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On lifting and extending properties on direct sums of hollow uniform modules

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Abstract. A module M is said to be *lifting* if, for any submodule N of M, there exists a direct summand X of M contained in N such that N/X is small in M/X. A module M is said to satisfy the *finite internal exchange property* if, for any direct summand X of M and any finite direct sum decomposition $M = \bigoplus_{i=1}^{n} M_i$, there exists a direct summand M_i' of M_i $(i = 1, 2, \ldots, n)$ such that $M = X \oplus (\bigoplus_{i=1}^{n} M_i')$. In this paper, we first give characterizations for the square of a hollow and uniform module to be lifting (extending). In addition, we solve negatively the question "Does any lifting module satisfy the finite internal exchange property?" as an application of this result.

1. Preliminaries

Throughout this paper, R is a ring with identity and modules are unitary right R-modules. Let M be a module and N, K submodules of M with $K \subseteq N$. N is said to be *small* in M (or a *small submodule* of M) if $N + X \neq M$ for any proper submodule X of M and we denote by $N \ll M$ in this case. A pair (Q, f) of a module Q and an epimorphism $f : Q \to M$ is said to be a *small cover* of M if $\ker f \ll Q$. K is said to be a *coessential submodule* of N in M if $N/K \ll M/K$ and we write $K \subseteq_c N$ in M in this case. A module M is said to satisfy the *finite internal exchange property*

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(or briefly, *FIEP*) if, for any direct summand X of M and any finite direct sum decomposition $M = \bigoplus_{i=1}^{n} M_i$, there exists a direct summand M_i' of M_i $(i = 1, 2, \dots, n)$ such that $M = X \oplus (\bigoplus_{i=1}^{n} M_i')$. Let $M = A \oplus B$ be a module and $h : A \to B$ a homomorphism. Then $\{a + h(a) \mid a \in A\}$ is called a *graph* of h and denoted by $\langle h \rangle$. It is clear that $M = \langle h \rangle \oplus B$, $M = A + \langle h \rangle$ if h is an epimorphism, and $A \cap \langle h \rangle = \ker h$.

A module M is said to be *extending* (or *CS*) if, for any submodule N of M, there exists a direct summand X of M such that N is an essential submodule of X. An indecomposable extending module is called *uniform*. A lifting module is defined as a dual concept of an extending module, that is, a module M is said to be *lifting* if, for any submodule N of M, there exists a direct summand X of M such that X is a coessential submodule of N in M. An indecomposable lifting module is called *hollow*. It is well-known that uniform modules (hollow modules, resp.) are closed under nonzero submodules and essential extensions (nonzero factor modules and small covers, resp.). A module M is said to be *uniserial* if its submodules are linearly ordered by inclusion. Clearly, any uniserial module is hollow and uniform. However the converse is not true. We consider

$$R = \begin{pmatrix} K & K & K & K \\ 0 & K & 0 & K \\ 0 & 0 & K & K \\ 0 & 0 & 0 & K \end{pmatrix}, \quad M_R = (K, K, K, K)$$

where K is a field. Then M has only 6 submodules

$$M,\ (0, K, K, K),\ (0, K, 0, K),\ (0, 0, K, K),\ (0, 0, 0, K),\ 0.$$

Hence M is hollow and uniform but not uniserial.

Extending modules and lifting modules are important because they characterize right noetherian rings, right perfect rings, semiperfect rings, right (co-)H-rings and Nakayama rings (cf. [2]).

Let A and B be modules. A is called *generalized B-injective* if, for any module X, any homomorphism $f : X \to A$ and any monomorphism $g : X \to B$, there exist direct sum decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, a homomorphism $h_1 : B_1 \to A_1$ and a monomorphism $h_2 : A_2 \to B_2$ such that $p_1 f = h_1 q_1 g$ and $q_2 g = h_2 p_2 f$, where $p_i : A = A_1 \oplus A_2 \to A_i$ and $q_i : B = B_1 \oplus B_2 \to B_i$ $(i = 1, 2)$ are canonical projections ([5]). A is called *generalized B-projective* if, for any module X, any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist direct sum decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$ and an epimorphism $h_2 : B_2 \to A_2$ such that $f|_{A_1} = gh_1$ and $g|_{B_2} = fh_2$. It is already known that, a finite direct sum of lifting modules (extending modules, resp.) with the FIEP $M = \bigoplus_{i=1}^{n} M_i$ is lifting (extending, resp.) with the FIEP if and only if K and L are relative generalized projective (relative generalized injective, resp.) for every $k = 1, 2, \ldots, n$, any direct summand K of M_k and any direct summand L of $\bigoplus_{i \neq k} M_i$ by [6, Theorem 3.7] ([5, Theorem 2.15], resp.).

In this paper, we first give characterizations for the square of a hollow and uniform module to be a lifting module (an extending module) which does not necessarily satisfy the FIEP, by certain projectivities (injectivities). Using this result, we give an example of a lifting module not satisfying the FIEP in order to solve the question "Does any lifting module satisfy the finite internal exchange property?" negatively.

For undefined terminologies, the reader is referred to [1], [2], [3], [7] and [8].

2. Main results

Lemma 1. *Let A and B be modules and put $M = A \oplus B$. For any nonzero proper direct summand X of M, the following holds:*

(1) *If A and B are hollow, then so is X.*

(2) *If A and B are uniform, then so is X.*

Proof. Let $p : M = A \oplus B \to A$ and $q : M = A \oplus B \to B$ be canonical projections.

(1) Since A and B are hollow and X is non-small, X satisfies either $p(X) = A$ or $q(X) = B$. Without loss of generality, we can take X with $p(X) = A$. By $X \neq M$, we see $X \cap B \ll B$ because B is hollow. Since X is a proper direct summand of M, we obtain $\ker p|_X = X \cap B \ll X$. Hence $(X, p|_X)$ is a small cover of A. Therefore X is hollow.

(2) Since A and B are uniform and X is non-essential, X satisfies either $X \cap A = 0$ or $X \cap B = 0$. Without loss of generality, we can take X with $X \cap A = 0$. Then $q|_X : X \to B$ is a nonzero monomorphism. Therefore X is uniform because it is isomorphic to a submodule of a uniform module B. $\square$

Now we give a key lemma in this paper.

Lemma 2. *Let U be a hollow and uniform module and put $M = U^2$, $U_1 = U \times 0$ and $U_2 = 0 \times U$. Then for any submodule N_1 of U_1 and any epimorphism h_1 from N_1 to U_2, $\langle h_1 \rangle$ is a direct summand of M.*

Proof. If $N_1 = U_1$ or $\ker h_1 = 0$, it is clear $M = \langle h_1\rangle \oplus U_2$ or $M = \langle h_1\rangle \oplus U_1$. We assume $N_1 \neq U_1$ and $\ker h_1 \neq 0$, and take a submodule N_2 of U_2 which is a natural isomorphic image of N_1 and an epimorphism h_2 from N_2 to U_1. Now we prove $M = \langle h_1\rangle \oplus \langle h_2\rangle$.

First we show $M = \langle h_1\rangle + \langle h_2\rangle$. Let $\iota_i : h_i^{-1}(N_j) \to U_i$ $(i \neq j)$ be the inclusion mapping. Then $\operatorname{Im}\iota_i = h_i^{-1}(N_j) \subseteq h_i^{-1}(U_j) = N_i \subsetneq U_i$ $(i \neq j)$. We define a homomorphism h_i' from $h_i^{-1}(N_j)$ to U_i by $h_i'(x) = h_j h_i(x)$ for $x \in h_i^{-1}(N_j)$ $(i \neq j)$. Clearly h_i' is onto $(i = 1, 2)$. Since U_i is hollow, we obtain that $\iota_i - h_i' : h_i^{-1}(N_j) \to U_i$ is onto $(i \neq j)$. For any element $u_1 + u_2$ of M $(u_i \in U_i)$, there exists an element x_i of $h_i^{-1}(N_j)$ such that $(\iota_i - h_i')(x_i) = u_i$ $(i \neq j)$. Hence $u_1 + u_2 = ((x_1 - h_2(x_2)) + h_1(x_1 - h_2(x_2))) + ((x_2 - h_1(x_1)) + h_2(x_2 - h_1(x_1))) \in \langle h_1\rangle + \langle h_2\rangle$. Therefore $M = \langle h_1\rangle + \langle h_2\rangle$.

Next we show $\langle h_1\rangle \cap \langle h_2\rangle = 0$. We see

$$(\langle h_1\rangle \cap \langle h_2\rangle) \cap \ker h_1 = (\langle h_1\rangle \cap \langle h_2\rangle) \cap (\langle h_1\rangle \cap N_1) \subseteq \langle h_2\rangle \cap N_1 = 0.$$

Since $\langle h_1\rangle \cong N_1$ is uniform and $\ker h_1 \neq 0$, we obtain $\langle h_1\rangle \cap \langle h_2\rangle = 0$. □

The following is one of our main results.

Theorem 1. *Let U be a hollow and uniform module and put $M = U^2$, $U_1 = U \times 0$ and $U_2 = 0 \times U$. Then the following conditions are equivalent:*

(a) *M is lifting,*

(b) *for any module X, any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, one of the following holds:*
  - (i) *there exists a homomorphism $h : U_1 \to U_2$ such that $f = gh$,*
  - (ii) *there exist a submodule N of U_2 and an epimorphism $h : N \to U_1$ such that $g|_N = fh$,*

(c) *for any module X, any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, one of the following holds:*
  - (i) *there exists a homomorphism $h : U_1 \to U_2$ such that $f = gh$,*
  - (ii) *there exist a submodule K of $\ker g$ and a monomorphism $h : U_1 \to U_2/K$ such that $g'h = f$, where $g' : U_2/K \to X$ is defined by $g'(\overline{u}) = g(u)$ for $\overline{u} \in U_2/K$.*

Proof. Let $p_i : M = U_1 \oplus U_2 \to U_i$ be the canonical projection $(i = 1, 2)$.

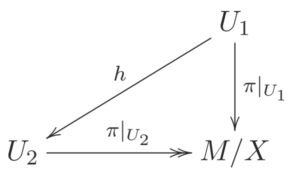
(a) ⇒ (b): Let $f : U_1 \to X$ be a nonzero homomorphism and $g : U_2 \to X$ an epimorphism. We define a homomorphism $\varphi : M \to X$ by $\varphi(u_1 + u_2) = f(u_1) - g(u_2)$ for $u_i \in U_i$ $(i = 1, 2)$. Since M is lifting, there exists a direct summand A of M such that $A \subseteq_c \ker\varphi$ in M. Then $M = \ker\varphi + U_2 = A + U_2$ because g is onto. So $p_1(A) = U_1$.

If $A \cap U_2 = 0$, we can define a homomorphism $h : U_1 = p_1(A) \to U_2$ by $h(p_1(a)) = p_2(a)$ for $a \in A$, and h satisfies $f = gh$. Therefore (i) holds.

Otherwise we see $A \cap U_1 = 0$ since U is uniform. Hence we can define an epimorphism $h : p_2(A) \to p_1(A) = U_1$ by $h(p_2(a)) = p_1(a)$ for $a \in A$, and h satisfies $g|_{p_2(A)} = fh$. Therefore (ii) holds.

(b) $\Rightarrow$ (a): Let X be a submodule of M. We may assume that X is a proper non-small submodule of M. Since U_1 and U_2 are hollow with $U_1 \cong U_2$, we only consider the case $p_1(X) = U_1$. Then $M = X + U_2$. Let $\pi : M \to M/X$ be the natural epimorphism. Since $\pi|_{U_2}$ is onto, one of the following (i) or (ii) holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $\pi|_{U_1} = \pi|_{U_2} h$, that is, the diagram



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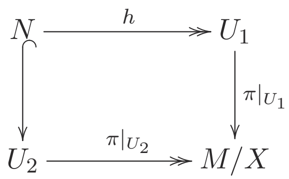
(ii) there exist a submodule N of U_2 and an epimorphism $h : N \to U_1$ such that $\pi|_N = \pi|_{U_1} h$, that is, the diagram



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In either case, we see $\langle -h \rangle$ is a direct summand of M by Lemma 2, and $\langle -h \rangle \subseteq X$ by the commutativity of the diagram. Put $M = \langle -h \rangle \oplus T$ using a direct summand T of M. Since T is hollow by Lemma 1, we obtain $T \cap X \ll T$. Hence $\langle -h \rangle \subseteq_c X$ in M. Therefore M is lifting.

(b) $\Rightarrow$ (c): It is enough to show (b)(ii) $\Rightarrow$ (c)(ii). For any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, we assume that there exist a submodule N of U_2 and an epimorphism $h : N \to U_1$ such that $g|_N = fh$. Then $\ker h \subseteq \ker g$, hence we can define an epimorphism $g' : U_2/\ker h \to X$ by $g'(\overline{u}) = g(u)$ for $\overline{u} \in U_2/\ker h$. Let $\overline{h} : N/\ker h \to U_1$ be the natural isomorphism and $\iota : N/\ker h \to U_2/\ker h$ the inclusion mapping, and put $h' = \iota\overline{h}^{-1}$. Clearly, h' is a monomorphism and $g'h' = f$.

(c) $\Rightarrow$ (b): We show (c)(ii) $\Rightarrow$ (b)(ii). For any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, we assume that there exist a submodule K of $\ker g$ and a monomorphism $h : U_1 \to U_2/K$ such that $f = g'h$, where $g' : U_2/K \to X$ is defined by $g'(\overline{u}) = g(u)$ for $\overline{u} \in U_2/\ker h$. We express $\operatorname{Im} h = N/K$. Let $\varphi : N/K \to U_1$ be the inverse map of h and $\pi : N \to N/K$ the natural epimorphism, and put $h' = \varphi\pi$. Then h' is onto and $g|_N = fh'$. □

Theorem 2. *Let U be a uniform and hollow module and put $M = U^2$, $U_1 = U \times 0$ and $U_2 = 0 \times U$. Then the following conditions are equivalent:*

(a) *M is extending,*

(b) *for any module X, any homomorphism $f : X \to U_2$ and any monomorphism $g : X \to U_1$, one of the following holds:*

  (i) *there exists a homomorphism $h : U_1 \to U_2$ such that $f = hg$,*

  (ii) *there exist a submodule K of U_1 and a monomorphism $h : U_2 \to U_1/K$ such that $hf = \pi g$, where π is the natural epimorphism from U_1 to U_1/K,*

(c) *for any module X, any homomorphism $f : X \to U_2$ and any monomorphism $g : X \to U_1$, one of the following holds:*

  (i) *there exists a homomorphism $h : U_1 \to U_2$ such that $f = hg$,*

  (ii) *there exist a submodule N of U_1 containing $\operatorname{Im} g$ and an epimorphism $h : N \to U_2$ such that $f = hg$.*

Proof. Let $p_i : M = U_1 \oplus U_2 \to U_i$ be the canonical projection ($i = 1, 2$).

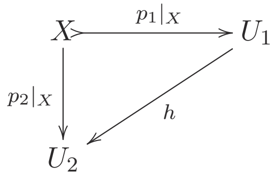
(a) $\Rightarrow$ (c): Let $f : X \to U_2$ be a nonzero homomorphism and $g : X \to U_1$ a monomorphism. We define a homomorphism $\varphi : X \to M$ by $\varphi(x) = g(x) + f(x)$ for $x \in X$. Since M is extending, there exists a direct summand A of M such that $\operatorname{Im} \varphi \subseteq_e A$. By $\operatorname{Im} \varphi \cap U_2 = 0$, $A \cap U_2 = 0$.

If $p_1(A) = U_1$, we can define a homomorphism $h : U_1 = p_1(A) \to U_2$ by $h(p_1(a)) = p_2(a)$ for $a \in A$, and h satisfies $f = hg$. Therefore (i) holds.

Otherwise, we see $p_2(A) = U_2$ since U is hollow. We see $\operatorname{Im} g \subseteq p_1(A)$, and we can define an epimorphism $h : p_1(A) \to p_2(A) = U_2$ by $h(p_1(a)) = p_2(a)$ for $a \in A$. Then h satisfies $f = hg$. Therefore (ii) holds.

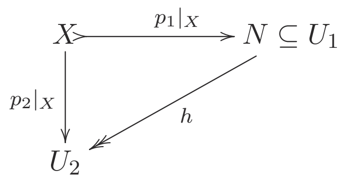
(c) $\Rightarrow$ (a): Let X be a submodule of M. We may assume that X is a nonzero non-essential submodule of M. Since U_1 and U_2 are uniform with $U_1 \cong U_2$, we only consider the case $X \cap U_2 = 0$ because U is uniform. Since $p_1|_X$ is a monomorphism, one of the following (i) or (ii) holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $p_2|_X = hp_1|_X$, that is, the diagram



commutes,

(ii) there exist a submodule N of U_1 containing $p_1(X)$ and an epimorphism $h : N \to U_2$ such that $p_2|_X = hp_1|_X$, that is, the diagram



commutes.

In either case, $\langle h\rangle$ is a direct summand of M by Lemma 2, and $X \subseteq \langle h\rangle$ by commutativity of the diagram. Since $\langle h\rangle$ is uniform by Lemma 1, we obtain $X \subseteq_e \langle h\rangle$. Therefore M is extending.

(c) $\Rightarrow$ (b): It is enough to show (c)(ii) $\Rightarrow$ (b)(ii). For any homomorphism $f : X \to U_2$ and any monomorphism $g : X \to U_1$, we assume that there exist a submodule N of U_1 containing $\operatorname{Im} g$ and an epimorphism $h : N \to U_2$ such that $f = hg$. Let $\overline{h} : N/\ker h \to U_2$ be the natural isomorphism and $\iota : N/\ker h \to U_1/\ker h$ the inclusion mapping, and put $h' = \iota\overline{h}^{-1}$. Then h' is a monomorphism and $h'f = \pi g$, where $\pi : U_1 \to U_1/\ker h$ is the natural epimorphism.

(b) $\Rightarrow$ (c): We show (b)(ii) $\Rightarrow$ (c)(ii). For any homomorphism $f : X \to U_2$ and any monomorphism $g : X \to U_1$, we assume that there exist a submodule K of U_1 and a monomorphism $h : U_2 \to U_1/K$ such that $hf = \pi g$, where $\pi : U_1 \to U_1/K$ is the natural epimorphism. We express $\operatorname{Im} h = N/K$. Let $\varphi : N/K \to U_2$ be the inverse map of h and $\eta : N \to N/K$ the natural epimorphism, and put $h' = \varphi\eta$. Then we see $\operatorname{Im} g \subseteq N$, h' is an epimorphism and $f = h'g$. □

Remark 1. In Theorem 2, the assumption "hollow" cannot be removed. In fact, $U_{\mathbb{Z}} = \mathbb{Z}$ is uniform and U^2 is extending. However it does not hold neither (i) nor (ii) in Theorem 2 (b) for a homomorphism $f : 2\mathbb{Z} \to U$ defined by $f(2n) = 3n$ and the inclusion mapping $g : 2\mathbb{Z} \to U$.

Lifting modules do not necessarily satisfy the FIEP. We can make an example of a lifting module without the FIEP, using Theorem 1.

Example 1. Let $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(q)}$ be the localizations of $\mathbb{Z}$ at two distinct prime numbers p and q respectively. We consider a semiperfect ring $R = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$ and its right ideal $L = \begin{pmatrix} 0 & \mathbb{Z}_{(q)} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$, and put $U_R = R/L$. Then U is uniserial whose the endomorphism ring is not local (see. [4]). According to [1, Proposition 12.10], U^2 does not satisfy the FIEP. We show U^2 is lifting. For any nonzero homomorphism $f : U \to U/X$ where X is a submodule of U, we can express

$$f(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}) = \overline{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}} + X$$

for some $x \in \mathbb{Z}_{(p)}$. If $x \in \mathbb{Z}_{(q)}$, we can define a homomorphism $h : U \to U$ with $h(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}) = \overline{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}}$, and h satisfies $\pi h = f$, where π is the natural epimorphism from U to U/X. Otherwise we can express $x = p^m \frac{1}{q^n} \frac{t}{s}$, where $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and $s, t \in \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$. Put $N = \overline{\begin{pmatrix} p^m & 0 \\ 0 & 0 \end{pmatrix}} R$. We can define an epimorphism $h : N \to U$ with $h(\overline{\begin{pmatrix} p^m & 0 \\ 0 & 0 \end{pmatrix}}) = \overline{\begin{pmatrix} q^n \frac{s}{t} & 0 \\ 0 & 0 \end{pmatrix}}$, and h satisfies $fh = \pi|_N$, where π is the natural epimorphism from U to U/X. Therefore U^2 is lifting by Theorem 1 (b) $\Rightarrow$ (a).

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