}}_{(c_z,b)}$. Also note that the case $z = 0$ corresponds to the algorithm of differentiation VII (see identities (11)).

If we denote $A_z(U) = D^{A_z}_{(c_z,b)}(U) = U'$, for z fixed, the output under the algorithm A_z of the representation $U \in \operatorname{rep} \mathscr{P}$ [2, 19], then A_z becomes a functor which acts on objects and morphisms of category $\operatorname{rep} \mathscr{P}$ as follows:

$$\begin{aligned} {}' = D_{(c_z,b)} &: \operatorname{rep} \mathscr{P} \longrightarrow \operatorname{rep} \dot{\mathscr{P}}_{(c_z,b)} \\ U'_0 &= U_0 \\ U'_{c_i^+} &= \begin{cases} U_{c_i} + F(U_{c_z}) & \text{if } i \geqslant z, \\ U_{c_i} & \text{if } 0 \leqslant i < z. \end{cases} \\ U'_{c_i^-} &= U_{c_i} \cap U_b, \text{ for } \ i \geqslant z, \\ \varphi' &= \varphi, \text{ for all F linear map-morphism } \varphi : U_0 \to V_0 \in \operatorname{rep} \mathscr{P}. \end{aligned} \tag{20}$$

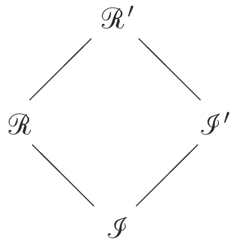
By construction, we can see that $D_{(c_z,b)}$ induces a categorical equivalence between quotient categories. In fact, we have the following results for algorithm A_z.

Lemma 6. *Let U and V be two fixed objects in the category $\operatorname{rep} \mathscr{P}$, $\mathscr{R} := \operatorname{rep} \mathscr{P}(U,V) = \operatorname{Hom}_F(U,V)$, $\mathscr{R}' := \operatorname{rep} \dot{\mathscr{P}}_{(c_z,b)}(U',V') = \operatorname{Hom}_F(U',V')$ and let $\mathscr{I}$ and $\mathscr{I}'$ be the ideals*

$$\mathscr{I}(U,V) = \mathscr{I} = \langle \{P(c_z), T(c_z, c_j), T(c_z),\ z < j \leqslant n\} \rangle \subseteq \operatorname{rep} \mathscr{P},$$

$$\mathscr{I}'(U',V') = \mathscr{I}' = \langle \{P(c_z)\} \rangle \subseteq \operatorname{rep} \dot{\mathscr{P}}_{(c_z,b)}.$$

Then, the following poset of subspaces is a lattice:



Proof. Firstly, we prove that $\mathscr{R}' = \mathscr{R} + \mathscr{I}'$. To do that, we choose a morphism $\psi \in \mathscr{R}'$. Then

$$\begin{aligned} \widetilde{\psi}(U_b) &\subseteq V_b, \\ \widetilde{\psi}(U_x) &\subseteq V_x, \text{ for any point } x \notin c_z^{\Upsilon}, \\ \widetilde{\psi}(F(U_x)) &\subseteq F(V_x), \text{ for any point } x \in \mathscr{P}'_{(c_z,b)}, \\ \widetilde{\psi}(\widetilde{U_x^-}) &\subseteq \widetilde{V_x^-}, \text{ for any point } x \in \mathscr{P}'_{(c_z,b)}. \end{aligned}$$

Note that, in general $\widetilde{\psi}(U_x) \nsubseteq V_x$, for any $x \in \mathscr{P}$, thus in general $\psi \notin \mathscr{R}$.

Let us now to define correction morphisms $w_0, w_1, w_2, \ldots, w_{\underline{n}}$ such that $\psi - \sum_{i=0}^{n} w_i \in \mathscr{R}$. To do that, we note that, if $\lambda \in Y_z^-$ then $\widetilde{\psi}(\lambda) = \lambda_{\widetilde{V_{c_z}^-}} + \lambda_{F(H_z)}$. Throughout the proof, we assume the same notation for subspaces of $\widetilde{U_0}$ and $\widetilde{V_0}$ (if not confusion). Thus, if $w_0 : U_0 \longrightarrow V_0$ is a linear map-morphism such that

$$\widetilde{w_0}(x) = \begin{cases} \lambda_{F(H_z)} & \text{if } x \in Y_z^-, \\ 0 & \text{otherwise,} \end{cases}$$

then $w_0 \in [(U_b + U_{c_{z-1}})^+, V_{c_z}^+]$ and $(\widetilde{\psi} - \widetilde{w_0})(H_j) = \widetilde{\psi}(H_j)$, for any $j > z$.

Suppose now that $\lambda = (\lambda_1, \lambda_2) \in \sum_{z<j\leqslant n} T_z^j \oplus N_z$. Then

$$\begin{aligned} &\widetilde{\psi}(\lambda) = (\psi(\lambda_1), \psi(\lambda_2)), \text{ where} \\ &\psi(\lambda_1) = \lambda^1_{V_{c_{z-1}}} + \lambda^1_{\widetilde{H_z^-}\cap V_b} + \lambda^1_{S_z\cap V_b} + \lambda^1_{Y_z^-} + \lambda^1_{\sum_{i<j\leqslant n} T_i^j} + \lambda^1_{N_z}, \\ &\psi(\lambda_2) = \lambda^2_{V_{c_{z-1}}} + \lambda^2_{\widetilde{H_z^-}\cap V_b} + \lambda^2_{S_z\cap V_b} + \lambda^2_{Y_z^-} + \lambda^2_{\sum_{i<j\leqslant n} T_i^j} + \lambda^2_{N_z}. \end{aligned}$$

Note that, for any subspace $L \in \{\overline{T_k}(z^j), X_k(z^j), N_{i(k)}, P_{i(k)}, Q_{i(k)},\ k \in \{1,2\}\}$, λ_L^1 and λ_L^2 have real and imaginary parts, thus, λ_L^1 and λ_L^2 can be written in the form:

$$\lambda_L^1 = (\lambda_L^{1,1}, \lambda_L^{1,2}), \quad \lambda_L^2 = (\lambda_L^{2,1}, \lambda_L^{2,2}).$$

Hence, if $w_1 : U_0 \longrightarrow V_0$ is a linear map-morphism such that

$$w_1(x) = \begin{cases} (\lambda_L^{2,1} + \frac{1}{\beta}\lambda_L^{1,2}, \lambda_L^{2,2} - \lambda_L^{1,1} - \frac{\alpha}{\beta}\lambda_L^{1,2}) & \text{if} x \in \sum_{j>z} \overline{T_2}(z^j) + N_{z(2)} \\ 0 & \text{otherwise.} \end{cases}$$

then $w_1 \in [(U_b + U_{c_z})^- + U_{c_{z-1}}^+, V_{c_z}^+] \subseteq [(U_b + U_{c_z})^-, V_{c_z}^+]$. And

$$\begin{aligned} (\widetilde{\psi} - \widetilde{w_1})(x) &\in N_z, \text{ for any } x \in P_z, \\ (\widetilde{\psi} - \widetilde{w_1})(x) &\in N_z, \text{ for any } x \in Q_z, \\ (\widetilde{\psi} - \widetilde{w_1})(x) &\in S_z, \text{ for any } x \in \sum_{z<j} T(z^j), \\ (\widetilde{\psi} - \widetilde{w_1})(x) = \widetilde{\psi}(x), &\text{ if } x \in \widetilde{U_{c_z}^-}. \end{aligned}$$

Therefore

$$(\widetilde{\psi} - \widetilde{w_0} - \widetilde{w_1})(x) \in \widetilde{V_{c_z}^-} \text{ if } x \in \widetilde{U_{c_z}^-}.$$

For each $i > z$, define now a linear map-morphism $w_i : U_0 \longrightarrow V_0$ such that $\widetilde{w_i}(H_i) = \lambda_{F(V_{c_z})}$, $\widetilde{w_i} = 0$, otherwise. Then $w_i \in [(U_b + U_{c_{z+i-2}})^- + U_{c_{z+i-3}}^+, V_{c_z}^+]$. Thus, if $w = \sum_{i=0}^{n+1} w_i$ then $(\widetilde{\psi} - \widetilde{w})(U_{c_i}) \subseteq V_{c_i}$, for $0 \leqslant i \leqslant n$ $(\widetilde{\psi} - \widetilde{w})(U_b) \subseteq V_b$, $(\widetilde{\psi} - \widetilde{w})(U_x) \subseteq V_x$. Therefore $\psi - w \in \mathscr{R}$ with

$$\mathscr{I}' = [U_b^- + U_{c_{z-1}}^+, V_{c_z}^+] + \sum_{i=2}^{n-z+2} [(U_b + U_{c_{z+i-2}})^- + U_{c_{z+i-3}}^+, V_{c_z}^+].$$

In order to prove that $\mathscr{R} \cap \mathscr{I}' = \mathscr{I}$, we note that $\mathscr{I} \subseteq \mathscr{I}'$ and $\mathscr{I} \subseteq \mathscr{R}$ by definition. Therefore $\mathscr{I} \subseteq \mathscr{R} \cap \mathscr{I}'$. On the other hand, we also note that in $\mathscr{R}$

$$\begin{aligned} \langle P(c_z) \rangle &= [U_{c_{z-1}}^+ + U_b^-, V_{c_z}^+], \\ \langle T(c_z) \rangle &= [U_{c_{z-1}}^+ + (U_b + U_{c_n})^-, V_{c_z}^+], \\ \langle T(c_z, cj) \rangle &= [U_{c_{z-1}}^+ + (U_b + U_{c_j-1})^-, V_{c_z}^+], \end{aligned}$$

then

$$\mathscr{R} \cap \mathscr{I} = [U_{c_{z-1}}^+ + U_b^-, V_{c_z}^+] \supseteq \mathscr{I}' = [U_b^- + U_{c_{z-1}}^+, V_{c_z}^+]. \qquad \square$$

Since Zavadskij proved the following lemma for DVII (case, $z = 0$) in [25]. It is enough to establish that the integration procedure holds for any other case (i.e., for $z \neq 0$), but this is guaranteed by the integration process of its short version VII$_s$, see [2, 19].

Lemma 7. *For each representation* $W \in \operatorname{rep} \dot{\mathscr{P}}_{(c_z,b)}$ *there exists a representation* $W^{\uparrow} \in \operatorname{rep} \mathscr{P}$, *such that* $(W^{\uparrow})' \simeq W \oplus P^m(c_z^+)$, *for some* $m \geqslant 0$.

Lemmas 6 and 7 prove the following result for the algorithm A_z.

Lemma 8. *Let* $\mathscr{P}$ *be an equipped poset with a pair of points* (c_z, b) A_z*-suitable (as described in Figure* 2.9 *and assignments* (19)*). Then the functor of differentiation*

$$D^{A_z}_{(c_z,b)} : \operatorname{rep} \mathscr{P} \longrightarrow \operatorname{rep} \dot{\mathscr{P}}_{(c_z,b)},$$

defined by formulas (20) *induces an equivalence between quotient categories*

$$\operatorname{rep} \mathscr{P} / \langle P(c_z), T(c_z), T(c_z, c_i) \mid 1 \leqslant i \leqslant n \rangle \simeq \operatorname{rep} \dot{\mathscr{P}}_{(c_z,b)} / \langle P(c_z^+) \rangle.$$

Lemmas 5, 7 and 8 establish the following corollary regarding the Gabriel quiver of the corresponding categories involved in the differentiation A_z.

Corollary 3. *If* $\Gamma(\mathscr{R})$ *and* $\Gamma(\mathscr{R}')$ *are the Gabriel's quivers of the categories* $\mathscr{R} = \operatorname{rep} \mathscr{P}$ *and* $\mathscr{R}' = \operatorname{rep} \dot{\mathscr{P}}_{(c_z,b)}$, *then*

$$\Gamma(\mathscr{R}) \setminus [P(c_z), T(c_z), T(c_z, c_i) \mid 1 \leqslant i \leqslant n] \simeq \Gamma(\mathscr{R}') \setminus [P(c_z^+)].$$

Remark 6. Note that, algorithm A_z can be also defined for equipped posets with a VII-suitable pair of points. Due that it can be defined in such a way that no action is allowed for the functor on the subspaces U_x associated with points $x \in a^{\triangledown} + b_{\vartriangle}$ in a representation $U \in \operatorname{rep} \mathscr{P}$. However, the interesting case happens whenever $a^{\blacktriangledown} = \varnothing$.

3.2. Categorical properties of the algorithm of differentiation VIII for equipped posets

In this section, we recall the definition of the *algorithm of differentiation* VIII and some of its categorical properties are proved [25].

A pair of weakly comparable points $a \prec b$ of an equipped poset $\mathscr{P}$ is *suitable* for differentiation VIII if $\mathscr{P}$ can be written in the form:
$\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + \Sigma + \{c, a, b\}$,

where Σ is the interior of the completely weak interval $[a, b]$ and c is a strong point incomparable with $[a, b]$.

The *derived poset* of the set $\mathscr{P}$ with respect to such a pair (a, b) is the equipped poset $\mathscr{P}' = \mathscr{P}'_{(a,b)}$,

which is obtained from $\mathscr{P}$ by replacing the point c for a three-point chain $c^- < c^0 < c^+$, where c^-, c^0 are weak points and c^+ is a strong point, $a \prec c^0 \prec b$ and the following conditions are satisfied:

1) each of three points c^-, c^+ and c^0 inherits all the previous order relations of the point c with the points of $\mathscr{P} \setminus \{c\}$;
2) the order relations in the whole set $\mathscr{P}'_{(a,b)}$ are induced by the initial relations in the subset $\mathscr{P} \setminus \{c\}$ and by the aforementioned relations.

The diagram in Figure 9 shows an equipped poset with a pair of points (a,b), VIII-suitable and its corresponding derived poset, in this case $A = a^{\blacktriangledown}$ and $B = b_{\blacktriangle}$:

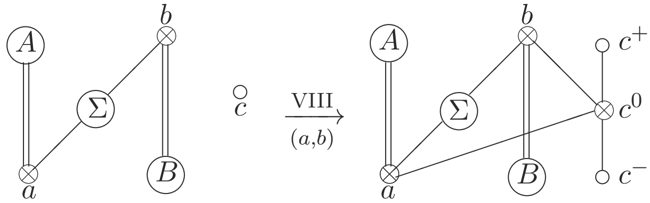


Figure 9. Diagrams of an equipped poset $\mathscr{P}$ and its corresponding derived poset ${\mathscr{P}'}_{(a,b)}$.

Let $\mathscr{P}$ be an equipped poset with a pair of points (a,b), VIII-suitable. The following formulas define the *differentiation functor* $D^{\mathrm{VIII}}_{(a,b)} : \operatorname{rep}\mathscr{P} \longrightarrow \operatorname{rep}\mathscr{P}'_{(a,b)}$, induced by the algorithm of differentiation VIII. Thus, for a representation given $U = (U_0; U_x \mid x \in \mathscr{P}) \in \operatorname{rep}\mathscr{P}$, we get the *derived representation* $U' = (U'_0; U'_x \mid x \in \mathscr{P}'_{(a,b)})$, where:

$$\begin{gathered} U'_0 = U_0, \qquad U'_{c^-} = U_c \cap \widetilde{U_b^-}, \\ U'_{c^+} = U_c + F(U_a), \qquad U'_{c^0} = U_a + U_c \cap U_b, \\ U'_x = U_x, \quad \text{for the remaining points } x \in \mathscr{P}', \\ \varphi' = \varphi, \quad \text{for all F linear map} - \text{morphism } \varphi : U_0 \to V_0. \end{gathered} \tag{21}$$

Note that, the following identities hold for indecomposable representations of an equipped poset with a pair of points (a,b) VIII-suitable.

$$T'(a) = G'_1(a,c) = G'_2(a,c) = T(a).$$

Lemmas 9 and 10 below were proved by Zavadskij in [25].

Let $U^{\downarrow}$ be a *reduced* (i.e., without direct summands of type $T(a)$, $G_1(a,c)$, $G_2(a,c)$) representations of a poset $\mathscr{P}'_{(a,b)}$ for which $U' = U^{\downarrow} \oplus T^m(a)$, where $2m = \dim(U_a^+ + U_b^-)/U_b^-$. In this case, if (E_0, W_0) is a (U_a^+, U_b^-)-cleaving pair, then $U^{\downarrow} = W$, where $W_x = U'_x \cap \widetilde{W_0}$. In this case, $U^{\downarrow}$ is a representation of the completed (by the relation $a \lhd b$) derived poset $\overline{\mathscr{P}}'_{(a,b)}$. Obviously, $T^{\downarrow}(a) = G_1^{\downarrow}(a,c) = G_2^{\downarrow}(a,c) = 0$.

Lemma 9. *For each representation* $W \in \operatorname{rep} \overline{\mathscr{P}}'_{(a,b)}$ *there exists a representation* $W^{\uparrow} \in \operatorname{rep} \mathscr{P}$*, such that* $(W^{\uparrow})' \simeq W \oplus T^m(a)$*, for some* $m \geqslant 0$*.*

Lemma 10. *In the case of the differentiation* VIII*, the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections* $\operatorname{Ind} \mathscr{P} \setminus [T(a), G_1(a,c), G_2(a,c)] \rightleftarrows \operatorname{Ind} \overline{\mathscr{P}}'_{(a,b)} = \operatorname{Ind} \mathscr{P}' \setminus [T(a)]$.

The following lemma characterizes the ideal $\mathscr{I} = \langle T(a), G_1(a,c), G_2(a,c)\rangle \subset \operatorname{rep} \mathscr{P}$, where $\mathscr{P}$ is an equipped poset with a pair of points (a,b), VIII-suitable.

Lemma 11. *If* $U = (U_0; U_x \mid x \in \mathcal{P})$ *and* $V = (V_0; V_x \mid x \in \mathscr{P})$ *are representations of an equipped poset* $\mathscr{P}$ *with a pair of points* (a,b)*,* VIII-*suitable, then the following equivalences hold for a linear map* $\varphi : U_0 \to V_0$*:*

1) $\varphi \in \langle T(a)\rangle$ *if and only if* $\varphi \in [(U_b + U_c)^-, V_a^+]$, $\quad \widetilde{\varphi}(U_b) \subset V_a$.
2) $\varphi \in \langle G_1(a,c)\rangle$ *if and only if* $\varphi \in [U_b^-, V_a^+]$, $\quad \widetilde{\varphi}(U_b) \subset V_a$, $\quad \widetilde{\varphi}(U_c) \subset V_c$.
3) $\varphi \in \langle G_2(a,c)\rangle$ *if and only if* $\varphi \in [U_b^-, V_a^- \cap V_c^-]$, $\quad \operatorname{Im}\widetilde{\varphi} \subset \widetilde{V_a^-} \cap \widetilde{V_c^-}$.

Proof. It is enough to assume $U_b^+ = U_0 \neq 0$. We also assume $V_a^+ \neq 0$ throughout the proof. Furthermore, we adopt the following partitions of spaces U_b and V_a: $U_b = \widetilde{U_b^-} \oplus N_b$; $V_a = \widetilde{V_a^-} \oplus M_a \oplus N_a$, where $M_a = \{v = e_\alpha + \boldsymbol{\xi} e_\beta \in V_a \mid v \in \widetilde{V_b^-}\}$.

If $\varphi \in [(U_b + U_c)^-, V_a^+]$ with $\widetilde{\varphi}(U_b) \subseteq V_a$, then: $\widetilde{\varphi}(U_x) \subseteq F(V_a) \subseteq V_x$, if $x \in a^{\triangledown}$; $\widetilde{\varphi}(U_x) \subseteq \widetilde{\varphi}(U_b) \subseteq V_a \subseteq V_x$, for any point $x \in a^{\curlyvee}$.

Since $\widetilde{\varphi}(U_c) = 0$, the arguments described above allow us to conclude that $\varphi \in \operatorname{rep} \mathscr{P}$.

This part of the proof can be finished by considering the cases for which $\widetilde{U_b^-} = 0$ or $N_b = 0$.

If $\widetilde{U_b^-} = 0$ and $N_b \neq 0$, then $U_0 = N_b^+$ and $\dim_G N_b = m$. Therefore, it is possible to define a representation $W \in \operatorname{rep} \mathscr{P}$ such that $W_0 = N_b^+$.

$$W_x = \begin{cases} F(N_b) & \text{if } x \in a^{\triangledown}, \\ N_b & \text{if } x \in a^{\curlyvee}, \\ 0 & \text{otherwise.} \end{cases}$$

We also define linear maps $f_0 : N_b^+ \to W_0$, $f_1 : W_0 \to V_0$ such that: $f_0(v) = v$ for all $v \in N_b^+$ and $f_1 = \varphi$. Since $W \simeq T_a^m$ then $\varphi_1 = U \xrightarrow{f_0} W \xrightarrow{g_0} T^m(a) \in \operatorname{rep} \mathscr{P}$, $\quad \varphi_2 = T^m(a) \xrightarrow{g_0^{-1}} W \xrightarrow{f_1} V \in \operatorname{rep} \mathscr{P}$ and $\varphi_2 \varphi_1 = \varphi$, where $g_0 : W \to T^m(a)$ is an isomorphism.

In the case $N_b = 0$, we observe that $\varphi = 0$. Thus $\varphi \in [(U_b + U_c)^-, V_a^+]$ and $\widetilde{\varphi}(U_b) \subseteq V_a$ imply $\varphi \in \langle T(a) \rangle$.

On the other hand, if $\varphi \in \langle T(a) \rangle$ then there exist morphisms $\varphi_1 : U \to T^m(a) \in \operatorname{rep} \mathscr{P}$ and $\varphi_2 : T^m(a) \to V \in \operatorname{rep} \mathscr{P}$, such that $\varphi = \varphi_2 \varphi_1$. Since, $\widetilde{\varphi_1}(U_b) \subseteq T_a^m(a)$ then $\varphi_1((U_b + U_c)^-) \subseteq (T_a^m(a))^-$, in particular, $\varphi_1(U_c^+) = \varphi_1(U_b^-) = 0$. Therefore, $\varphi((U_b + U_c)^-) = 0$ thus $(U_b + U_c)^- \subseteq \operatorname{Ker} \varphi$. Furthermore, since $T_a^m(a) = T_b^m(a)$ with $(T_a^m(a))^+ = F^{2m}$ it follows $\widetilde{\varphi_2}(T_b^m) \subseteq V_a$. Therefore, $\widetilde{\varphi}(U_b) = \widetilde{\varphi_2}(\widetilde{\varphi_1}(U_b)) \subseteq \widetilde{\varphi_2}(T_b^m) \subseteq V_a$, thus, $\operatorname{Im} \varphi \subseteq V_a^+$. With this argument, we conclude $\varphi \in [(U_b + U_c)^-, V_a^+]$ and $\widetilde{\varphi}(U_b) \subseteq V_a$ if and only if $\varphi \in \langle T(a) \rangle$.

Arguments used above with the additional condition $\widetilde{\varphi}(U_c) \subseteq V_c$ allow us to conclude the second item, i.e., $\varphi \in [U_b^-, V_a^+]$ and $\widetilde{\varphi}(U_b) \subseteq V_a$ if and only if $\varphi \in \langle G_1(a,c) \rangle$.

The following arguments prove the third item.

If $\varphi \in \langle G_2(a,c) \rangle$ then there exist morphisms $\varphi_1 : U \to G_2^m(a,c) \in \operatorname{rep} \mathscr{P}$ and $\varphi_2 : G_2^m(a,c) \to V \in \operatorname{rep} \mathscr{P}$, such that $\varphi = \varphi_2 \varphi_1$. Therefore, $\varphi_2 \varphi_1(U_b^-) = \varphi_2(\varphi_1(U_b^-)) = 0$, due that $\varphi_1(U_b^-) \subseteq ((G_2^m(a,c))_b)^- = 0$. Furthermore, $\widetilde{\varphi}(N_b) = \widetilde{\varphi_2}\widetilde{\varphi_1}(N_b) \subseteq V_a \cap \widetilde{V_c^-} = \widetilde{V_a^-} \cap \widetilde{V_c^-}$ thus $\operatorname{Im}\varphi \subseteq V_a^- \cap V_c^-$.

On the other hand, if $\varphi \in [U_b^-, V_a^- \cap V_c^-]$ and $\operatorname{Im}\widetilde{\varphi} \subseteq \widetilde{V_a^-} \cap \widetilde{V_c^-}$, then $\widetilde{\varphi}(U_x) \subseteq \widetilde{V_a^-} \cap \widetilde{V_c^-} \subseteq F(V_a) \subseteq V_x$ if $x \in a^{\triangledown}$.

$$\widetilde{\varphi}(U_x) \subseteq \widetilde{V_a^-} \cap \widetilde{V_c^-} \subseteq V_x \quad \text{if} \quad x \in a^{\curlyvee}.$$

Finally, $\widetilde{\varphi}(U_c) \subseteq \widetilde{V_a^-} \cap \widetilde{V_c^-} \subseteq V_c$, therefore, $\varphi \in \operatorname{rep} \mathscr{P}$.

Now we can use arguments as above to find out morphisms $\varphi_1 : U \to G_2^m(a,c), \varphi_2 : G_2^m(a,c) \to V \in \operatorname{rep} \mathscr{P}$, such that $\varphi = \varphi_2 \varphi_1$. Note that, the representation $W \simeq G_2^m(a,c)$ defined for the case $U_b^- = 0$, $N_b \neq 0$ has the form $(W_0; W_x \mid x \in \mathscr{P})$, where $W_0 = N_b^+$ and

$$W_x = \begin{cases} F(N_b) & \text{if } x \in a^{\triangledown} + c^{\triangledown}, \\ N_b & \text{if } x \in a^{\curlyvee}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\varphi \in [U_b^-, V_a^- \cap V_c^-]$ and $\operatorname{Im}\widetilde{\varphi} \subseteq \widetilde{V_a^-} \cap \widetilde{V_c^-}$ if and only if $\varphi \in \langle G_2(a,c) \rangle$. □

The following lemma can be proved by using arguments described in the proof of Lemma 11.

Lemma 12. *If U' and V' are representations of a poset $\mathscr{P}'_{(a,b)}$ and $\varphi : U_0 \to V_0$ is a linear morphism, then $\varphi \in [U_b^-, V_a^+]$ and $\widetilde{\varphi}(U_b) \subseteq V_a$ if and only if $\varphi \in \langle T(a) \rangle$ in $\operatorname{rep}\mathscr{P}'$.*

Remark 7. Denote by $\mathscr{R} = \operatorname{rep}\mathscr{P}$ and $\mathscr{R}' = \operatorname{rep}\mathscr{P}'$ the categories of representations associated with the equipped posets $\mathscr{P}$ and $\mathscr{P}'_{(a,b)}$, respectively. Due to the fact that $\varphi' = \varphi$, we obtain the natural inclusions $\mathscr{R}(U,V) \subset \mathscr{R}'(U',V')$ for all objects $U, V \in \mathscr{R}$. $\mathscr{I}$ denotes the ideal in the category $\mathscr{R}$ consisting of morphisms which pass through the objects $T(a)$, $G_1(a,c)$ and $G_2(a,c)$. $\mathscr{I}'$ denotes the ideal in the category $\mathscr{R}'$ consisting of morphisms which pass through the object $T(a)$. Taking into account that $T'(a) = G'_1(a,c) = G'_2(a,c) = T(a)$, we get also inclusions $\mathscr{I}(U,V) \subset \mathscr{I}'(U',V')$ for all objects $U, V \in \mathscr{R}$. Thus, for each pair of representations $U, V \in \mathscr{R}$ it is possible to obtain the lattice of subspaces shown in Figure 10.

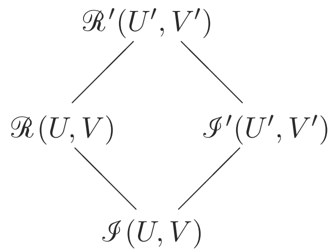


Figure 10. The lattice associated with the ideals $\mathscr{I}$, $\mathscr{I}'$ and categories $\mathscr{R}$, $\mathscr{R}'$ defined by the differentiation VIII.

Lemma 13. *Let U, V be an arbitrary pair of representations in $\mathscr{R}$. Then, the following identity holds.*

$$\mathscr{R}(U,V) \cap \mathscr{I}'(U',V') = \mathscr{I}(U,V).$$

Proof. The Remark 7 allows us to conclude $\mathscr{I}(U,V) \subseteq \mathscr{R}(U,V) \cap \mathscr{I}'(U',V')$. So, it is enough to prove $\mathscr{R}(U,V) \cap \mathscr{I}'(U',V') \subseteq \mathscr{I}(U,V)$ in order to obtain the identity proposed. To do that, we suppose that a morphism $\psi : U_0 \to V_0 \in \mathscr{R}(U,V) \cap \mathscr{I}'(U',V')$ and define the following partition for the space U_0:

$$U_0 = (U_c^+ \cap U_b^-) \oplus T_b^- \oplus (U_c^+ \cap N_b^+) \oplus T_b^+ \oplus T_c^+ \oplus W_0,$$

where $T_b^- \subseteq U_b^-$, $T_b^- \cap U_c^+ = 0$, $T_b^+ \subseteq N_b^+$, $T_b^+ \cap U_c^+ = 0$, $T_c^+ \cap U_b^+ = 0$ and W_0 is a complementary subspace in U_0.

Since by Lemma 11, $\mathrm{Im}\psi \subseteq V_a^+$, we can assume $V_a^+ = V_0$ and define a partition of the form:

$$V_a^+ = (V_a^- \cap V_c^-) \oplus T_a^- \oplus (V_c^+ \cap N_a^+) \oplus T_a^+,$$

where $T_a^- \subseteq V_a^-$, $T_a^- \cap V_c^+ = 0$, $T_a^+ \subseteq N_a^+$ and $T_a^+ \cap V_c^+ = 0$.

Lemma 12 allows building the following linear maps induced by ψ, and by the partition of the spaces U_0 and V_a^+:

$$\psi_1 = e_{(V_a^- \cap V_c^-)}\psi e_{(U_c^+ \cap N_b^+)}, \qquad \psi_2 = e_{(V_c^- \cap N_a^+)}\psi e_{(U_c^+ \cap N_b^+)}, \tag{22}$$

$$\begin{aligned} \psi_3 &= e_{(V_c^- \cap V_a^-)}\psi e_{(T_b^+)}, & \psi_4 &= e_{(T_a^-)}\psi e_{(T_b^+)}, \\ \psi_5 &= e_{(V_c^- \cap N_a^+)}\psi e_{(T_b^+)}, & \psi_6 &= e_{(T_a^+)}\psi e_{(T_b^+)}, \end{aligned} \tag{23}$$

$$\begin{aligned} \psi_7 &= e_{(V_c^- \cap V_a^-)}\psi e_{(T_c^+)}, & \psi_8 &= e_{(T_a^-)}\psi e_{(T_c^+)}, \\ \psi_9 &= e_{(V_c^- \cap N_a^+)}\psi e_{(T_c^+)}, & \psi_{10} &= e_{(T_a^+)}\psi e_{(T_c^+)}, \end{aligned} \tag{24}$$

$$\begin{aligned} \psi_{11} &= e_{(V_c^- \cap V_a^-)}\psi e_{(W_0)}, & \psi_{12} &= e_{(T_a^-)}\psi e_{(T_c^+)}, \\ \psi_{13} &= e_{(V_c^- \cap N_a^+)}\psi e_{(W_0)}, & \psi_{14} &= e_{(T_a^+)}\psi e_{(W_0)}, \end{aligned} \tag{25}$$

Then Lemma 11 allows concluding that $\psi_1, \psi_3, \psi_7, \psi_8, \psi_{11} \in \langle G_2(a,c)\rangle$, $\psi_2, \psi_5, \psi_9, \psi_{10} \in \langle G_1(a,c)\rangle$, and $\psi_4, \psi_6, \psi_{12}, \psi_{13}, \psi_{14} \in \langle T(a)\rangle$. As $\mathscr{I} = \langle T(a), G_1(a,c), G_2(a,c)\rangle_{\mathscr{R}}$, thus $\psi = \sum_{i=1}^{14}\psi_i \in \mathscr{I}(U,V)$. Therefore,

$$\mathscr{R}(U,V) \cap \mathscr{I}'(U',V') = \mathscr{I}(U,V). \qquad \square$$

Lemma 14. *Let U, V be an arbitrary pair of representations in $\mathscr{R}$. Then, the following identity holds.*

$$\mathscr{R}(U,V) + \mathscr{I}'(U',V') = \mathscr{R}'(U',V').$$

Proof. The Remark 7 allows us to conclude that $\mathscr{R}(U,V) + \mathscr{I}'(U',V') \subseteq \mathscr{R}'(U',V')$. In order to prove the equality, we proceed as follows:

From definition of the functor $D_{(a,b)}^{\mathrm{VIII}}$, we can note that for $\varphi' \in \mathscr{R}'(U',V')$, and for $x \in \{A \cup B \cup \Sigma \cup \{a,b\}\} \subset \mathscr{P}$, $\widetilde{\varphi}'(U_x) \subset V_x$, then $\widetilde{\varphi}(U_x) \subset V_x$. Therefore, for $x \in \mathscr{P} \setminus \{c\}$ and $\varphi \in \mathscr{R}$, $\widetilde{\varphi}(U_x) \subset V_x$, and $\widetilde{\varphi}(U_c) \subset V_c + F(V_a) \nsubseteq V_c$, then in general $\varphi \notin \mathscr{R}$ and $\mathscr{R}'(U',V') \nsubseteq \mathscr{R}(U,V)$.

The following procedure allows obtaining a morphism $\varphi \in \mathscr{R}(U,V)$ from a morphism $\psi \in \mathscr{R}'(U',V')$. To get this morphism, we need to do a partition of the vector space U_0, as follows.

$$U_0 = U_b^- \cap U_c^+ \oplus N_b^+ \cap U_c^+ \oplus T_c^+ \oplus T_b^- \oplus T_b^+ \oplus W_0,$$

where $T_b^- \subseteq U_b^-$, $T_b^- \cap U_c^+ = 0$, $T_b^+ \subseteq N_b^+$, $T_b^+ \cap U_c^+ = 0$, $T_c^+ \subseteq U_c^+$, $T_c^+ \cap U_b^+ = 0$, W_0 is a complementary subspace of T_c^+ in U_0. Actually, this partition is induced by the (U_c^+, U_b^+)-cleaving pair (T_c^+, W_0). Furthermore, $T_b^+ = T_{b_1}^+ \oplus T_{b_2}^+$.

We assume $e_\gamma \in T_{b_1}^+$, if there exists $e_\delta \in N_b^+ \cap U_c^+$, such that $e_\gamma + \boldsymbol{\xi} e_\delta \in N_x$ for some $x \in U_{b_\curlywedge \setminus \{b\}}$. In this case, $T_{b_2}^+$ is a complementary subspace.

The following partition of the space V_0 is induced by the (V_c^+, V_a^+)-cleaving pair (X_c^+, Y_0).

$$V_0 = V_a^- \cap V_c^+ \oplus X_a^- \oplus X_c^+ \oplus N_b^+ \cap V_c^+ \oplus X_a^+ \oplus Y_0,$$

where $X_a^- \subseteq V_a^-$, $X_c^+ \subseteq V_c^+$, $X_a^+ \subseteq N_a^+$ and Y_0 is a complementary subspace.

Note that $X_a^+ = (X_a^+)_1 \oplus (X_a^+)_2$, where if $N_a = G\{v = e_{\gamma_j} + \boldsymbol{\xi} e_{\delta_j}\}_{1 \leqslant j \leqslant k}$, for some positive integer k, then $(N_a^+)_1 = F\{e_{\gamma_j}\}$, $(N_a^+)_2 = F\{e_{\delta_j}\}$.

We use the same notation for any subspace N_x associated with a point $x \in \mathscr{P}^\otimes$. Furthermore, if X is a subspace of a F-vector space with a fixed basis $\{e_1, e_2, \ldots, e_t\}$, then a vector of the form $\gamma_1 e_1 + \gamma_2 e_2 + \cdots + \gamma_t e_t$ will be denoted $\{\gamma_r\}_X$, $1 \leqslant r \leqslant t$. Therefore, if $v = e_\gamma + \boldsymbol{\xi} e_\delta \in \widetilde{U_0}$ and $\psi : U_0 \to V_0 \in \mathscr{R}'(U',V')$, then $\psi(e_\gamma)$ and $\psi(e_\delta)$ can be written in the following form for suitable sets of indexes:

$$\begin{aligned}
\psi(e_\gamma) &= \{\gamma_i\}_{V_a^- \cap V_c^-} + \{\delta_j\}_{X_a^-} + \{\gamma_k\}_{X_c^+} + \{\delta_l\}_{N_a^+ \cap V_c^+} + \{\varepsilon_m^1\}_{(X_a^+)_1} \\
&\quad + \{\varepsilon_m^2\}_{(X_a^+)_2} + \{\eta_n\}_{Y_0}, \\
\psi(e_\delta) &= \{\gamma_i'\}_{V_a^- \cap V_c^-} + \{\delta_j'\}_{X_a^-} + \{\gamma_k'\}_{X_c^+} + \{\delta_l'\}_{N_a^+ \cap V_c^+} + \{\varepsilon_m'^1\}_{(X_a^+)_1} \\
&\quad + \{\varepsilon_m'^2\}_{(X_a^+)_2} + \{\eta_n'\}_{Y_0},
\end{aligned}$$

with ε_m^1, $\varepsilon_m'^1$, ε_m^2, $\varepsilon_m'^2 \in F$.

Let $w_1, w_2 : U_0 \to V_0$ be linear maps induced by ψ, defined in such a way that:

If e_γ is a vector of a fixed basis of $N_b^+ \cap U_c^+$, then:

$$w_1(e_\gamma) = \{\delta_j\}_{X_a^-} + \{\varepsilon_m^1\}_{(X_a^+)_1} + \{\varepsilon_m^2\}_{(X_a^+)_2}.$$

If $v = e_\gamma + \boldsymbol{\xi} e_\delta$ belongs to a fixed basis of N_b, with $e_\delta \in T^+_{b_1}$, then:

$$w_1(e_\delta) = \{\delta'_j\}_{X^-_a} + \{\varepsilon'^1_m\}_{(X^+_a)_1} + \{\varepsilon'^2_m\}_{(X^+_a)_2},$$

where if $\boldsymbol{\xi}^2 + \alpha\boldsymbol{\xi} + \beta = 0$, then

$$\begin{aligned} \varepsilon'^1_m &= -\frac{1}{\beta}\varepsilon^2_m, \quad \text{for each } m, \\ \varepsilon'^2_m &= \varepsilon^1_m + \frac{\alpha}{\beta}\varepsilon^2_m, \quad \text{for each } m, \end{aligned} \tag{26}$$

$w_1(t) = 0$ for the other basic vectors $t \in U_0$.

$w_2(e_\gamma) = \{\delta_j\}_{X^-_a} + \{\varepsilon^1_m\}_{(X^+_a)_1} + \{\varepsilon^2_m\}_{(X^+_a)_2}$ if e_γ is a vector of a fixed basis of T^+_c,

$w_2(t) = 0$ for the other basic vectors $t \in U_0$.

Note that, $w_1, w_2 : U_0 \to V_0 \in [U^-_b, V^+_a]$, with $\widetilde{w_1}(U_b) \subseteq V_a$ and $\widetilde{w_2}(U_b) \subseteq V_a$. Thus, $w = w_1 + w_2 \in \langle T(a)\rangle_{\mathscr{R}'}$, therefore, by Lemma 12, $w \in \mathscr{I}'(U', V')$.

If $U = (U_0; U_x \mid x \in \mathscr{P})$ is a representation of an equipped poset $\mathscr{P}$, then:

if $x \in a^\triangledown$ then $(\widetilde{\psi} - \widetilde{w})(U_x) = \widetilde{\psi}(U_x) - \widetilde{w}(U_x) \subseteq V_x + F(V_a) = V_x$;

if $x \in a^\curlyvee$ then $(\widetilde{\psi} - \widetilde{w})(U_x) = \widetilde{\psi}(U_x) - \widetilde{w}(U_x) \subseteq U_x + V_a = V_x$;

if $x \in b_\vartriangle$ then $\widetilde{w}(U_x) = 0$ and $(\widetilde{\psi} - \widetilde{w})(U_x) = \widetilde{\psi}(U_x) \subset V_x$.

$(\widetilde{\psi} - \widetilde{w})(U_c) = (\widetilde{\psi} - \widetilde{w})(U^+_c \cap U_b \oplus N^+_b \cap U^+_c \oplus T^+_c) \subseteq V_c$. Therefore, $\varphi = \psi - w \in \mathscr{R}(U, V)$, and $\psi = \varphi + w \in \mathscr{R}(U, V) + \mathscr{I}'(U', V')$, hence $\mathscr{R}'(U', V') = \mathscr{R}(U, V) + \mathscr{I}'(U', V')$. □

Lemma 15. *Let $\mathscr{P}$ be an equipped poset with a pair of points (a, b),* VIII-*suitable. Then, the functor $D^{\mathrm{VIII}}_{(a,b)} \colon \operatorname{rep}\mathscr{P} \to \operatorname{rep}\mathscr{P}'_{(a,b)}$, defined by formulas (21), induces an equivalence between quotient categories:*

$$\mathscr{R}/\mathscr{I} \xrightarrow{\sim} \mathscr{R}'/\mathscr{I}',$$

where $\mathscr{R} = \operatorname{rep}\mathscr{P}$, $\mathscr{R}' = \operatorname{rep}\mathscr{P}'_{(a,b)}$, $\mathscr{I} = \langle T(a), G_1(a,c), G_2(a,c)\rangle_{\mathscr{R}}$ and $\mathscr{I}' = \langle T(a)\rangle_{\mathscr{R}'}$.

Proof. The density of the functor $D^{\mathrm{VIII}}_{(a,b)}$ is guaranteed by Lemmas 9 and 10. Besides, Lemmas 13 and 14 allow us to conclude that the functor $D^{\mathrm{VIII}}_{(a,b)}$ is faithful and full, respectively. □

The following result holds as a direct consequence of Lemmas 9, 10 and 15.

Corollary 4. *If $\Gamma(\mathscr{R})$ and $\Gamma(\mathscr{R}')$ are the Gabriel's quivers of the categories $\mathscr{R}$ and $\mathscr{R}'$, then $\Gamma(\mathscr{R}) \setminus [T(a), G_1(a,c), G_2(a,c)] \simeq \Gamma(\mathscr{R}') \setminus [T(a)]$.*

3.3. Categorical properties of the algorithm of differentiation IX for equipped posets

In this section, we present the definition of the *algorithm of differentiation* IX giving a proof of some of its categorical properties [25].

A pair of comparable weak points $a \prec b$ of an equipped poset $\mathscr{P}$ is called IX-*suitable* if $\mathscr{P}$ can be written in the form:

$$\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + \Sigma + \{p, a, b\},$$

where Σ is the interior of the completely weak interval $[a, b]$ and p is a weak point incomparable with a, and $p \prec b$ [25].

The *derived poset* of the set $\mathscr{P}$, with respect to the pair (a, b), is the equipped poset $\mathscr{P}' = \mathscr{P}'_{(a,b)}$, obtained from $\mathscr{P}$ by replacing the point p by a weak two-point chain $p^- \prec p^+$ with the additional relations $a \prec p^+ \prec b$ and $p^- \lhd b$ (plus all the induced relations). The points p^-, p^+ inherits all the previous order relations of the point p with the points in $\mathscr{P} \setminus \{p\}$.

The following diagram shows an equipped poset with a pair of points (a, b), IX-*suitable*, and its corresponding derived poset:

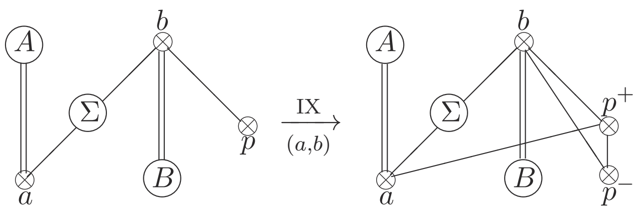


FIGURE 11. Diagrams of an equipped poset $\mathscr{P}$ and its corresponding derived poset $\mathscr{P}'_{(a,b)}$.

Let $\mathscr{P}$ be an equipped poset with a pair of points (a, b), IX-suitable. The following formulas define the *differentiation functor* $D^{\mathrm{IX}}_{(a,b)} : \operatorname{rep} \mathscr{P} \longrightarrow \operatorname{rep} \mathscr{P}'_{(a,b)}$ induced by the algorithm of differentiation IX. Thus for a given representation $U = (U_0; U_x \mid x \in \mathscr{P}) \in \operatorname{rep} \mathscr{P}$, we get the *derived representation* $U' = (U'_0; U'_x \mid x \in \mathscr{P}'_{(a,b)})$:

$$\begin{gathered} U'_0 = U_0, \qquad U'_{p^-} = U_p \cap \widetilde{U_b^-}, \qquad U'_{p^+} = U_p + U_a, \\ U'_x = U_x, \quad \text{for the remaining points.} \\ \varphi' = \varphi, \quad \text{for all } F \text{ linear map-morphism } \varphi : U_0 \to V_0. \end{gathered} \tag{27}$$

Note that, for the functor $D^{\mathrm{IX}}_{(a,b)}$ and for indecomposable representations $T(a)$ and $T(a, p)$, we have $T'(a) = T'(a, p) = T(a)$. The following arguments were used by Zavadskij in order to describe the integration procedure for the algorithm IX [25].

Representations U in rep$\mathscr{P}$ without direct summands $T(a)$ and $T(a,p)$ will be called *reduced*. A reduced representation $U^{\downarrow}$, for which $U' = U^{\downarrow} \oplus T^m(a)$, is defined evidently, analogously to the previous cases. Take some complementing pair of subspaces (E_0, W_0) in U_0, with respect to the pair (U_a^+, U_b^-), and set $U^{\downarrow} = W$, where $W_x = U'_x \cap W_0$ ($W_x = U'_x \cap \widetilde{W_0}$) for a strong (weak) point $x \in \mathscr{P}'$. Obviously, $T^{\downarrow}(a) = T^{\downarrow}(a,p) = 0$.

The representation $U^{\downarrow}$ does not depend, up to isomorphism, on the choice of E_0 and W_0 and, due to the inclusion $W_a^+ \subset W_b^-$, is a representation of the set $\overline{\mathscr{P}}'_{(a,b)}$ completed by the relation $a \lhd b$.

Lemma 16. *For each representation $W \in \operatorname{rep} \overline{\mathscr{P}}'_{(a,b)}$ there exists a representation $W^{\uparrow} \in \operatorname{rep}\mathscr{P}$ such that $(W^{\uparrow})' \simeq W \oplus T^m(a)$, for some $m \geqslant 0$.*

Lemma 17. *In the case of the differentiation* IX*, the operations $\downarrow$ and $\uparrow$ induce mutually inverse bijections*

$$\operatorname{Ind}\mathscr{P} \setminus [T(a), T(a,p)] \rightleftarrows \operatorname{Ind}\overline{\mathscr{P}}'_{(a,b)} = \operatorname{Ind}\mathscr{P}' \setminus [T(a)].$$

The following lemma characterizes morphisms which pass through the objects from the ideal $\mathscr{I} = \langle T(a), T(a,p)\rangle \subset \operatorname{rep}\mathscr{P}$, where $\mathscr{P}$ is an equipped poset with a pair of points (a,b), IX-suitable. In Lemmas 18, 19, 20, and 21, we assume the following partitions for the subspaces U_x, $x \in a^{\curlyvee}$:

$U_x = \widetilde{U_x^-} \oplus M_x \oplus N_x$, for all $x \in a^{\curlyvee} \setminus \{b\}$, $\quad M_x \subset \widetilde{U_b^-}$, $\quad M_x \cap U_x^- = 0$, for all $x \in a^{\curlyvee} \setminus \{b\}$, $\quad M_b = \sum\limits_{x \in a^{\curlyvee}\setminus\{b\}} M_x$, $\quad \widetilde{U_b^-} = \widetilde{H_b} \oplus M_b$, $\quad N_x \cap U_b^- = 0$, for all $x \in a^{\curlyvee}$.

Lemma 18. *If $U = (U_0; U_x \mid x \in \mathscr{P})$ and $V = (V_0; V_x \mid x \in \mathscr{P})$ are representations of an equipped poset $\mathscr{P}$ with a pair of points (a,b),* IX-*suitable, then the following equivalences hold for a linear map $\varphi : U_0 \to V_0$:*

1) *$\varphi \in \langle T(a)\rangle$ if and only if $\varphi \in [H_b, V_a^+]$, $\widetilde{\varphi}(U_b) \subset V_a$;*
2) *$\varphi \in \langle T(a,p)\rangle$ if and only if $\varphi \in [H_b, V_a^+ \cap V_p^+]$, $\widetilde{\varphi}(U_b) \subset V_a \cap V_p$.*

Proof. In order to prove the first item, it is enough to adapt arguments used to prove the first item of Lemma 11. In fact, the same arguments can be used if $M_b = 0$.

For the second item, we assume $U_b^+ = U_0 \neq 0$.

If $\varphi \in [H_b, V_a^+ \cap V_p^+]$, with $\widetilde{\varphi}(U_b) \subseteq V_a \cap V_p$ then:

$\widetilde{\varphi}(U_x) \subseteq F(V_a) \cap F(V_p) \subseteq F(V_a) \subseteq V_x$, if $x \in a^{\triangledown}$;

$\widetilde{\varphi}(U_x) \subseteq \widetilde{\varphi}(U_b) \subseteq V_a \cap V_p \subseteq V_a \subseteq V_x$, for any point $x \in a^{\curlyvee}$.

Since $\widetilde{\varphi}(U_p) \subseteq \widetilde{\varphi}(U_b) \subseteq V_a \cap V_p \subseteq V_p$, the arguments described above allow us to conclude that $\varphi \in \operatorname{rep}\mathscr{P}$.

This part of the proof can be finished by considering the cases in which $N_b = 0$ or $N_b \neq 0$ in U_0.

If $\widetilde{U_b^-} = 0$ and $N_b \neq 0$, then $U_0 = N_b^+$ and $\dim_G N_b = m$, for some $m > 0$. Therefore, it is possible to define a representation $W \in \operatorname{rep}\mathscr{P}$ such that $W_0 = N_b^+$ and

$$W_x = \begin{cases} F(N_b) & \text{if } x \in a^{\triangledown}, \\ N_b & \text{if } x \in b_{\curlywedge}, \\ 0 & \text{otherwise.} \end{cases}$$

We also define the linear maps $f_0 : N_b^+ \to W_0$ and $f_1 : W_0 \to V_0$ such that: $f_0(v) = v$ for all $v \in N_b^+$ and $f_1 = \varphi$.

Since $W \simeq T^m(a,p)$, then $\varphi_1 = U \xrightarrow{f_0} W \xrightarrow{g_0} T^m(a,p) \in \operatorname{rep}\mathscr{P}$, $\varphi_2 = T^m(a,p) \xrightarrow{g_0^{-1}} W \xrightarrow{f_1} V \in \operatorname{rep}\mathscr{P}$ and $\varphi_2\varphi_1 = \varphi$, where $g_0 : W \to T^m(a,p)$ is an isomorphism.

If $N_b = M_b = 0$ in U_0 or $N_a^+ \cap N_p^+ = 0$ in V_0, we note that $\varphi = 0$.

If $H_b = N_b = 0$ in U_0, we define a representation $W = (W_0; W_x \mid x \in \mathscr{P})$ such that $W_0 = \sum_{x \in b_{\curlywedge}} M_x^+$ and:

$$W_x = \begin{cases} F\left(\sum_{x \in b_{\curlywedge}} M_x\right) & \text{if } x \in a^{\triangledown}, \\ \sum_{x \in b_{\curlywedge}} M_x & \text{if } x \in b_{\curlywedge}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\dim \widetilde{W_0} = m$, then $W \simeq T^m(a,p)$. Therefore, we can apply the arguments used above to find morphisms $\varphi_1, \varphi_2 \in \operatorname{rep}\mathscr{P}$ such that $\varphi = \varphi_2\varphi_1$.

Thus, $\varphi \in [H_b, V_a^+ \cap V_p^+]$ and $\widetilde{\varphi}(U_b) \subseteq V_a \cap V_p$ implies $\varphi \in \langle T(a,p) \rangle$.

On the other hand, if $\varphi \in \langle T(a,p) \rangle$, there exist morphisms $\varphi_1 : U \to T^m(a,p) \in \operatorname{rep}\mathscr{P}$ and $\varphi_2 : T^m(a,p) \to V \in \operatorname{rep}\mathscr{P}$ such that $\varphi = \varphi_2\varphi_1$, for some $m > 0$. Since, $\widetilde{\varphi_1}(U_b) \subseteq T_a^m(a,p)$, then $\varphi_1(H_b) \subseteq (T_a^m(a,p))^-$, in fact, $\varphi_1(H_b) = 0$. Therefore, $\varphi(H_b) = 0$, thus $H_b \subseteq \operatorname{Ker}\varphi$. Furthermore, since $T_a^m(a,p) = T_b^m(a,p) = T_p^m(a,p)$ with $(T_a^m(a,p))^+ = F^{2m}$ it follows $\widetilde{\varphi_2}(T_b^m) \subseteq V_a \cap V_p$, therefore $\widetilde{\varphi}(U_b) = \widetilde{\varphi_2}(\widetilde{\varphi_1}(U_b)) \subseteq \widetilde{\varphi_2}(T_b^m) \subseteq V_a \cap V_p$ and $\operatorname{Im}\varphi \subseteq V_a^+ \cap V_p^+$. With this argument, we conclude $\varphi \in [H_b, V_a^+ \cap V_p^+]$ and $\widetilde{\varphi}(U_b) \subseteq V_a \cap V_p$ if and only if $\varphi \in \langle T(a,p) \rangle$. We are done. □

The following lemma can be proved by using arguments described in the proof of Lemma 18.

Lemma 19. *If U' and V' are representations of a poset $\mathscr{P}'_{(a,b)}$ and $\varphi : U_0 \to V_0$ is a linear morphism, then $\varphi \in [H_b, V_a^+]$ and $\widetilde{\varphi}(U_b) \subseteq V_a$ if and only if $\varphi \in \langle T(a) \rangle$ in $\operatorname{rep}\mathscr{P}'$.*

Remark 8. Denote by $\mathscr{R} = \operatorname{rep}\mathscr{P}$ and $\mathscr{R}' = \operatorname{rep}\mathscr{P}'$, the categories of representations associated with the equipped posets $\mathscr{P}$ and $\mathscr{P}'_{(a,b)}$, respectively. Due to the fact that $\varphi' = \varphi$, we obtain the natural inclusions $\mathscr{R}(U, V) \subset \mathscr{R}'(U', V')$ for all objects $U, V \in \mathscr{R}$. $\mathscr{I} = \langle T(a), T(a,p) \rangle_{\mathscr{R}}$ and $\mathscr{I}' = \langle T(a) \rangle_{\mathscr{R}'}$ denote ideals in the category $\mathscr{R}$ and $\mathscr{R}'$, respectively. We get also inclusions $\mathscr{I}(U, V) \subset \mathscr{I}'(U', V')$ for all objects $U, V \in \mathscr{R}$, taking into consideration that $T'(a) = T'(a,p) = T(a)$. Thus, for each pair of representations $U, V \in \mathscr{R}$, we obtain the diagram of inclusions shown in Figure 12.

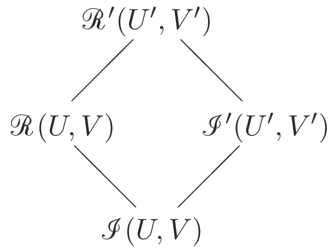


Figure 12. The lattice associated with the ideals $\mathscr{I}$, $\mathscr{I}'$ and categories $\mathscr{R}$, $\mathscr{R}'$ defined by the differentiation IX.

Lemma 20. *Let U, V be an arbitrary pair of representations in $\mathscr{R}$. Then the following identity holds*

$$\mathscr{R}(U, V) \cap \mathscr{I}'(U', V') = \mathscr{I}(U, V).$$

Proof. Let U, V be arbitrary representations in the category $\mathscr{R}$, and let φ be a morphism in $\mathscr{R}(U, V) \cap \mathscr{I}'(U', V')$. Then $\varphi \in [U_b^-, V_a^+]$ with $\widetilde{\varphi}(U_b) \subseteq V_a$. Now we define the following partitions of the spaces U_0 and V_a^+ (we assume $V_0 = V_a^+$):

$$U_0 = U_a^- \cap U_p^- \oplus U_a^- \cap N_p^+ \oplus T_a^- \oplus T_p^- \oplus (N_a \cap N_p)^+ \oplus T_a^+ \oplus T_p^+ \oplus U_p^- \cap N_a^+ \oplus X_0,$$

where $T_a^- \subseteq U_a^-$, $T_a^- \cap U_p^+ = 0$, $T_p^- \subseteq U_p^-$, $U_a^+ \cap T_p^- = 0$.

$\widetilde{U_a^-} \cap T_a = 0$, $\widetilde{U_p^-} \cap T_p = 0$, $T_a^+ \subseteq N_a^+ \oplus M_a^+$, $T_p^+ \subseteq N_p^+ \oplus M_p^+$ and X_0 is a complementary subspace in U_0. Furthermore, $T_a^+ \cap U_p^+ = 0$ and $T_p^+ \cap U_a^+ = 0$.

Now, we consider the next suitable partition to the space V_0

$$V_0 = V_a^- \cap V_p^- \oplus V_a^- \cap N_p^+ \oplus X_a^- \oplus (N_a \cap N_p)^+ \oplus T_a^+ \oplus Y_0.$$

The spaces $T_x^{\pm}$ are defined as for the space U_0, and Y_0 is a complementary subspace in V_0.

We assume the notations $X_1 = T_a^+$, $X_2 = T_p^+$, $X_3 = (N_a \cap N_p)^+$, $X_4 = X_0$. In V_a, $Y_1 = V_a^- \cap V_p^-$, $Y_2 = V_a^- \cap N_p^+$, $Y_3 = X_a^-$, $Y_4 = (N_a \cap N_p^+)$, $Y_5 = T_a^+$ $Y_6 = Y_0$, and $\varphi_{ij} = e_{Y_j}\varphi e_{X_i}$. Then

$$\varphi = \sum_{j=1}^{6}\sum_{i=1}^{4}\varphi_{ij} = \sum_{j=1}^{6}\sum_{i=1}^{4} e_{Y_j}\varphi e_{X_i}.$$

By Lemma 18, $\varphi_{ij} \in \langle T(a)\rangle_{\mathscr{R}}$ if $j = 1$, $\varphi_{ij} \in \langle T(a,p)\rangle_{\mathscr{R}}$ otherwise. Therefore $\varphi \in \mathscr{I}(U,V)$, thus $\mathscr{R}(U,V) \cap \mathscr{I}'(U',V') \subseteq \mathscr{I}(U,V)$.

The Remark 8 allows us to conclude $\mathscr{I}(U,V) \subseteq \mathscr{R}(U,V) \cap \mathscr{I}'(U',V')$. This result proves the desired identity. □

Lemma 21. *Let U, V be an arbitrary pair of representations in $\mathscr{R}$. Then, the following identity holds*

$$\mathscr{R}(U,V) + \mathscr{I}'(U',V') = \mathscr{R}'(U',V').$$

Proof. From definition of the functor $D^{\mathrm{IX}}_{(a,b)}$, we can note that for ψ in $\mathscr{R}'(U',V')$, and for $x \in \{A \cup B \cup \Sigma \cup \{a,b\}\} \subset \mathscr{P}$, $\widetilde{\psi}(U_x) \subset V_x$, then $\widetilde{\psi}(U_x) \subset V_x$. Therefore, for $x \in \mathscr{P} \setminus \{p\}$ and $\psi \in \mathscr{R}$, $\widetilde{\psi}(U_x) \subset V_x$, since $\widetilde{\psi}(U_p) \subset V_p + V_a \nsubseteq V_p$, then in general $\psi \notin \mathscr{R}$ and $\mathscr{R}'(U',V') \nsubseteq \mathscr{R}(U,V)$. The following procedure allows us to obtain a morphism $\varphi \in \mathscr{R}(U,V)$ from a morphism $\psi \in \mathscr{R}'(U',V')$. To get this morphism, we assume the same partition, as above for the space U_0, and define the following partition for the space V_0:

$$V_0 = V_a^- \cap V_p^- \oplus V_a^- \cap N_p^+ \oplus X_a^- \oplus (N_a \cap N_p)^+ \oplus X_a^+ \oplus X_p^+ \oplus X_p^- \oplus V_p^- \cap N_a^+ \oplus Y_0$$

where Y_0 is a complementary subspace in V_0. The spaces X_x are defined as the spaces T_x in U_0, whereas $N_a, N_p \subseteq \widetilde{V_0}$ are defined as for space U_0. Furthermore, $X_p^+ = X_{p_1} \oplus X_{p_2}$ ($X_a^+ = X_{a_1} \oplus X_{a_2}$), where $e_\lambda \in X_{p_1}$ ($e_\lambda \in X_{a_1}$) if and only if there exists $e_\zeta \in M_p^+ \cap V_a^-$ ($e_\zeta \in M_a^+ \cap V_p^-$) such that $v = e_\zeta + \boldsymbol{\xi} e_\lambda \in M_p$ ($v = e_\zeta + \boldsymbol{\xi} e_\lambda \in M_a$).

If $N_a = G\{v = e_{\zeta_j} + \boldsymbol{\xi} e_{\lambda_j}\}_{1\leqslant j\leqslant k}$, for some positive integer k, then $(N_a^+)_1 = F\{e_{\zeta_j}\}$, $(N_a^+)_2 = F\{e_{\lambda_j}\}$. We use the same notation for any subspace N_x associated with a point $x \in \mathscr{P}^{\otimes}$. Furthermore, if X is a subspace of a k-vector space with a fixed basis $\{e_1, e_2, \dots, e_t\}$, then a vector of the form $\zeta_1 e_1 + \zeta_2 e_2 + \cdots + \zeta_t e_t$ will be denoted $\{\zeta_r\}_X$, $1 \leqslant r \leqslant t$. Therefore, if $\psi : U_0 \to V_0 \in R'(U', V')$, then $\widetilde{\psi}(U_a + U_p) \subset V_a + V_p$; and for any vector $e_\zeta \in U_p^-$, we have:

$$\widetilde{\psi}(e_\zeta) = \{\zeta_i\}_{\widetilde{V_a^-} \cap \widetilde{V_p^-}} + \{\lambda_j\}_{\widetilde{V_a^-} \cap F(N_p)} + \{\gamma_k\}_{\widetilde{V_p^-} \cap F(N_a)} + \{\delta_l\}_{\widetilde{X_a^-}} + \{\mu_m\}_{\widetilde{X_p^-}},$$

for suitable index sets. In fact, $\widetilde{\psi}(\widetilde{U_p^-} \oplus M_p) = \widetilde{\psi}(\widetilde{U_b^-} \cap U_p) \subseteq V_p \cap \widetilde{V_b^-} \subseteq V_p$.

If $e_\zeta + \boldsymbol{\xi} e_\lambda \in N_a \cap N_p$ then:

$$\begin{aligned}\psi(e_\zeta) = {} & \{\zeta_i\}_{V_a^- \cap V_p^-} + \{\lambda_j^1\}_{V_a^- \cap N_p^+} + \{\delta_l\}_{X_a^-} + \{\gamma_k^1\}_{V_p^- \cap N_a^+} + \{\gamma_k^2\}_{X_{a_1}} \\ & + \{\varepsilon_n^1\}_{(N_a \cap N_p)^+} + \{\varepsilon_n^2\}_{(N_a \cap N_p)^+} + \{\varpi_t^1\}_{X_{a_2}} + \{\varpi_t^2\}_{X_{a_2}}, \\ \psi(e_\lambda) = {} & \{\zeta_i'\}_{V_a^- \cap V_p^-} + \{\lambda_j'^1\}_{V_a^- \cap N_p^+} + \{\delta_l'\}_{X_a^-} + \{\gamma_k'^1\}_{V_p^- \cap N_a^+} + \{\gamma_k'^2\}_{X_{a_1}} \\ & + \{\varepsilon_n'^1\}_{(N_a \cap N_p)^+} + \{\varepsilon_n'^2\}_{(N_a \cap N_p)^+} + \{\varpi_t'^1\}_{X_{a_2}} + \{\varpi_t'^2\}_{X_{a_2}}.\end{aligned}$$

If $e_\zeta \in T_p^+$ then:

$$\begin{aligned}\psi(e_\zeta) = {} & \{\zeta_i\}_{V_a^- \cap V_p^-} + \{\lambda_j^1\}_{V_a^- \cap N_p^+} + \{\lambda_j^2\}_{X_{p_1}} + \{\delta_l\}_{X_a^-} + \{\mu_m\}_{X_p^-} \\ & + \{\gamma_k^1\}_{V_p^- \cap N_a^+} + \{\gamma_k^2\}_{X_{a_1}} + \{\varepsilon_n^1\}_{(N_a \cap N_p)^+} + \{\varepsilon_n^2\}_{(N_a \cap N_p)^+} \\ & + \{\varpi_t^1\}_{X_{a_2}} + \{\varpi_t^2\}_{X_{a_2}} + \{\nu_s^1\}_{X_{p_2}} + \{\nu_s^2\}_{X_{p_2}}.\end{aligned}$$

We define the F-linear morphisms w_1 and w_2, as follows:

$$w_1 : U_0 \to V_0,$$

such that, for all basic vector $e_\theta \in (N_a \cap N_p)^+$, $w_1(e_\theta) = \{\lambda_j^1\}_{V_a^- \cap N_p^+} + \{\delta_l\}_{X_a^-} + \{\gamma_k^1\}_{V_p^- \cap N_a^+} + \{\gamma_k^2\}_{X_{a_1}} + \{\varpi_t^1\}_{X_{a_2}} + \{\varpi_t^2\}_{X_{a_2}}$, $w_1(t) = 0$, for the other basic vectors t in U_0.

$$w_2 : U_0 \to V_0,$$

such that, for all basic vector $e_\theta \in T_p^+$, $w_2(e_\theta) = \{\delta_l\}_{X_a^-} + \{\gamma_k^2\}_{X_{a_1}} + \{\varpi_t^1\}_{X_{a_2}} + \{\varpi_t^2\}_{X_{a_2}}$, $w_2(t) = 0$, for the other basic vectors $t \in U_0$. Thus, if $w = w_1 + w_2$ then $w \in [H_b, V_a^+]$ and $\widetilde{w}(U_b) \subset V_a$. Therefore, $w \in \mathscr{I}'(U', V')$ by Lemma 19.

Note that,
$(\widetilde{\psi}-\widetilde{w})(U_x) \subseteq F(V_a) \oplus V_x = V_x$ if $x \in a^{\triangledown}$;
$(\widetilde{\psi}-\widetilde{w})(U_x) \subseteq V_x + V_a = V_x$, if $x \in a^{\curlyvee}$;
$(\widetilde{\psi}-\widetilde{w})(\widetilde{U_p^-} \oplus M_p) \subseteq \widetilde{V_b^-} \cap V_p \subseteq V_p$.
If a basic vector $v = e_\zeta + \boldsymbol{\xi} e_\lambda \in N_a \cap N_p \subseteq U_b$, then:

$$(\widetilde{\psi}-\widetilde{w})(v) = \{\zeta_i\}_{V_a^- \cap V_p^-} + \{\varepsilon_n^1\}_{(N_a \cap N_p)^+} + \{\varepsilon_n^2\}_{(N_a \cap N_p)^+} \\ + \boldsymbol{\xi}(\{\zeta_i'\}_{V_a^- \cap V_p^-} + \varepsilon_n'^1\}_{(N_a \cap N_p)^+} + \{\varepsilon_n'^2\}_{(N_a \cap N_p)^+}) \in V_p.$$

For a basic vector $v = e_\zeta + \boldsymbol{\xi} e_\lambda \in T_p$, we have:

$$(\widetilde{\psi}-\widetilde{w})(v) = \{\zeta_i\}_{V_a^- \cap V_p^-} + \{\lambda_j^1\}_{V_a^- \cap N_p^+} + \{\lambda_j^2\}_{X_{p_1}} + \{\mu_m\}_{X_p^-} \\ + \{\gamma_k^1\}_{V_p^- \cap N_a^+} + \{\varepsilon_n^1\}_{(N_a \cap N_p)^+} + \{\varepsilon_n^2\}_{(N_a \cap N_p)^+} + \{\nu_s^1\}_{X_{p_2}} + \{\nu_s^2\}_{X_{p_2}} \\ + \boldsymbol{\xi}(\{\zeta_i'\}_{V_a^- \cap V_p^-} + \{\lambda_j'^1\}_{V_a^- \cap N_p^+} + \{\lambda_j'^2\}_{X_{p_1}} + \{\mu_m'\}_{X_p^-} \\ + \{\gamma_k'^1\}_{V_p^- \cap N_a^+} + \{\varepsilon_n'^1\}_{(N_a \cap N_p)^+} + \{\varepsilon_n'^2\}_{(N_a \cap N_p)^+} \\ + \{\nu_s'^1\}_{X_{p_2}} + \{\nu_s'^2\}_{X_{p_2}}) \in V_p.$$

Therefore, $(\widetilde{\psi}-\widetilde{w})(U_p) \subseteq V_p$ and $\varphi = \psi - w \in \mathscr{R}(U,V)$. Thus, $\psi = \varphi + w \in \mathscr{R}(U,V) + \mathscr{I}'(U',V')$, hence $\mathscr{R}'(U',V') \subseteq \mathscr{R}(U,V) + \mathscr{I}'(U',V')$.
The Remark 8 allows us to conclude that $\mathscr{R}(U,V) + \mathscr{I}'(U',V') \subseteq \mathscr{R}'(U',V')$ with this inclusion, we are done. □

Since Zavadskij proved in [25] that $\operatorname{Ind}\mathscr{P} \setminus [T(a), T(a,p)] \rightleftarrows \operatorname{Ind}\overline{\mathscr{P}}' = \operatorname{Ind}\mathscr{P}' \setminus [T(a)]$. Then we have automatically the following fact from Lemmas 16, 17, 20, and 21:

Lemma 22. *Let $\mathscr{P}$ be an equipped poset with a pair of points (a,b),* IX-*suitable. Then, the functor $D^{\mathrm{IX}}_{(a,b)}\colon \operatorname{rep}\mathscr{P} \to \operatorname{rep}\mathscr{P}'_{(a,b)}$, defined by formulas* (27)*, induces an equivalence between quotient categories:*

$$\mathscr{R}/\mathscr{I} \xrightarrow{\sim} \mathscr{R}'/\mathscr{I}',$$

where $\mathscr{R} = \operatorname{rep}\mathscr{P}$, $\mathscr{R}' = \operatorname{rep}\mathscr{P}'_{(a,b)}$, $\mathscr{I} = \langle T(a), T(a,p)\rangle_{\mathscr{R}}$ and $\mathscr{I}' = \langle T(a)\rangle_{\mathscr{R}'}$.

The following corollary holds as a consequence of Lemma 22.

Corollary 5. *If $\Gamma(\mathscr{R})$ and $\Gamma(\mathscr{R}')$, are the Gabriel's quivers of the categories $\mathscr{R}$ and $\mathscr{R}'$, then $\Gamma(\mathscr{R}) \setminus [T(a), T(a,p)] \simeq \Gamma(\mathscr{R}') \setminus [T(a)]$.*

3.4. Categorical properties of the algorithm of differentiation X for equipped posets with involution

Let $(\mathscr{P}, \Phi) = \mathscr{P}$ be an equipped poset with involution $*$ and Φ be the set of all the equivalence classes of its points with respect to this involution. We denote by $\mathrm{rep}(\mathscr{P}, \Phi)$ the category of all the representations of $(\mathscr{P}, \Phi)$ or simply by $\mathrm{rep}\,\mathscr{P}$ if there is no doubt about the involution and their classes [8, 24, 25].

Let $U = (U_0; U_\kappa \mid \kappa \in \Phi)$ be a representation in $\mathrm{rep}\,\mathscr{P}$. If $x \neq x^*$ then $x \sim x^*$ and we assume the notation (x, x^*) for a class $\kappa \in \Phi$.

Let (F, G) be the pair of fields we are working on. Let U_0 be some finite-dimensional F-vector space, $\widetilde{U}_0$ its complexification and $\kappa \in \Phi$ be some class. We assume the notation, U_0^κ $(\widetilde{U}_0^\kappa)$ for direct sum of $|\kappa|$-copies of U_0 $(\widetilde{U}_0)$ numbered by the points $x \in \kappa$. In this case, the copy of U_0 $(\widetilde{U}_0)$ in U_0^κ $(\widetilde{U}_0^\kappa)$ corresponding to a point x is denoted by U_0^x $(\widetilde{U}_0^x)$ and usually is identified with U_0 $(\widetilde{U}_0)$. So, $U_0^\kappa = U_0^x = U_0$ $(\widetilde{U}_0^\kappa = \widetilde{U}_0^x = \widetilde{U}_0)$ if x is small (weak) and $U_0^\kappa = U_0^x \oplus U_0^{x^*} = U_0^2$ $(\widetilde{U}_0^\kappa = \widetilde{U}_0^x \oplus \widetilde{U}_0^{x^*} = \widetilde{U}_0^2)$ if x is big (biweak).

For each class $\kappa \in \Phi$ and each point $x \in \kappa$, we consider natural injections and projections:

$$\begin{aligned} i_x :& U_0 = U_0^x \longrightarrow U_0^\kappa \text{ if } x \text{ is a small or big point,} \\ i_x :& \widetilde{U}_0 = \widetilde{U}_0^x \longrightarrow \widetilde{U}_0^\kappa \text{ if } x \text{ is a weak or biweak point,} \\ \pi_x :& U_0^\kappa \longrightarrow U_0^x = U_0 \text{ if } x \text{ is a small or big point,} \\ \pi_x :& \widetilde{U}_0^\kappa \longrightarrow \widetilde{U}_0^x = \widetilde{U}_0 \text{ if } x \text{ is a weak or biweak point.} \end{aligned} \tag{28}$$

Choosing a subspace $U_\kappa \subset U_0^\kappa$ $(U_\kappa \subset \widetilde{U}_0^\kappa)$ if κ correspond to a small or big (weak or biweak) point, we attach to it two subspaces in U_0 $(\widetilde{U}_0)$ of the form:

$$U_x^- := i_x^{-1}(U_\kappa), \quad U_x^+ := \pi_x(U_\kappa). \tag{29}$$

Identifying U_0^x $(\widetilde{U}_0^x)$ with U_0 $(\widetilde{U}_0)$, we also can assume $U_x^- = U_\kappa \cap U_0^x$ $(U_x^- = U_\kappa \cap \widetilde{U}_0^x)$. Let x be a small or weak point, then $\kappa = \{x\}$. Therefore $U_x^- = U_x^+$, for which points x we will omit the notations $\pm$ and write simply U_x, for a big point a set $U_x^+ = \{s \in U_0 \mid (s, t) \in U_{(x,x^*)}\}$.

Let $\underline{U_\kappa} = \sum i_y(U_x^+) = \sum e_{xy}(U_{(x,x^*)})$, where $x < y$ and $y \in \kappa$. The dimension of a representation U is the vector $\underline{\dim}\, U = (h_0, h_\kappa)_{\kappa \in \Phi}$, where $h_0 = \dim\, U_0$ over the field F and $h_\kappa = \dim\, (U_\kappa / \underline{U_\kappa})$ over the field G.

Zavadskij defined the *algorithm of differentiation* X in [25], afterwards, he presented in [28] the following modified version of this differentiation:

A pair of incomparable points (a,b) in $\mathscr{P}$ where a is big (i.e. $a \neq a^*$) and b is weak is called X-*suitable* (i.e. suitable for differentiation X), if $\mathscr{P} = a^\triangledown + b_\vartriangle$.

The *derived equipped poset* with involution $(\mathscr{P}', \Phi') = \mathscr{P}'$, with respect to the pair (a,b) is obtained from $(\mathscr{P}, \Phi)$ in the following way:

(a) the point a^* is replaced by a three-point chain $a^* < q < a_0$, where a^*, a_0 are big points and q is weak;

(b) the point b is replaced by a two-point chain $b_0 < b$, where b_0 is big and b is weak;

(c) an order relation $a < b_0$ is added;

(d) Φ' is obtained from Φ by adding two new classes: a non-trivial one $\{a_0, b_0\}$ and a trivial one $\{q\}$.

Naturally, all the order relations induced by those in $\mathscr{P}$ and by those aforementioned are added as well.

Figure 13 shows an equipped poset with involution $(\mathscr{P}, \Phi)$ with a pair of points (a,b) X-*suitable* and its corresponding derived poset $(\mathscr{P}', \Phi')$.

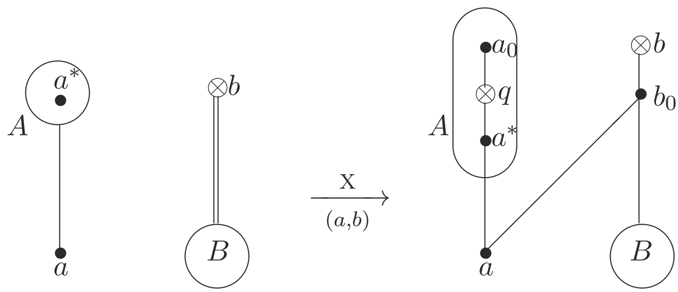


FIGURE 13. Diagrams of equipped posets with involution $(\mathscr{P}, \Phi)$ and $(\mathscr{P}', \Phi')$.

Set $A = a^\blacktriangledown$, $B = b_\blacktriangle$ in $\mathscr{P}$ and $\widehat{a} = a^\blacktriangledown$, $B' = \mathscr{P}' \setminus a^\triangledown$ in $\mathscr{P}'$. Let $U = (U_0; U_\kappa \mid \kappa \in \Phi)$ be a representation of the set $(\mathscr{P}, \Phi)$, where U_0 is a finite-dimensional F-space. Consider an ordered sum $U_0^2 = U_0 \oplus U_0$, we can define the *coupling* of a sequence of n subspaces $X_1, \ldots, X_n \subset U_0^2$ being a subspace in U_0^2 of the form:

$$[X_1 - X_2 - \cdots - X_n] = \{(t_0, t_n) \mid (t_{i-1}, t_i) \in X_i \text{ for some } t_i\}.$$

The categories $\mathscr{R}_\Phi$ and $\mathscr{R}'_{\Phi'}$ are described as follows:

$$\begin{aligned} \mathscr{R}_\Phi &= \{\mathrm{rep}(\mathscr{P}, \Phi) \mid U_A^- = U_a^+ \subset U_b^+,\ U_b^- = U_B^+\}. \\ \mathscr{R}'_{\Phi'} &= \{\mathrm{rep}(\mathscr{P}', \Phi') \mid U_a^+ \subset U_{B'}^+,\ U_{a_0}^- = U_q^+,\ U_b^- = U_{b_0}^+\}. \end{aligned} \tag{30}$$

Let $\mathscr{P}$ be an equipped poset with involution, and a pair of points (a,b), X-suitable. The following formulas define the *differentiation functor* $D^{\mathrm{X}}_{(a,b)}$:

$\mathscr{R}_\Phi \longrightarrow \mathscr{R}'_{\Phi'}$ induced by the algorithm of differentiation X. Thus, for a given representation $U = (U_0; U_\kappa \mid \kappa \in \Phi) \in \mathscr{R}_\Phi$, we define the *derived representation* $U' = (U'_0; U'_\kappa \mid \kappa \in \Phi')$ in such a way that

$$\begin{gathered} U'_0 = U_0, \qquad U'_b = U_b + \widetilde{U_a^+}, \\ U'_{(a_0,b_0)} = [U_{(a^*,a)} - U_b] + (0, U_a^+), \qquad U'_q = [U_{(a^*,a)} - U_b - U_{(a,a^*)}], \\ U'_{(a,a^*)} = U_{(a,a^*)} \cap (U_B^+, U_0), \\ U'_\kappa = U_\kappa \quad \text{for the remaining classes } \kappa \in \Phi', \\ \varphi' = \varphi \quad \text{for all F linear map} - \text{morphism } \varphi : U_0 \to V_0. \end{gathered} \tag{31}$$

Following [28], if (E_0, W_0) is a (U_a^+, U_B^+)-cleaving pair of U_0, then the *reduced* derived representation $U^\downarrow$ is defined (uniquely up to isomorphism) by the equality $U' = U^\downarrow \oplus P^m(\widehat{a})$, where $m = \dim E_0 = \dim(U_a^+, U_B^+)/U_B^+$ its evident form is $U^\downarrow = W$, with W_0 taken from the cleaving pair and $W_\kappa = U'_\kappa \cap {W_0}^\kappa$.

Obviously, $G'_1(b,a) = P(\widehat{a}) \oplus P(b_0)$ and $G'_2(b,a) = P^2(\widehat{a})$, hence $G_1^\downarrow(b,a) = P(b_0)$ and $G_2^\downarrow(b,a) = 0$.

Let W be an object in $\mathscr{R}'_{\Phi'}$. To construct the *primitive object* $W^\uparrow \in \mathscr{R}_\Phi$, we represent the spaces $W_{(a_0,b_0)}$, W_q and W_b, respectively, in the form

$$\begin{aligned} W_{(a_0,b_0)} &= \underline{W}_{(a_0,b_0)} \oplus F_1, \quad F_1 = \{(f_{11}, f'_{11}), \dots, (f_{1p_1}, f'_{1p_1})\}; \\ W_q &= \widetilde{W_{a^*}^+} \oplus F_2, \quad F_2 = \{(f_{21}, f'_{21}), \dots, (f_{2p_2}, f'_{2p_2})\}; \\ W_b &= \widehat{W_{b_0}^+} \oplus H; \end{aligned}$$

where F_i and H are some complements with the choosen bases for F_i. Consider a new F-space E_0 with a base

$$\{e_{11}, \dots, e_{1p_1}\} \cup \{e_{21}, e'_{21}, \dots, e_{2p_2}, e'_{2p_2}\}$$

of dimension $m = p_1 + 2p_2$. Then, set $W^\uparrow = (U_0; U_\kappa \mid \kappa \in \Phi)$ where

$$\begin{aligned} U_0 &= W_0 \oplus E_0; \\ \dot{U}_\kappa &= W_\kappa \oplus E_0^{\kappa\cap A} \text{ for } \kappa \neq \{a, a^*\},\ \{b\}; \\ \dot{U}_{(a,a^*)} &= W_{(a,a^*)} + \{(e_{11}, f_{11}), \dots, (e_{1p_1}, f_{1p_1})\} \\ &\qquad + \{(e_{2j}, f_{2j}), (e'_{2j}, f'_{2j}) : j = 1, \dots, p_2\}; \\ \dot{U}_b &= \widetilde{W_{B'}^+} + \{(e_{11}, f'_{11}), \dots, (e_{1p_1}, f'_{1p_1})\} + H. \end{aligned} \tag{32}$$

The desired isomorphisms $(U^{\downarrow})^{\uparrow} \simeq U$, for a reduced object $U \in \mathscr{R}_{\Phi}$ (without direct summands $G_2(b,a)$) and $(W^{\uparrow})^{\downarrow} \simeq W$, for a reduced object $W \in \mathscr{R}'_{\Phi'}$ (without direct summands $P(\widehat{a})$) hold. Then the following two lemmas are given as a consequence of the previous construction of the primitive object (also called the integration process).

Lemma 23. *For each representation $W \in \mathscr{R}'_{\Phi'}$, there exists a representation $W^{\uparrow} \in \mathscr{R}_{\Phi}$ such that $(W^{\uparrow})' \simeq W \oplus P^m(\widehat{a})$, for some $m \geqslant 0$.*

Lemma 24. *In the case of the differentiation* X, *the operations $\downarrow$ and $\uparrow$ induce mutually inverse bijections*

$$\operatorname{Ind}\mathscr{R}_{\Phi} \setminus [G_2(b,a)] \rightleftarrows \operatorname{Ind}\mathscr{R}'_{\Phi'} \setminus [P(\widehat{a})].$$

Remark 9. Let $\mathscr{R}_{\Phi}$ and $\mathscr{R}'_{\Phi'}$ be the categories described in the equation (30), associated with the equipped posets with involution $\mathscr{P}$ and $\mathscr{P}'_{(a,b)}$, respectively. Due to the fact that $\varphi' = \varphi$, we obtain the natural inclusions $\mathscr{R}_{\Phi}(U,V) \subset \mathscr{R}'_{\Phi'}(U',V')$ for all objects $U, V \in \mathscr{R}_{\Phi}$. Let $\mathscr{I} = \langle G_2(b,a)\rangle_{\mathscr{R}_{\Phi}}$ and $\mathscr{I}' = \langle P(\widehat{a})\rangle_{\mathscr{R}'_{\Phi'}}$ be ideals in the category $\mathscr{R}_{\Phi}$ and $\mathscr{R}'_{\Phi'}$, respectively. We get also inclusions $\mathscr{I}(U,V) \subset \mathscr{I}'(U',V')$ for all objects $U, V \in \mathscr{R}_{\Phi}$, taking into consideration that $G'_2(b,a) = P^2(\widehat{a})$. Thus, for each pair of representations $U, V \in \mathscr{R}_{\Phi}$, we obtain the following diagram of inclusions

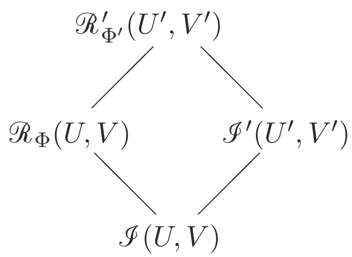


FIGURE 14. The lattice associated with the ideals $\mathscr{I}$, $\mathscr{I}'$ and vector spaces $\mathscr{R}_{\Phi}(U,V)$, $\mathscr{R}'_{\Phi'}(U',V')$ defined by differentiation X.

The following lemmas allow us to establish that the differentiation X induces a categorical equivalence.

Lemma 25. *Let U and V be arbitrary representations in $\mathscr{R}_{\Phi}$. Then the following identities hold*

$$\mathscr{R}_{\Phi}(U,V) + \mathscr{I}'(U',V') = \mathscr{R}'_{\Phi'}(U',V')$$

and

$$\mathscr{R}_{\Phi}(U,V) \cap \mathscr{I}'(U',V') = \mathscr{I}(U,V).$$

Proof. The inclusions $\mathscr{R}_{\Phi}(U,V)+\mathscr{I}'(U',V')\subseteq\mathscr{R}'_{\Phi'}(U',V')$ and $\mathscr{I}(U,V)\subseteq\mathscr{R}_{\Phi}(U,V)\cap\mathscr{I}'(U',V')$ follow from Remark 9. Thus, it suffices to prove $\mathscr{R}_{\Phi}(U,V)+\mathscr{I}'(U',V')\subseteq\mathscr{R}'_{\Phi'}(U',V')$ and $\mathscr{R}_{\Phi}(U,V)\cap\mathscr{I}'(U',V')\subseteq\mathscr{I}(U,V)$ in order to obtain the identities.

Firstly, we prove that $\mathscr{R}_{\Phi}(U,V)+\mathscr{I}'(U',V')\subseteq\mathscr{R}'_{\Phi'}(U',V')$, with $\mathscr{I}'=[U_B^+ +(U'_a)^+,(V'_A)^-]$. We note that in general, if $(x,y)\in U_{(a,a^*)}$ and $(r,s)\in U_b$, then not necessarily $(\psi(x),\psi(y))\in V_{(a,a^*)}$ and $(\psi(r),\psi(s))\in V_b$. However, for any $(x,y)\in U_{(a,a^*)}\cap(U_B^+,U_0)$ it holds that $(\psi(x),\psi(y))\in V_{(a,a^*)}\cap(V_B^+,V_0)\subset V_{(a,a^*)}$, provided that $\psi:U_0\longrightarrow V_0\in\mathscr{R}'_{\Phi'}(U',V')$. Thus, for any pair of vectors of the form $(x,y)\in U_{(a,a^*)}$, it is necessary to define a linear map-morphism which can be used to adjust the corresponding images to subspaces $V_{(a,a^*)}$ and V_b. To do that, we consider the following partitions of the vector spaces U_0^2 and V_0^2

$$U_0^2=U_{(a,a^*)}\cap\widetilde{U_b^-}\oplus U_{(a,a^*)}\cap N_b\oplus T_{(a,a^*)}\oplus T_b\oplus T_0,$$

where $U_b=\widetilde{U_b^-}\oplus N_b$, $N_b=\langle(1,\boldsymbol{\xi})^t\rangle_G$, $N_{b_1}=\langle(1,0)^t\rangle_F$, $N_{b_2}=\langle(0,1)^t\rangle_F$, then $N_b^+=N_{b_1}+N_{b_2}$,

$$U_{(a,a^*)}\cap\widetilde{U_b^-}\subseteq U'_{(a,a^*)},\qquad U_a^+=U_a^+\cap U_B^+\oplus M_B,$$
$$T_b=\widetilde{T_b^-}\oplus H_b,\qquad \widetilde{T_b^-}\subseteq\widetilde{U_b^-},\ H_b\subset N_b,\qquad U_{a^*}=U_a^+\oplus L_{a^*},$$

where $T_{(a,a^*)}$, T_b and T_0 are complementary subspaces of $U_0^2=U_{(a,a^*)}\cap\widetilde{U_b^-}\oplus U_{(a,a^*)}\cap N_b$ and $U_{(a,a^*)}+U_b$ in $U_{(a,a^*)}$, U_b and U_0^2, respectively. The same notation is keeping for subspace V_0^2 and the corresponding partition.

Now, we consider the following cases.

(i) Suppose that $(x,y)\in U_{(a,a^*)}\cap(U_B^+,U_0)$. Then $(\psi(x),\psi(y))\in V'_{(a,a^*)}=V_{(a,a^*)}\cap(V_B^+,V_0)\subset V_{(a,a^*)}$.

(ii) If $(x,y)\in T_{(a,a^*)}$, then there exists $z\in U_b^+$ such that $(z,x)\in U_b$. Thus, $(y,z)\in U'_{(a_0,b_0)}$, $y\notin U_b^+$ and $(\psi(y),\psi(z))\in V'_{(a_0,b_0)}$. Assume that vectors $\{(t_{i_1}^j,t_{i_2}^j):1\leqslant j\leqslant k\}$ constitute a basis of $[U_{(a,a^*)}-U_b]$ and that $\{t_a^L\}:a\leqslant L\leqslant m$ is a basis of subspace V_a^+. In this case, $\lambda_{V_a^+}$ denotes a linear combination of the form $\sum_{h=1}^m\lambda_h t_a^h,\ \lambda_h\in G$. Therefore,

$$(\psi(y),\psi(z))=\sum_{j=1}^k\lambda_j(t_{i_1}^j,t_{i_2}^j)+(0,\lambda_{V_a^+}),$$

$$\psi(y)=\sum_{j=1}^k\lambda_j t_{i_1}^j,\qquad \psi(z)=\sum_{j=1}^k\lambda_j t_{i_2}^j+\lambda_{V_a^+}.$$

Then, there exists a unique vector s such that $(\psi(y), s) \in V_{(a^*,a)}$ and $(s, \psi(z) - \lambda_{V_a^+}) \in V_b$, where $y \notin V_b^+$. Thus, if the F-linear map-morphism $w_1 : U_0 \longrightarrow V_0$ is defined in such a way that

$$w_1(x) = \begin{cases} \psi(x) - s, & \text{if } x \in M_B, \\ 0, & \text{otherwise;} \end{cases}$$

then $w_1 \in [U_B^+ + (U_a')^+, (V_A')^-]$. Note that, $\psi(U_{A'}^-) \subseteq V_{A'}^-$ besides, if $(x, y) \in T_{(a,a^*)}$ then

$$((\psi - w_1)(x), (\psi - w_1)(y)) = (\psi(x) - \psi(x) + s, \psi(y)) = (s, \psi(y)) \in V_{(a,a^*)}.$$

(iii) If $(x, y) \in H_b$, it holds that

$$(\psi(x), \psi(y)) = \left(\sum_{j=1}^{k} \delta_j t_{i_1}^j, \sum_{j=1}^{k} \delta_j t_{i_2}^j + \lambda_{V_a^+} \right).$$

If $w_2 : U_0 \longrightarrow V_0$ is a linear map-morphism such that

$$w_2(y) = \begin{cases} \lambda_{V_a^+}, & \text{if } y \in H_b^+, \\ 0, & \text{otherwise,} \end{cases}$$

then $w_2 \in [U_B^+ + (U_a')^+, (V_A')^-]$. Note that, $\widetilde{\psi}(\widetilde{U_b^-}) \subseteq \widetilde{V_b^-}$, and for $(x, y) \in H_b$, it holds that

$$\begin{aligned} ((\psi - w_2)(x), (\psi - w_2)(y)) &= \left(\psi(x), \sum_{j=1}^{k} \delta_j t_{i_2}^j + \lambda_{V_a^+} - \lambda_{V_a^+} \right) \\ &= \left(\sum_{j=1}^{k} \delta_j t_{i_1}^j, \sum_{j=1}^{k} \delta_j t_{i_2}^j \right) \in V_b. \end{aligned}$$

(iv) Suppose now, that $(x, y) \in U_{(a,a^*)} \cap N_b$, with $y \in L_{a^*}$. Then $(y, x) \in U_{(a^*,a)}$ and $(x, y) \in U_b$. Thus, $(y, y) \in U'_{(a_0,b_0)}$ and $(\psi(y), \psi(y)) \in V'_{(a_0,b_0)}$, $(\psi(x), \psi(y)) \in V_b'$.

$$(\psi(x), \psi(y)) = \left(\sum_{j=1}^{k} \gamma_j t_{i_1}^j, \sum_{j=1}^{k} \gamma_j t_{i_2}^j + \lambda_{V_a^+} \right)$$

with $\left(\sum_{j=1}^{k} \gamma_j t_{i_1}^j, \sum_{j=1}^{k} \gamma_j t_{i_2}^j \right) \in V_b$ and $(\psi(y), \psi(y) - \lambda_{V_a^+}) \in [V_{(a^*,a)} - V_b]$. Hence, there exist t_1 such that (t_1 unique) $(\psi(y), t_1) \in V_{(a^*,a)}$ and $(t_1, \psi(y) - \lambda_{V_a^+}) \in V_b$, we write (in V)

$$V_{(a,a^*)} \cap \widetilde{U_b^-} = T_1, \qquad V_{(a,a^*)} \cap \widetilde{N_b} = T_2, \qquad T_{(a,a^*)} = T_3,$$

then $(t_1, \psi(y)) = \lambda_{T_1} + \lambda_{T_2} + \lambda_{T_3}$, where

$$\lambda_{T_1} = (r_1^1, r_1^2), \qquad \lambda_{T_2} = (r_2^1, r_2^2), \qquad \lambda_{T_3} = (r_3^1, r_3^2),$$

$r_i^1 \in \operatorname{Re} T_i$; $r_i^2 \in \operatorname{Im} T_i$ are linear combinations of all elements of the basis of the corresponding subspace ($\operatorname{Re} T_i$= real part of T_i, $\operatorname{Im} T_i$= imaginary part of T_i). Define the linear map-morphism $w_3 : U_0 \longrightarrow V_0$ such that

$$w_3(x) = \begin{cases} \psi(x) - r_1^1 - r_2^1 & \text{if } x \in M_b, \\ \psi(x) - r_1^2 - r_2^2 & \text{if } x \in L_{a^*} \cap N_b^+, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}(\psi^\kappa - w^\kappa)(x, y) &= ((\psi - w)(x), (\psi - w)(y)) \\ &= (\psi(x) - \psi(x) + r_1^1 + r_2^1, \psi(y) - \psi(y) + r_1^2 + r_2^2) \\ &\in V_{(a,a^*)} \cap \widetilde{V_b^-} + V_{(a,a^*)} \cap N_b \text{ if } x \in M_b \text{ and } y \in L_{a^*}.\end{aligned}$$

Thus $(\psi^\kappa - w^\kappa)(x, y) \in V_{(a,a^*)} \cap V_b$, with $w_3 \in [U_B^+ + (U_a')^+, (V_A')^-]$.

(v) Define $w = w_1 + w_2 + w_3 \in [U_B^+ + (U_a')^+, (V_A')^-]$. It is easy to see that $[U_B^+ + (U_a')^+, (V_A')^-] \simeq \langle P(a^\blacktriangledown)\rangle_{\mathscr{R}'}$. Then, by construction, the linear morphism $(\psi^\kappa - w^\kappa)(U_\kappa) \subseteq V_\kappa$, for any class $k \in \Phi$. In particular, $\underbrace{(\psi^\kappa - w^\kappa)}(U_{(a,a^*)}) = (\psi - w)^\kappa(U_{(a,a^*)}) \subseteq V_{(a,a^*)}$ and $(\widetilde{\psi} - \widetilde{w})(U_b) = (\psi - w)(U_b) \subseteq V_b$. Therefore, $\varphi = \psi - w \in \mathscr{R}_\Phi(U, V)$, which proves that $\psi \in \mathscr{R}_\Phi(U, V) + \mathscr{I}'(U', V')$, thus $\mathscr{R}_\Phi(U, V) + \mathscr{I}'(U', V') = \mathscr{R}'_{\Phi'}(U', V')$.

In order to prove that $\mathscr{R}_\Phi(U, V) \cap \mathscr{I}'(U', V') \subseteq \mathscr{I}(U, V)$, with $\mathscr{I} = [U_B^+, V_A^-]$, (it is easy to see that $[U_B^+, V_A^-] \simeq \langle G_2(b, a)\rangle_{\mathscr{R}_\Phi}$), we take a morphism $\varphi \in \mathscr{R}_\Phi(U, V) \cap \mathscr{I}'(U', V')$. Then as $\varphi \in \mathscr{I}'(U', V')$, φ can be factored through morphisms $\varphi_1 : U' \longrightarrow P^m(\widehat{a})$ and $\varphi_2 : P^m(\widehat{a}) \longrightarrow V'$ that pass through sums of the representation $P(\widehat{a})$. Thus $\varphi = \varphi_2\varphi_1$ with $\varphi = \varphi_1$, and $\varphi_2 = id$. Note that since $P_a^+ = P_B = 0$ then $\varphi_2\varphi_1(U_B^+) = 0$, besides we have that $\operatorname{Im} \varphi \subset (V_A')^-$ provided that $\varphi \in [U_B^+ + (U_a')^+, (V_A')^-]$. Then $\operatorname{Im} \varphi \subset V_A^-$, therefore $\varphi \in [U_B^+, V_A^-] = \mathscr{I}(U, V)$ and with this argument, we are done. □

Lemma 26. *Let $\mathscr{P}$ be an equipped poset with involution, with a pair of points (a, b),* X*-suitable. Then, the functor $D^{\mathrm{X}}_{(a,b)} : \mathscr{R}_\Phi \longrightarrow \mathscr{R}'_{\Phi'}$, defined by formulas (31), induces an equivalence between quotient categories*

$$\mathscr{R}_\Phi/\mathscr{I} \xrightarrow{\sim} \mathscr{R}'_{\Phi'}/\mathscr{I}',$$

where $\mathscr{I} = \langle G_2(b, a)\rangle_{\mathscr{R}_\Phi}$ and $\mathscr{I}' = \langle P(\widehat{a})\rangle_{\mathscr{R}'_{\Phi'}}$.

Proof. The density of the functor $D^{\mathrm{X}}_{(a,b)}$ is guaranteed by Lemmas 23 and 24. Lemma 25 allows us to conclude that the functor $D^{\mathrm{X}}_{(a,b)}$ is faithful and full. □

As a consequence of Lemmas 23, 24 and 26, we obtain the following corollary regarding the Gabriel quiver of the corresponding categories.

Corollary 6. *If* $\Gamma(\mathscr{R}_\Phi)$ *and* $\Gamma(\mathscr{R}'_{\Phi'})$ *are the Gabriel quivers of the categories* $\mathscr{R}_\Phi$ *and* $\mathscr{R}'_{\Phi'}$*, then* $\Gamma(\mathscr{R}_\Phi) \setminus [G_2(b,a)] \simeq \Gamma(\mathscr{R}'_{\Phi'}) \setminus [P(\widehat{a})]$.

Remark 10. The main Theorem 1 is proved by Lemmas 4, 5, 7–10, 15–17, 22–24 and 26.

Remark 11 (Historical remark; a relationship between the theory of representation of equipped posets and Krawtchouk matrices)**.** It is worth recalling the way that Zavadskij rediscovered the famous Krawtchouk matrices in his paper [28]. In such a work, he defined for two rings A, B and an (A,B)-bimodule W the ${}_AW_B$-matrix problem which consists of reducing to some canonical form one rectangular matrix M over W by elementary transformations of its rows over A and columns over B.

The particular case when $A = F$ is a field admitting quadratic extensions G_1, G_2 (which may coincide) in the algebraic closure $\overline{F}$ the $G_1 \underset{F}{\otimes} G_2$-problem is called the *biquadratic matrix problem* (which in general is still an open problem) over the triple (G_1, F, G_2), the problem is named *homogeneous* whenever $G_1 \simeq G_2$. Zavadskij proved that the $G \underset{F}{\otimes} G$-problem is equivalent to the $(1,\sigma)$-pencil problem over G, where $\sigma(a + \boldsymbol{\xi} b) = a - \boldsymbol{\xi} b$.

In the page 43 of [28] Zavadskij wrote the following sentence to justify the use of matrices of type Θ in his description of the indecomposable representations of the $G \underset{F}{\otimes} G$-bimodule:

"Before to prove Theorem 17, we need to introduce an integer matrix sequence Θ_n which expresses in a perfect way a precise relationship between polynomial invariants for the $G \underset{F}{\otimes} G$-problem and the $(1,\sigma)$-pencil problem".

Section 8 of that work is devoted to give many properties of matrices Θ_n which now we know were introduced in the late 1920s by Krawtchouk [10]. In the current notation for Krawtchouk matrices $\Theta_{n+1} = K^{(n)}$ where $\Theta_n^{i,j} = \sum_k (-1)^k \binom{j-1}{k} \binom{n-j}{i-k-1}$.

He also wrote that the problem of classifying indecomposable representations of the critical equipped poset $M_1 = \{\otimes \quad \otimes\}$ can be reduced to the $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$-problem and therefore to the $(1, \sigma)$-pencil problem over the complex field $\mathbb{C}$.

References

[1] V.M. Bondarenko and A.G. Zavadskij, *Posets with an equivalence relation of tame type and of finite growth*, Can. Math. Soc. Conf. Proc. **11** (1991), 67–88.

[2] A.M. Cañadas, *Descripción categórica de algunos algoritmos de diferenciación*, (Dissertation) Universidad Nacional de Colombia (2007), 157p.

[3] A.M. Cañadas, *Morphisms in categories of representations of equipped posets*, JP Journal of Algebra, Number Theory and Applications **25** (2012), no. 2, 145–176.

[4] A.M. Cañadas, *Some categorical properties of the algorithm of differentiation VII for equipped posets*, JP Journal of Algebra, Number Theory and Applications **25** (2012), no. 2, 177–213.

[5] A.M. Cañadas, *The school of Kiev in Colombia; the legacy of Alexander Zavadskij*, São Paulo Journal of Mathematical Sciences **7** (2013), no. 1, 105–126.

[6] A.M. Cañadas, I.D.M. Gaviria, and P.F.F. Espinosa, *Categorical properties of the algorithm of differentiation D-VIII for equipped posets*, JP Journal of Algebra, Number Theory and Applications **29** (2013), no. 2, 133–156.

[7] A.M. Cañadas, I.D.M. Gaviria, and P.F.F. Espinosa, *On the algorithm of differentiation D-IX for equipped posets*, JP Journal of Algebra, Number Theory and Applications **29** (2013), no. 2, 157–173.

[8] A.M. Cañadas and A.G. Zavadskij, *Categorical description of some differentiation algorithms*, Journal of Algebra and Its Applications **5** (2006), no. 5, 629–652.

[9] A.M. Cañadas and V.C. Vargas, *On the apparatus of differentiation DI-DV for posets*, São Paulo Journal of Mathematical Sciences **9** (2019), no. 1, 249–286.

[10] P. Feinsilver and J. Kocik, *Krawtchouk Polynomials and Krawtchouk Matrices*, Recent Advances in Applied Probability (2005), 115–141.

[11] P. Gabriel, *Représentations indécomposables des ensemblés ordonnés*, Semin. P. Dubreil, 26 annee 1972/73, Algebre, Expose **13** (1973), 301–304.

[12] M.M. Kleiner, *Partially ordered sets of finite type*, Zap. Nauchn. Semin. LOMI **28** (1972), 32–41 (in Russian); English transl., J. Sov. Math **3** (1975), no. 5, 607–615.

[13] L.A. Nazarova and A.V. Roiter, *Representations of partially ordered sets*, Zap. Nauchn. Semin. LOMI **28** (1972), 5–31 (in Russian); English transl., J. Sov. Math. **3** (1975), 585–606.

[14] L.A. Nazarova and A.V. Roiter, *Categorical matrix problems and the Brauer-Thrall conjecture*, Preprint Inst. Math.AN UkSSR, Ser. Mat. **73.9** (1973), 1–100 (in Russian); English transl. in Mitt.Math. Semin. Giessen **115** (1975).

[15] L.A. Nazarova and A.V. Roiter, *Representations of posets with involution*, Preprint Inst. Math.AN UkSSR. **86.80** (1986).

[16] L.A. Nazarova and A.G. Zavadskij, *Partially ordered sets of tame type*, Akad. Nauk Ukrain. SSR Inst. Mat., Kiev (1977), 122-143 (Russian).

[17] L.A. Nazarova and A.G. Zavadskij, *Partially ordered sets of finite growth*, Function. Anal. i Prilozhen., **19** (1982), no. 2, 72–73 (in Russian); English transl., Functional. Anal. Appl., **16** (1982), 135–137.

[18] C.M. Ringel, *Tame Algebras and Integral Quadratic Forms*, LNM, vol. 1099, Springer-Verlag, 1984. 371p.

[19] C. Rodriguez and A.G. Zavadskij, *On corepresentations of equipped posets and their differentiation*, Revista Colombiana de Matemáticas **41** (2007), 117–142.

[20] D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Gordon and Breach, London, 1992. 499p.

[21] K. Spindler, *Abstract Algebra with Applications*, Vol. I, Marcel Dekker, Inc, New York, Basel, Hong Kong, 1994. 791p.

[22] A.G. Zavadskij, *Differentiation with respect to a pair of points*, Matrix problems, Collect. sci. Works. Kiev (1977), 115–121 (in Russian).

[23] A.G. Zavadskij, *The Auslander-Reiten quiver for posets of finite growth*, Topics in Algebra, Banach Center Publ. **26** (1990), Part 1, 569–587.

[24] A.G. Zavadskij, *An algorithm for posets with an equivalence relation*, Canad. Math. Soc. Conf. Proc. **11** (1991), 299–322.

[25] A.G. Zavadskij, *Tame equipped posets*, Linear Algebra Appl. **365** (2003), 389–465.

[26] A.G. Zavadskij, *Equipped posets of finite growth*, Representations of Algebras and Related Topics, AMS, Fields Inst. Comm. Ser. **45** (2005), 363–396.

[27] A.G. Zavadskij, *On Two Point Differentiation and its Generalization*, Algebraic Structures and their Representations, AMS, Contemporary Math. Ser. **376** (2005), 413–436.

[28] A.G. Zavadskij, *On the Kronecker Problem and related problems of Linear Algebra*, Linear Algebra and Its Applications **425** (2007), 26–62.

[29] A.G. Zavadskij, *Representations of generalized equipped posets and posets with automorphisms over Galois field extensions*, Journal of Algebra **332** (2011), 386–413.

[30] A.V. Zabarilo and A.G. Zavadskij, *One-Parameter Equipped Posets and Their Representations*, Functional Analysis and Its Applications **34** (2000), no. 2, 138–140.

[31] A.V. Zabarilo and A.G. Zavadskij, *Representations of one-parameter equipped posets I, II*, Matematychni studii (in Russian) **11** (1999), 3–16 and No. 2, 119–134; Unabridge version in: Preprint 98.1 (in Russian) KSTUCA, Kiev, (1998), 1–44.

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