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On the nilpotence of the prime radical in module categories

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Abstract. For $M \in R$-Mod and τ a hereditary torsion theory on the category $\sigma[M]$ we use the concept of prime and semiprime module defined by Raggi et al. to introduce the concept of τ-pure prime radical $\mathfrak{N}_\tau(M) = \mathfrak{N}_\tau$ as the intersection of all τ-pure prime submodules of M. We give necessary and sufficient conditions for the τ-nilpotence of $\mathfrak{N}_\tau(M)$. We prove that if M is a finitely generated R-module, progenerator in $\sigma[M]$ and $\chi \neq \tau$ is FIS-invariant torsion theory such that M has τ-Krull dimension, then $\mathfrak{N}_\tau$ is τ-nilpotent.

Introduction

It is known [11], [12] that, for any ring R having right Krull dimension, the prime radical $N(R)$ is nilpotent. Later Albu, Teply and Krause proved in [1] the relative version of this theorem for an ideal invariant hereditary torsion theory τ. More precisely, let $\tau = (\mathbb{T}_\tau, \mathbb{F}_\tau)$ be a hereditary torsion theory on Mod-R such that the ring R has τ-Krull dimension, and let $N_\tau = N_\tau(R)$ denote the τ-pure prime radical, that is, the intersection of the τ-pure prime ideals. They proved that N_τ is a τ-nilpotent ideal, i.e., $N_\tau^n \in \mathbb{T}_\tau$ for some integer $n > 0$.

In this paper we use the product of submodules of modules given by L. Bican et al. [4] and the concept of prime and semiprime modules given

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in [14] and [15], to define the τ-pure prime radical $\mathfrak{N}_\tau(M)$ (this is the intersection of the τ-pure prime submodules of M) of an R-module M, where $\tau \in M$-tors. With these definitions, we proceed to make a good case of how natural the framework is, and how ring-theoretic results can be extended to the module-theoretic. So with this tool in hand the Theorem 2.16 presents necessary and sufficient conditions on M and on τ, for the τ-pure prime radical $\mathfrak{N}_\tau(M)$ to be τ-nilpotent, i.e, $\mathfrak{N}^n_\tau(M)$ is τ-torsion for $n > 0$. When M is projective in $\sigma[M]$, we prove in Corollary 3.9, that any FIS-invariant (as defined in [6]) hereditary torsion theory $\tau \in M$-tors satisfies the condition $ii)$ of the Theorem 2.16. The results obtained in this paper generalize the results given in [1]. The FIS-invariant hereditary torsion theories on the category $\sigma[M]$ are the natural extension of the ideal invariant torsion theories on R-Mod.

In order to do this, we organized the paper in three sections. In section 1, we provide the necessary material that is needed for the reading of the next sections. In section 2, we give the main results. In section 3 we give interesting results about the FIS-invariant hereditary torsion theories on the category $\sigma[M]$.

Throughout this paper R will denote an associative ring with unity and R-Mod will denote the category of unitary left R-modules. Let M and X be R-modules. X is said to be M-generated if there exists an R-epimorphism from a direct sum of copies of M onto X. The category $\sigma[M]$ is defined as the full subcategory of R-Mod consisting of all R-modules X which are isomorphic to a submodule of an M-generated module.

Let M-$tors$ be the frame of all hereditary torsion theories on $\sigma[M]$. For a family $\{M_\alpha\}$ of left R-modules in $\sigma[M]$, let $\chi(\{M_\alpha\})$ be the greatest element of M-$tors$ for which all the M_α are torsion free, and let $\xi(\{M_\alpha\})$ denote the least element of M-$tors$ for which all the M_α are torsion. $\chi(\{M_\alpha\})$ is called the torsion theory cogenerated by the family $\{M_\alpha\}$, and $\xi(\{M_\alpha\})$ is the torsion theory generated by the family $\{M_\alpha\}$. In particular, the greatest element of M-$tors$ is denoted by χ and the least element of M-$tors$ is denoted by ξ. If τ is an element of M-$tors$, $gen(\tau)$ denotes the interval $[\tau, \chi]$.

Let $\tau \in M$-$tors$. By $\mathbb{T}_\tau, \mathbb{F}_\tau, t_\tau$ we denote the torsion class, the torsion free class and the torsion functor associated to τ, respectively. For $N \in \sigma[M]$, N is called τ-cocritical if $N \in \mathbb{F}_\tau$ and for all $0 \neq N' \subseteq N$, $N/N' \in \mathbb{T}_\tau$. We say that N is cocritical if N is τ-cocritical for some $\tau \in M$-$tors$. For $N \in \sigma[M]$, let $\widehat{N}$ denote the injective hull of N in $\sigma[M]$. If N is an essential submodule of M, we write $N \subseteq_{ess} M$. If N is a fully invariant submodule of M we write $N \subseteq_{FI} M$. For $\tau \in M$-tors and $M' \in \sigma[M]$, a submodule N

of M' is τ-dense in M' if $M'/N \in \mathbb{T}_\tau$ and N is τ-pure in M' if $M'/N \in \mathbb{F}_\tau$. For $N \subseteq M$ we denote $\overline{N} = \bigcap\{C \subseteq M \mid N \subseteq C \text{ and } M/C \in \mathbb{F}_\tau\}$. The module $\overline{N}$ is called the τ-purification of N in M. If $M \in R$-Mod we denote by $\mathrm{Sat}_\tau(M)$ the set of all τ-pure submodules of M.

Let $M \in R$-Mod. In [5] the annihilator in M of a class $\mathcal{C}$ of R-modules is defined as $\mathrm{Ann}_M(\mathcal{C}) = \underset{K\in\Omega}{\cap} K$, where

$\Omega = \{K \subseteq M \mid \text{there exists } W \in \mathcal{C} \text{ and } f \in \mathrm{Hom}(M, W) \text{ with } K = \ker f\}$. Also in Beachy [3, Definition 1.5] a product is defined in the following way. Let N be a submodule of M. For each module $X \in R$-Mod, $N \cdot X = \mathrm{Ann}_X(\mathcal{C})$, where $\mathcal{C}$ is the class of modules W, such that $f(N) = 0$ for all $f \in \mathrm{Hom}(M, W)$. For $M \in R$-Mod and K, L submodules of M, in [4] the product $K_M L$ as $K_M L = \sum\{f(K) \mid f \in \mathrm{Hom}(M, L)\}$ is defined, when M is projective in $\sigma[M]$ this product coincides with the product given by Beachy in [3]. We prove in [5, Proposition 1.9] that if $M \in R$-Mod and $\mathcal{C}$ is a class of left R-modules, then $\mathrm{Ann}_M(\mathcal{C}) = \sum\{N \subset M \mid N_M X = 0 \text{ for all } X \in \mathcal{C}\}$. If $N \in R$-Mod, then we denote $\mathrm{Ann}_X(N)$ instead of $\mathrm{Ann}_X(\{N\})$.

Let $M \in R$-Mod and $N \neq M$ a fully invariant submodule of M. N is prime in M (or N is a prime submodule of M) if for any K, L fully invariant submodules of M we have that $K_M L \subseteq N$ implies that $K \subseteq N$ or $L \subseteq N$. We say that M is a prime module if 0 is prime in M see [14, Definition 13 and Definition 16]. For $X \in R$-Mod we denote $\mathrm{ann}(X) = \{r \in R \mid rX = 0\}$.

For $N \in \sigma[M]$ a proper fully invariant submodule K of M is said to be associated to N, if there exists a non-zero submodule L of N such that $\mathrm{Ann}_M(L') = K$ for all non-zero submodules L' of L. By [5, Proposition 1.16] we have that K is prime in M. We denote by $\mathrm{Ass}_M(N)$ the set of all submodules prime in M associated to N. Also note that, if N is a uniform module, then $\mathrm{Ass}_M(N)$ has at most one element.

The τ-Krull dimension $k_\tau(N)$ of a module $N \in \sigma[M]$ is the Krull dimension (or deviation) $K\dim(\mathrm{Sat}_\tau(N))$ of the poset $\mathrm{Sat}_\tau(N)$. Thus $k_\tau(N) = -1$ if and only if $N \in \mathbb{T}_\tau$, and $k_\tau(N) = \alpha$ for an ordinal $\alpha \geqslant 0$ if $k_\tau(N) \nless \alpha$ and, given any descending chain

$$N \supset N_1 \supset N_2 \supset \cdots \supset N_i \supset N_{i+1} \supset \cdots$$

of τ-pure submodules, $k_\tau(N_i/N_{i+1}) < \alpha$ for all but finitely many i. Note that if $M = R$ and $\tau = \xi$, then $k_\tau(N) = K\dim(N) = K\dim(\mathcal{L}(N))$ is the Krull dimension of the R-module N in the sense of Gordon and Robson [11]. A module $0 \neq N \in \sigma[M]$ is called τ-critical if it is τ-torsion free with τ-Krull dimension and $k_\tau(N/L) < k_\tau(N)$ for every $0 \neq L \subseteq N$. A τ-critical module N with $k_\tau(N) = \alpha$ is called α-τ-critical.

For details about concepts and terminology concerning torsion theories in $\sigma[M]$, see [17] and [18]. For torsion theoretic dimensions, the reader is referred to Golan [1], [10], [16]

1. Preliminaries

In this section we provide the necessary material that is needed for the reading of the next sections. We use the product of modules defined in [5] and the concept of prime and semiprime module defined in [14] and [15] respectively. We show some properties of τ-pure and τ-dense modules for a hereditary torsion theory on the category $\sigma[M]$. We show that if M is projective in $\sigma[M]$ and $\mathrm{Spec}(M) \neq \varnothing$, then $\mathrm{Fnil}(M) = \mathfrak{N}$, where $\mathrm{Fnil}(M) = \{a \in M \mid a \text{ is } M\text{-strongly nilpotent }\}$, $\mathrm{Spec}(M) = \{P \mid P \text{ is prime submodule of } M\}$ and $\mathfrak{N} = \cap\{P \mid P \in \mathrm{Spec}(M)\}$.

We require a goodly number of results from the literature. We include here those results for convenience of the reader.

Definition 1.1 (see [7]). *Let $M \in R$-Mod and $N \subsetneq_{FI} M$. We say that N is semiprime in M (or a semiprime submodule of M) if for any $K \subseteq_{FI} M$ such that $K_M K \subseteq N$, then $K \subseteq N$. We say M is semiprime if 0 is semiprime in M.*

Remark 1.2 (see [7]). Notice that if M is projective in $\sigma[M]$ and $N \subseteq_{FI} M$, then N is semiprime in M if and only if for any submodule K of M such that $K_M K \subseteq N$ implies that $K \subseteq N$.

Remark 1.3 (see [7]). Let $M \in R$-Mod be projective in $\sigma[M]$. Similarly we can prove that a fully invariant submodule P of M is prime in M if and only if for any submodules K and L of M containing P, such that $K_M L \subseteq P$, then $K = P$ or $L = P$.

Definition 1.4 (see [8]). *Let $M \in R$-Mod. If N is a submodule of M, then successive powers of N are defined as follows: First, $N^2 = N_M N$. Then by induction, for any integer $k > 2$, we define $N^k = N_M(N^{k-1})$.*

Lemma 1.5 (see [8]). *Let $M \in R$-Mod be projective in $\sigma[M]$ and let N be a semiprime submodule of M. If J is a submodule of M such that $J^n \subseteq N$ for some positive positive integer n, then. $J \subseteq N$.*

Proposition 1.6 (see [8]). *Let $M \in R$-Mod and $\tau \in M$-tors. If P is a prime τ-pure submodule of M, then there exists $P' \subseteq P$ such that P' is a minimal τ-pure prime submodule of M.*

Proposition 1.7 (see [8]). *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\tau \in M$-tors. If U is a uniform submodule of M, such that $U \in \mathbb{F}_\tau$, then* $\mathrm{Ann}_M(U)$ *is a τ-pure prime submodule of M.*

Definition 1.8 (see [8]). *Let $M \in R$-Mod and $\tau \in M$-tors. A fully invariant submodule N of M is τ-nilpotent if $N^n = \underbrace{N_M N_M \dots {}_M N}_{n\text{-times}} \in \mathbb{T}_\tau$ or, equivalently, if $N^n \subseteq t_\tau(M)$ for some positive integer n; it is τ-idempotent if $\frac{N}{N_M N} \in \mathbb{T}_\tau$.*

Remark 1.9. Notice that N is τ-idempotent if and only if $\overline{N} = \overline{N_M N}$. In fact we know that $\frac{\overline{N}}{N_M N}$ is the τ-purification of $\frac{N}{N_M N}$ in $\frac{M}{N_M N}$.

Thus $\left(\frac{\overline{N}}{N_M N}\right) \Big/ \left(\frac{N}{N_M N}\right) \in \mathbb{T}_\tau$. Since $\frac{N}{N_M N} \in \mathbb{T}_\tau$, then $\frac{\overline{N}}{N_M N} \in \mathbb{T}_\tau$. Hence $\frac{\overline{N}}{\overline{N_M N}} \in \mathbb{T}_\tau$. But $\frac{\overline{N}}{\overline{N_M N}} \subseteq \frac{M}{\overline{N_M N}} \in \mathbb{F}_\tau$. Therefore $\frac{\overline{N}}{\overline{N_M N}} = 0$. So $\overline{N} = \overline{N_M N}$. Inversely if $\overline{N} = \overline{N_M N}$, then $\frac{\overline{N}}{\overline{N_M N}} = 0$. Hence $\frac{\overline{N}}{N_M N} \Big/ \frac{\overline{N_M N}}{N_M N} = 0 \in \mathbb{T}_\tau$. As $\overline{N_M N}$ is the τ-purification of $N_M N$ in M, then $\frac{\overline{N_M N}}{N_M N} \in \mathbb{T}_\tau$. Thus $\frac{\overline{N}}{N_M N} \in \mathbb{T}_\tau$. So $\frac{N}{N_M N} \in \mathbb{T}_\tau$.

Lemma 1.10. *Let $M \in R$-Mod and $\tau \in M$-tors. If L and N are a submodules of M such that $L \subseteq N$ and $N/L \in \mathbb{T}_\tau$, then $\frac{N_M K}{L_M K} \in \mathbb{T}_\tau$ for all $K \in \sigma[M]$.*

Proof. Let $y \in N_M K$. As $N_M K = \sum\{f(N) \mid f \in \mathrm{Hom}(M, K)\}$, then $y = f_1(n_1) + f_2(n_2) + \cdots + f_r(n_r)$ with $n_i \in N$ and $f_i \in \mathrm{Hom}(M, K)$. Put $y' = y + L_M K$. Then $y' = f_1(n_1) + f_2(n_2) + \cdots + f_r(n_r) + L_M K$.

Let $n_i' = n_i + L$. Thus $0 = \mathrm{ann}(n_i')(n_i + L)$. Hence $\mathrm{ann}(n_i')n_i \subseteq L$. So $\mathrm{ann}(n_i')f_i(n_i) = f_i(\mathrm{ann}(n_i')n_i) \subseteq f_i(L) \subseteq L_M K$.

Therefore $\mathrm{ann}(n_i')\,[f_i(n_i) + L_M K] = 0$, and hence $\mathrm{ann}(n_i') \subseteq \mathrm{ann}(f_i(n_i) + L_M K)$. On the other hand we have that $N/L \in \mathbb{T}_\tau$. Hence $\frac{R}{\mathrm{ann}(n_i')} \cong Rn_i' \in \mathbb{T}_\tau$. So $\frac{R}{\mathrm{ann}(f_i(n_i)+L_M K)} \in \mathbb{T}_\tau$ for all $1 \leqslant i \leqslant r$. Thus $\frac{N_M K}{L_M K} \in \mathbb{T}_\tau$.

Corollary 1.11. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\tau \in M$-tors. If I is a submodule of M such that $\frac{I}{I_M I} \in \mathbb{T}_\tau$, then $\frac{I}{I^n} \in \mathbb{T}_\tau$ for any integer $n \geqslant 2$.*

Proof. For $n = 2$ we have the result. Now let $n > 2$. By Lemma 1.10 we have that $\frac{I_M I}{(I^n)_M I} \in \mathbb{T}_\tau$. Now we consider the following short exact sequence $0 \to \frac{I_M I}{(I^n)_M I} \to \frac{I}{(I^n)_M I} \to \frac{I}{I_M I} \to 0$. Hence we obtain that $\frac{I}{(I^n)_M I} \in \mathbb{T}_\tau$. As M is projective in $\sigma[M]$, then $\frac{I}{(I^n)_M I} = \frac{I}{I_M (I^n)} = \frac{I}{I^{n+1}} \in \mathbb{T}_\tau$.

Proposition 1.12. *Let $M \in R$-Mod be projective in $\sigma[M]$. If N is a submodule of M and $\{L_i\}_{i\in I}$ is a family of submodules of a module $X \in \sigma[M]$, then $N_M\left(\sum\limits_{i\in I} L_i\right) = \sum\limits_{i\in I} N_M L_i$.*

Proof. By [5, Proposition 1.3] we have that $\sum\limits_{i\in I} N_M L_i \subseteq N_M\left(\sum\limits_{i\in I} L_i\right)$. We consider the canonical epimorphism $\varphi : \bigoplus\limits_{i\in I} L_i \to \sum\limits_{i\in I} L_i$ and let $f : M \to \sum\limits_{i\in I} L_i$ be a morphism. As M is projective in $\sigma[M]$ there exists $\widehat{f} : M \to \bigoplus\limits_{i\in I} L_i$ such that the following diagram commutes:

$$\begin{tikzcd}
& M \arrow[dl, "\widehat{f}"'] \arrow[d, "f"] & \\
\bigoplus\limits_{i\in I} L_i \arrow[r, "\varphi"] & \sum\limits_{i\in I} L_i \arrow[r] & 0
\end{tikzcd}$$

So $\varphi \circ \widehat{f} = f$. Thus $f(N) = \varphi(\widehat{f}(N))$. If $x \in N$, then $f(x) = \varphi(\widehat{f}(x))$. But $\widehat{f}(x) \in \bigoplus\limits_{i\in I} L_i$. So there exists $r \in \mathbb{N}$ such that $\widehat{f}(x) = (l_1, l_2, \ldots, l_r)$. Hence we obtain $\pi_j(\widehat{f}(x)) = l_j$ where $\pi_j : \bigoplus\limits_{i\in I} L_i \to L_j$ is the projection on L_j. Thus we have $f(x) = \varphi(\widehat{f}(x)) = \varphi(\pi_1 \circ \widehat{f}(x), \pi_2 \circ \widehat{f}(x), \ldots, \pi_r \circ \widehat{f}(x)) = \pi_1 \circ \widehat{f}(x) + \pi_2 \circ \widehat{f}(x) + \cdots + \pi_r \circ \widehat{f}(x)$ with $\pi_j \circ \widehat{f} : M \to L_j$ and $1 \leqslant j \leqslant r$. Therefore $N_M\big(\sum\limits_{i\in I} L_i\big) \subseteq \sum\limits_{i\in I} N_M L_i$.

Definition 1.13. *Let $M \in R$-Mod and $N \in \sigma[M]$. If Y is a submodule of M, we define $\mathrm{Ann}_M^N(Y) = \sum\{L \subseteq N \mid Y_M L = 0\}$.*

Proposition 1.14. *Let $M \in R$-Mod be projective in $\sigma[M]$ and let Y be a submodule of M. If N is a fully invariant submodule of M, then $\mathrm{Ann}_M^N(Y)$ is a fully invariant submodule of M.*

Proof. Let $f : M \to M$ be a morphism and L be a submodule of N such that $Y_M L = 0$. We claim that $Y_M[f(L)] = 0$. In fact, we know that $Y_M[f(L)] = \sum\{h(Y) \mid h : M \to f(L)\}$. Now let $h : M \to f(L)$ be a morphism. As M is projective in $\sigma[M]$, then there exists a morphism

$g: M \to L$ such that the following diagram commutes

$$\begin{tikzcd} & M \arrow[dl, "g"'] \arrow[d, "h"] & \\ L \arrow[r, "f"] & f(L) \arrow[r] & 0 \end{tikzcd}$$

So $f(g(Y)) = h(Y)$. As $Y_M L = 0$, then $g(Y) = 0$. Thus $h(Y) = 0$. Hence $Y_M [f(L)] = 0$. On the other hand as N is a fully invariant submodule of M, then $f(L) \subseteq N$. So $f(L) \subseteq \mathrm{Ann}_M^N(Y)$. Therefore $h\left(\mathrm{Ann}_M^N(Y)\right) \subseteq \mathrm{Ann}_M^N(Y)$. Hence $\mathrm{Ann}_M^N(Y)$ is a fully invariant submodule of M.

Notice that if M is projective in $\sigma[M]$ in Definition 1.13, then by Proposition 1.12 $\mathrm{Ann}_M^N(Y)$ is the largest submodule of N such that $Y_M\left[\mathrm{Ann}_M^N(Y)\right] = 0$.

For $M \in R$-Mod and $\tau \in M$-tors, we will also use the notation.

- $\mathrm{Spec}(M) = \{P \mid P \text{ is a prime submodule of } M\}$;
- $\mathrm{Spec}_\tau(M) = \{P \in \mathrm{Spec}(M) \mid P \text{ is } \tau\text{-pure in } M\}$.

Notice that if $\mathrm{Spec}(M) \neq \varnothing$, then

$$\mathfrak{N} = \mathfrak{N}(M) = \bigcap\{P \mid P \text{ is a prime submodule of } M\}$$

and if $\mathrm{Spec}(M) = \varnothing$, then $\mathfrak{N} = M$.

Analogously if $\mathrm{Spec}_\tau(M) \neq \varnothing$, then

$$\mathfrak{N}_\tau = \mathfrak{N}_\tau(M) = \bigcap\{P \in \mathrm{Spec}(M) \mid P \text{ is } \tau\text{-pure in } M\}$$

and if $\mathrm{Spec}_\tau(M) = \varnothing$, then $\mathfrak{N}_\tau = M$.

Definition 1.15. *Let $M \in R$-Mod. An element $a \in M$ is M-strongly nilpotent if any sequence $a = a_0, a_1, a_2 \ldots$ such that $Ra_{i+1} \subseteq (R_{a_i})_M (R_{a_i})$ is ultimately zero.*

We put $\mathrm{Fnil}(M) = \{a \in M \mid a \text{ is } M\text{-strongly nilpotent}\}$.

Proposition 1.16. *Let $M \in R$-Mod. If M is projective in $\sigma[M]$ and $\mathrm{Spec}(M) \neq \varnothing$, then $\mathrm{Fnil}(M) = \mathfrak{N}$.*

Proof. Suppose that $a \notin \mathfrak{N}$. Hence there exists a prime submodule P of M such that $a \notin P$. Thus $(Ra)_M (Ra) \nsubseteq P$. Therefore there exists $a_1 \in (Ra)_M (Ra)$ with $a_1 \notin P$. Thus $(Ra_1)_M (Ra_1) \nsubseteq P$. Repeating this process we obtain a sequence of non zero elements $a = a_0, a_1, a_2 \ldots$ such that $Ra_{i+1} \subseteq (Ra_i)_M (Ra_i)$. Hence a is not M-strongly nilpotent. So $a \notin \mathrm{Fnil}(M)$. Thus $\mathrm{Fnil}(M) \subseteq \mathfrak{N}$. Now let $a \notin \mathrm{Fnil}(M)$, then there exists a sequence $a = a_0, a_1, a_2 \ldots$ such that $Ra_{i+1} \subseteq (Ra_i)_M (Ra_i)$ for all i. If $A = \{a = a_0, a_1, a_2 \ldots . a_n \ldots\}$, then $A \cap \{0\} = \varnothing$. Using Zorn's lemma we may choose a fully invariant submodule P of M maximal with respect to having $A \cap P = \varnothing$. We claim that P is prime in M. In fact, let K and L be fully invariant submodules of M such that $P \subsetneq K$ and $P \subsetneq L$. Therefore there exist $a_i \in K \cap A$ and $a_j \in L \cap A$. So we may suppose that $j \geqslant i$. So $j = i + r$ for some integer $r \geqslant 0$. Since $Ra_{i+1} \subseteq (Ra_i)_M (Ra_i)$, then $Ra_j \subseteq (Ra_i)_M (Ra_i) \subseteq K$. Thus $a_j \in K \cap L$. Hence $a_{j+1} \in Ra_{j+1} \subseteq (Ra_j)_M (Ra_j) \subseteq K_M L$. Therefore $K_M L \nsubseteq P$. So P is prime in M. Moreover $A \cap P = \varnothing$. Thus $a \notin P$. So $a \notin \mathfrak{N}$. Thus $\mathfrak{N} \subseteq \mathrm{Fnil}(M)$.

Lemma 1.17. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\tau \in M$-tors. If C and B are fully invariant submodules of M, then $\overline{\overline{C}_M B} = \overline{C_M B}$.*

Proof. By [5, Proposition 1.3] we have that $C_M B \subseteq \overline{C}_M B$. So $\overline{C_M B} \subseteq \overline{\overline{C}_M B}$. As $C_M B \subseteq \overline{C_M B}$, then by Proposition 1.12. $C_M \left(B + \overline{C_M B}\right) = C_M B + C_M \left(\overline{C_M B}\right) \subseteq \overline{C_M B}$. By [3, Proposition 5.5] we have that $C_M \left(\frac{B + \overline{C_M B}}{\overline{C_M B}}\right) = 0$. We claim that $\overline{C}_M \left(\frac{B + \overline{C_M B}}{\overline{C_M B}}\right) = 0$. In fact let $f : M \to \frac{B + \overline{C_M B}}{\overline{C_M B}}$ be a morphism, then $f(C) = 0$. So we can define the morphism $\widehat{f} : \frac{M}{C} \to \frac{B + \overline{C_M B}}{\overline{C_M B}}$ such that $\widehat{f}(x + C) = f(x)$. As $\overline{C}/C \in \mathbb{T}_\tau$ and $\frac{B + \overline{C_M B}}{\overline{C_M B}} \subseteq \frac{M}{\overline{C_M B}} \in \mathbb{F}_\tau$. Thus $\widehat{f}\left(\overline{C}/C\right) = 0$. Hence $0 = \widehat{f}(x + C) = f(x)$ for all $x \in \overline{C}$. Hence $\widehat{f}\left(\overline{C}\right) = 0$. Therefore $\overline{C}_M \left(\frac{B + \overline{C_M B}}{\overline{C_M B}}\right) = 0$. Hence by [3, Proposition 5.5] we have that $\overline{C}_M \left(B + \overline{C_M B}\right) \subseteq \overline{C_M B}$. Thus $\overline{C}_M B \subseteq \overline{C_M B}$. Hence $\overline{\overline{C}_M B} \subseteq \overline{C_M B}$. Therefore $\overline{\overline{C}_M B} = \overline{C_M B}$.

2. τ-Nilpotence of $\mathfrak{N}_\tau$

In this section we use the τ-noetherian modules and we give equivalent conditions for $\mathfrak{N}_\tau$ to be τ-nilpotent. Although much more difficult we give necessary and sufficient conditions for the τ-nilpotence under the hypothesis that M is a finitely generated R-module with τ-Krull dimension. These results extend the results given by Albu, Krause and Teply in [1, Proposition 3.3 and Theorem 4.5].

τ-Noetherian modules

The following definition was given in [6, Definition 1.1].

We say that an R-module M is τ-noetherian if the lattice $\mathrm{Sat}_\tau(M)$ satisfies the ascending chain condition.

Lemma 2.1. *Let $M \in R$-Mod, $\tau \in M$-tors and $N \in \sigma[M]$. If $N \in \mathbb{F}_\tau$, then $\mathrm{Ann}_M(N)$ is τ-pure in M.*

Lemma 2.2. *Let $M \in R$-Mod be a projective generator in $\sigma[M]$ and $\tau \in M$-tors. Suppose that P is maximal in the set of all τ-pure fully invariant submodules of M. Then P is a prime submodule of M.*

Proof. Let L and K be fully invariant submodules of M such that $P \subseteq L$, $P \subseteq K$ and $L_M K \subseteq P$. As M is projective, then by [3, Proposition 5.5] $L_M(K/P) = 0$. By [5, Proposition 1.9] we have that $L \subseteq \mathrm{Ann}_M(K/P)$. Since M is generator of $\sigma[M]$, then $\mathrm{Ann}_M(K/P) \neq M$. On the other hand we know that $P \subseteq L \subseteq \mathrm{Ann}_M(K/P)$. Moreover K/P is a τ-torsion free module, then by Lemma 2.1 we have that $\mathrm{Ann}_M(K/P)$ is a fully invariant τ-pure submodule of M. By the maximality of P we must have $P = L = \mathrm{Ann}_M(K/P)$. So by [8, Remark 1.4] we have that P is a prime submodule of M.

Notice that the lemma 2.2 is not true in general. Consider the Example given in [7, Example 3.15]. The maximal fully invariant submodules L, K and N of $M = E(S)$ are ξ-pure in M. But they are not prime submodules of M.

Remark 2.3. If $M \in R - Mod$, $\tau \in M$-*tors*, and N is a fully invariant submodule of M, then τ^N denotes the direct image of τ in the category $\sigma[M/N]$, where $\mathbb{T}_{\tau^N} = \{L \in \sigma[M/N] \mid L \in \mathbb{T}_\tau\}$. It is clear that $\tau^N \in M/N$-*tors*. Moreover, if $L \in \sigma[M/N]$, then $\{K \subseteq L \mid L/K \in \mathbb{F}_{\tau^N}\} = \{K \subseteq L \mid L/K \in \mathbb{F}_\tau\}$. In fact let $K \subseteq L$ such that $L/K \in \mathbb{F}_{\tau^N}$. If $L/K \notin \mathbb{F}_\tau$, then there exists $0 \neq L'/K \subseteq L/K$ such that $L'/K \in \mathbb{T}_\tau$. As $L/K \in \sigma[M/N]$, then $L'/K \in \mathbb{T}_{\tau^N}$. So $L/K \notin \mathbb{F}_{\tau^N}$ a contradiction. Therefore, $L/K \in \mathbb{F}_\tau$ and $\{K \subseteq L \mid L/K \in \mathbb{F}_{\tau^N}\} \subseteq \{K \subseteq L \mid L/K \in \mathbb{F}_\tau\}$.

Analogously we prove that

$$\{K \subseteq L \mid L/K \in \mathbb{F}_\tau\} \subseteq \{K \subseteq L \mid L/K \in \mathbb{F}_{\tau^N}\}.$$

Notice that If M is projective in $\sigma[M]$, $\tau \in M$-*tors* and $I \subseteq A$ both are fully invariant submodules of M, then by Remark 2.3, $\overline{\left(\frac{A}{I}\right)} = \frac{\overline{A}}{I}$, where

$\overline{\left(\frac{A}{I}\right)}$ is τ^I-purification of $\frac{A}{I}$ in $\frac{M}{I}$ and $\overline{A}$ is the τ-purification of A in M. So we will denote by $\frac{\overline{A}}{I}$ the τ^I-purification of $\frac{A}{I}$ in $\frac{M}{I}$.

Proposition 2.4. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\chi \neq \tau \in M$-tors. If M is τ-noetherian, then the following conditions are equivalent.*

i) $\mathfrak{N}_\tau^k \subseteq t_\tau(M)$ for some integer $k > 0$.

ii) For each fully invariant submodule A of M, there exists an integer $k > 0$ such that $\left(\mathfrak{N}_\tau^k\right)_M \overline{A} \subseteq \overline{(\mathfrak{N}_\tau)_M A}$.

iii) For each fully invariant submodule A of M that is contained in $\mathfrak{N}_\tau$, there exists an integer $k > 0$ such that $\left(\mathfrak{N}_\tau^k\right)_M \overline{A} \subseteq \overline{(\mathfrak{N}_\tau)_M A}$.

Proof. i) $\Rightarrow$ *ii)* As $\mathfrak{N}_\tau^k \subseteq t_\tau(M)$, then by [5, Proposition 1,3], we have that $\left(\mathfrak{N}_\tau^k\right)_M \overline{A} \subseteq (t_\tau(M))_M \overline{A}$. Since $t_\tau(M)$ is fully invariant submodule of M, then $(t_\tau(M))_M \overline{A} \subseteq t_\tau(M)$. Thus $\left(\mathfrak{N}_\tau^k\right)_M \overline{A} \subseteq t_\tau(M)$. We know that $\overline{0} = t_\tau(M)$. As $\overline{(\mathfrak{N}_\tau)_M A}$ is τ-pure in M, then $t_\tau(M) \subseteq \overline{(\mathfrak{N}_\tau)_M A}$. Hence $\left(\mathfrak{N}_\tau^k\right)_M \overline{A} \subseteq \overline{(\mathfrak{N}_\tau)_M A}$.

ii) $\Rightarrow$ *iii)* It is clear.

iii) $\Rightarrow$ *i)* Since $\tau \neq \chi$, then $t_\tau(M) \subsetneq M$. Moreover $t_\tau(M)$ is a τ-pure fully invariant submodule of M. As M is τ-noetherian, then there exists a maximal fully invariant submodule Q of M such that Q is τ-pure in M. By Lemma 2.2. we have that Q is a prime submodule of M. Hence $\mathfrak{N}_\tau \subseteq Q \subsetneq M$. Suppose that there exists a τ-pure fully invariant submodule I of M such that $I \subseteq \mathfrak{N}_\tau$, but $\mathfrak{N}_\tau^j \nsubseteq I$ for all $j > 0$. As M is τ-netherian we can choose I maximal with respect to this property. It is clear that $I \subsetneq \mathfrak{N}_\tau$. So $\mathfrak{N}_\tau/I \neq 0$. As $\mathfrak{N}_\tau/I \in \mathbb{F}_\tau$ and M is τ noetherian, then by [6, Proposition 2.7] we have that $\mathrm{Ass}_M(\mathfrak{N}_\tau/I) \neq \varnothing$. Let $P \in \mathrm{Ass}_M(\mathfrak{N}_\tau/I)$, then $\mathrm{Ann}_M^{(\mathfrak{N}_\tau/I)}(P) = \sum\{\frac{N'}{I} \subseteq \frac{\mathfrak{N}_\tau}{I} \mid P_M \frac{N'}{I} = 0\} \neq 0$. So we put $\mathrm{Ann}_M^{(\mathfrak{N}_\tau/I)}(P) = \frac{A}{I}$. Thus $P_M \frac{A}{I} = 0$. By [3, Proposition 5.5] we have that $P_M A \subseteq I$. We claim that A is a fully invariant submodule of M. In fact as $A \subseteq \mathfrak{N}_\tau$, then by [5, Proposition 1.3] we have that $A_M M \subseteq (\mathfrak{N}_\tau)_M M = \mathfrak{N}_\tau$ since $\mathfrak{N}_\tau$ is fully invariant submodule of M. As $I \subseteq A$ and $I_M M = I$, then $I = I_M M \subseteq A_M M$. Hence $\frac{A_M M}{I} \subseteq \frac{\mathfrak{N}_\tau}{I}$. By [3, Proposition 5.6] we have that $P_M(A_M M) = (P_M A)_M M \subseteq I_M M = I$. Hence by [3, Proposition 5.5] $P_M\left(\frac{A_M M}{I}\right) = 0$. Thus $\frac{A_M M}{I} \subseteq \mathrm{Ann}_M^{(\mathfrak{N}_\tau/I)}(P) = \frac{A}{I}$. Hence $A_M M \subseteq A$. Therefore A is a fully invariant submodule of M. Since $P \in \mathrm{Ass}_M(\mathfrak{N}_\tau/I)$, then there exists $N'/I \subseteq \mathfrak{N}_\tau/I$ such that $P = \mathrm{Ann}_M(N''/I)$ for all $0 \neq N''/I \subseteq N'/I$. By lemma 2.1 we have that P is τ-pure in M. As P is prime, then $\mathfrak{N}_\tau \subseteq P$. So by [5, Proposition 1.3] we have that $(\mathfrak{N}_\tau)_M A \subseteq P_M A$. Thus $(\mathfrak{N}_\tau)_M A \subseteq I$. On the other hand we know that $A \subseteq \mathfrak{N}_\tau$. By hypothesis, there exists an integer $k > 0$ such that $\left(\mathfrak{N}_\tau^k\right)_M \overline{A} \subseteq \overline{(\mathfrak{N}_\tau^k)_M A} \subseteq \overline{I} = I$. Moreover as $\mathfrak{N}_\tau$ is τ-pure, then $\overline{A} \subseteq \mathfrak{N}_\tau$. Since $t_\tau(M/A) = \overline{A}/A$, then $\overline{A}/A$ is a fully invariant submodule of M/A. Hence by [14, Lemma 17] we have that $\overline{A}$ is a fully invariant submodule of M. Since $I \subsetneq A \subseteq \overline{A} \subseteq \mathfrak{N}_\tau$, our choice of I implies $\mathfrak{N}_\tau^q \subseteq \overline{A}$ for some $q > 0$. By [5, Proposition 1.3] we have that $\left(\mathfrak{N}_\tau^k\right)_M \mathfrak{N}_\tau^q \subseteq \left(\mathfrak{N}_\tau^k\right)_M \overline{A} \subseteq I$. Hence $\mathfrak{N}_\tau^{k+q} \subseteq I$ which is a contradiction. Therefore, every τ-pure fully invariant submodule of M contains a power of $\mathfrak{N}_\tau$, so in particular, $\mathfrak{N}_\tau^k \subseteq t_\tau(M)$ for some $k > 0$.

Modules with τ-Krull dimension

Let $M \in R$-Mod and $\tau \in M$-tors. For $X \subseteq \operatorname{Hom}(M, M)$, let $\mathcal{A}_{X_\tau} = \bigcap_{f \in X}\{\ker f \mid \ker f \text{ is } \tau\text{-pure in } M\}$. We consider the set $\tau\text{-}\mathcal{A}_M = \{\mathcal{A}_{X_\tau} \mid X \subseteq \operatorname{Hom}(M, M)\}$.

The following definition was given in [7, Definition 3.1].

Definition 2.5. *Let $M \in R$-Mod and $\tau \in M$-tors. We say M satisfies the ascending chain condition (ACC) on τ-pure annihilators, if $\tau\text{-}\mathcal{A}_M$ satisfies ACC. If $\tau = \xi$, we say that M satisfies ACC on annihilators.*

Proposition 2.6. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\tau \in M$-tors. If M is semiprime and M has τ-Krull dimension, then M satisfies ACC on annihilators.*

Proof. As M has τ-Krull dimension, then by [7, Corollary 2.13] we have that N contains a monoform submodule for all $0 \neq N \in \sigma[M]$. By [7, Proposition 3.8] we have that M is non M-singular, moreover as M has finite uniform dimension. So by [7, Proposition 3.6] we have that M satisfies ACC on annihilators.

Corollary 2.7. *Let $M \in R$-Mod be projective in $\sigma[M]$ and let $\tau \in M$-tors. If M is semiprime and M has τ-Krull dimension, then M has a finite number of minimal prime submodules $P_1, P_2, \ldots, P_n$ and $P_1 \cap P_2 \cap \cdots \cap P_n = 0$.*

Proof. By Proposition 2.6. we have that M satisfies ACC on annihilators. So by [8, Theorem 2.2.] the result is clear.

Proposition 2.8. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\tau \in M$-tors. If $M \in \mathbb{F}_\tau$ and M has τ-Krull dimension such that $\mathfrak{N} = 0$, then $\mathfrak{N}_\tau = 0$.*

Proof. We suppose that $0 = \mathfrak{N} = \bigcap\{P \mid P$ is prime submodule of $M\}$. So M is semiprime module. Moreover by the proof of [9, Proposition 1.12] we have that every prime submodule P of M contains a minimal prime submodule. Hence by Corollary 2.7 we have that $P_1 \cap P_2 \cap \cdots \cap P_n = 0$ where P_i is a minimal prime submodule of M. Moreover, by [9, Lemma 1.23] we have that $P_i = \mathrm{Ann}_M\left(\cap_{j\neq i} P_j\right)$. As $M \in \mathbb{F}_\tau$, then by Lemma 2.1 P_i is τ-pure in M for $i = 1, 2, \ldots, n$. Thus $\mathfrak{N}_\tau \subseteq P_1 \cap P_2 \cap \cdots \cap P_n = 0$. So $\mathfrak{N}_\tau = 0$.

Notice that if M is as in Proposition 2.8, then every minimal τ-pure prime submodule P of M is a minimal prime submodule of M. In fact let P be a minimal prime submodule of M. By [9, Proposition 1.12] we have that there exists a minimal prime submodule P' of M such that $P' \subseteq P$. As P' is τ-pure, then $P' = P$.

Lemma 2.9. *Let $M \in R$-Mod be projective in $\sigma[M]$ and let $\tau \in M$-tors. If $M \in \mathbb{F}_\tau$ and M has τ-Krull dimension such that $\mathfrak{N}_\tau \neq 0$, then $\mathfrak{N}_\tau$ contains a non zero fully invariant submodule I of M such that $I_M I = 0$.*

Proof. By Proposition 2.8 we have that $\mathfrak{N} \neq 0$. Moreover $\mathfrak{N} \subseteq \mathfrak{N}_\tau$. If $\mathfrak{N}_M\mathfrak{N} = 0$, then we have finished. Suppose that $\mathfrak{N}_M\mathfrak{N} \neq 0$, then there exists $f : M \to \mathfrak{N}$ such that $f(\mathfrak{N}) \neq 0$. So there exists $n \in \mathfrak{N}$ such that $0 \neq a = f(n)$. Hence $0 \neq Ra$. If $(Ra)_M (Ra) \neq 0$, then there exists $0 \neq a_1 \in (Ra)_M (Ra)$. Thus $(Ra_1)_M (Ra_1) \subseteq (Ra)_M (Ra)$. So we obtain a succession $a = a_0, a_1, \ldots, a_i, a_{i+1}$ such that $a_{i+1} \in (Ra_i)_M (Ra_i)$. Since $a \in \mathfrak{N}$, then by Proposition 1.16 we have that $a \in \mathrm{Fnil}(M)$. Hence there exits n such that $a_n \neq 0$ such that $(Ra_n)_M (Ra_n) = 0$. As $Ra_n \subseteq \mathfrak{N}$, then $(Ra_n)_M M \subseteq \mathfrak{N}$ and $(Ra_n)_M M$ is a fully invariant submodule of M. Moreover as M is projective in $\sigma[M]$ by [3, Proposition 5.5] we have that $[(Ra_n)_M M]_M [(Ra_n)_M M] = [(Ra_n)_M (M_M Ra_n)]_M M \subseteq [(Ra_n)_M Ra_n]_M M = 0$. So $I = (Ra_n)_M M$.

Lemma 2.10. *Let $M \in R$-Mod be projective in $\sigma[M]$. Suppose that A, B are submodules of M and I is a fully invariant submodule of M such that $I \subseteq A$, $I \subseteq B$, then $\left(\frac{A}{I}\right)_{M/I}\left(\frac{B}{I}\right) = \frac{A_M B + I}{I}$.*

Proof. Let $F : M/I \to B/I$ be a morphism. As M is projective in $\sigma[M]$, then there exists a morphism $f : M \to B$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\pi_1"] \arrow[d, "f"] & M/I \arrow[d, "F"] & \\ B \arrow[r, "\pi_2"] & B/I \arrow[r] & 0 \end{tikzcd}$$

So $\pi_2 \circ f = F \circ \pi_1$. As $(\pi_2 \circ f)(A) = \pi_2(f(A)) = \frac{f(A)+I}{I}$ and $(F \circ \pi_1)(A) = F(\pi_1(A)) = F(A/I)$, then $F(A/I) = \frac{f(A)+I}{I} \subseteq \frac{A_M B+I}{I}$. Thus $\left(\frac{A}{I}\right)_{\frac{M}{I}} \left(\frac{B}{I}\right) \subseteq \frac{A_M B+I}{I}$. Inversely let $f : M \to B$ be a morphism. As I is fully invariant submodule of M, then we can define the morphism $\widehat{f} : M/I \to B/I$ such that $\widehat{f}(x+I) = f(x) + I$. So $\widehat{f}(A/I) = \frac{f(A)+I}{I} \subseteq \left(\frac{A}{I}\right)_{\frac{M}{I}} \left(\frac{B}{I}\right)$. Thus $\frac{A_M B+I}{I} \subseteq \left(\frac{A}{I}\right)_{\frac{M}{I}} \left(\frac{B}{I}\right)$. So $\left(\frac{A}{I}\right)_{\frac{M}{I}} \left(\frac{B}{I}\right) = \frac{A_M B+I}{I}$.

Corollary 2.11. *Let $M \in R$-Mod be projective in $\sigma[M]$. Suppose that A is a submodule of M and I is a fully invariant submodule of M such that $I \subseteq A$, then $\left(\frac{A}{I}\right)^r = \frac{A^r+I}{I}$.*

We consider the following condition:

($\mathfrak{A}$) If A and B are fully invariant submodules of M, such that $B \subseteq \mathfrak{N}_\tau$, then $B^r_M \overline{A} \subseteq \overline{B_M A}$ for some integer $r > 0$.

Proposition 2.12. *Let $M \in R$-Mod be projective in $\sigma[M]$ and let $\tau \in M$-tors. If M has τ-Krull dimension and M satisfies the property ($\mathfrak{A}$), then the following conditions hold.*

i) For any fully invariant submodule I of M such that $I \subseteq \mathfrak{N}_\tau$, the factor module M/I satisfies ($\mathfrak{A}$)

ii) The τ-purification of a τ-nilpotent fully invariant submodule I of M is τ-nilpotent.

Proof. i) Let I be a fully invariant submodule of M such that $I \subseteq \mathfrak{N}_\tau$. We put $\mathfrak{N}_{\tau^I}(M/I) = \bigcap\{P/I \mid P/I \in \operatorname{Spec}_{\tau^I}(M/I)$, where $\tau^I \in \frac{M}{I}$-*tors* $\}$. We claim that $\mathfrak{N}_{\tau^I}(M/I) = \frac{\mathfrak{N}_\tau}{I}$. In fact, since $I \subseteq \mathfrak{N}_\tau$, then $I \subseteq P$ for all $P \in \operatorname{Spec}_\tau(M)$. So by Remark 2.3 and [14, Proposition 1.8] we have that $\frac{\cap\{P \mid P \in \operatorname{Spec}_\tau(M)\}}{I} = \bigcap\{P/I \mid P/I \in \operatorname{Spec}_{\tau^I}(M/I)\}$. Thus $\mathfrak{N}_{\tau^I}(M/I) = \frac{\mathfrak{N}_\tau}{I}$. Now let$\mathbb{A} = A/I$ and $\mathbb{B} = B/I$ be fully invariant submodules of M/I such that $\mathbb{B} \subseteq \mathfrak{N}_{\tau^I}(M/I)$. Hence $\frac{B}{I} \subseteq \frac{\mathfrak{N}_\tau}{I}$. So $B \subseteq \mathfrak{N}_\tau$. By hypothesis $B^r_M \overline{A} \subseteq \overline{B_M A}$ for some integer $r > 0$. On the other hand we have that $(\mathbb{B})^r_{M/I} \overline{\mathbb{A}} = \left(\frac{B}{I}\right)^r_{M/I} \overline{\left(\frac{A}{I}\right)}$ by Remark 2.3. Now by Lemma 2.10 and Corollary 2.11 we have that

$$\left(\frac{B}{I}\right)^r_{M/I} \overline{\left(\frac{A}{I}\right)} = \left(\frac{B^r+I}{I}\right)_{M/I} \overline{\left(\frac{A}{I}\right)} = \left(\frac{B^r+I}{I}\right)_{M/I} \left(\frac{\overline{A}}{I}\right)$$
$$= \frac{(B^r+I)_M \overline{A} + I}{I}.$$

By [5, Proposition 1.3] we have that

$$\frac{(B^r+I)_M\overline{A}+I}{I}=\frac{B^r{}_M\overline{A}+I_M\overline{A}+I}{I}=\frac{B^r{}_M\overline{A}+I}{I}\subseteq\frac{\overline{B_MA}+I}{I}$$
$$\subseteq\overline{\frac{B_MA+I}{I}}=\overline{\left(\frac{B}{I}\right)_{M/I}\left(\frac{A}{I}\right)}=\overline{\mathbb{B}_{M/I}\mathbb{A}}.$$

Thus $(\mathbb{B})^r_{M/I}\,\overline{\mathbb{A}}\subseteq\overline{\mathbb{B}_{M/I}\mathbb{A}}$.

ii) Let A be a fully invariant submodule of M, such that A is τ-nilpotent, then there exists an integer $n>0$, such that $A^n\subseteq t_\tau(M)$. Hence $A^n\subseteq P$ for all $P\in\mathrm{Spec}_\tau(M)$. Then $A\subseteq P$ for all $P\in\mathrm{Spec}_\tau(M)$. So $A\subseteq\mathfrak{N}_\tau$. Now, we suppose that A has index of τ-nilpotency $\mathfrak{v}\,(A)=n$. If $n=1$, then $\overline{A^1}=\overline{A}\subseteq\overline{t_\tau(M)}=t_\tau(M)$. Thus $\overline{A}$ is τ-nilpotent. Let $n>1$, and suppose that the assertion has been established for any τ-nilpotent fully invariant submodule of M of τ-nilpotency index $<n$. As $A\subseteq\mathfrak{N}_\tau$, then $\overline{A}\subseteq\overline{\mathfrak{N}_\tau}=\mathfrak{N}_\tau$. On the other hand as $\frac{\overline{A}}{A}=t_\tau\,(M/A)$, then by [14, Lemma 17] we have that $\overline{A}$ is fully invariant submodule of M. So by hypothesis we have that there exists $r>0$ such that $\left(\overline{A}\right)^r_M\overline{A}\subseteq\overline{\overline{A}_MA}$. By Lemma 1.17 we have that $\overline{\overline{A}_MA}=\overline{A_MA}=\overline{A^2}$. Thus $\left(\overline{A}\right)^r_M\overline{A}\subseteq\overline{A^2}$. As $\mathfrak{v}\left(A^2\right)<\mathfrak{v}\,(A)=n$, then by induction hypothesis there exists an integer $m>0$, such that $\left(\overline{A^2}\right)^m\subseteq t_\tau(M)$. As $\left(\overline{A}\right)^{r+1}\subseteq\overline{A^2}$, then $\left[\left(\overline{A}\right)^{r+1}\right]^m\subseteq\left[\overline{A^2}\right]^m\subseteq t_\tau(M)$. Therefore $\left(\overline{A}\right)^{(r+1)m}\subseteq t_\tau(M)$.

For $M\in R$-Mod and $\tau\in M$-tors, we consider the following condition:

($\mathfrak{B}$) $\mathrm{Ann}^N_M\,(\mathfrak{N}_\tau)\neq 0$ for any nonzero $N\in\sigma[M]$ such that $N\in\mathbb{F}_\tau$.

Lemma 2.13. *Let $M\in R$-Mod be projective in $\sigma[M]$ and $\tau\in M$-tors. If M satisfies the property ($\mathfrak{B}$), then M/C satisfies the property ($\mathfrak{B}$) for all fully invariant submodule C of M with $C\subseteq\mathfrak{N}_\tau$.*

Proof. Let C be a fully invariant submodule of M with $C\subseteq\mathfrak{N}_\tau$. By [6, Proposition 1.5] we have that M/C is is projective in $\sigma\,[M/C]$. Moreover $\sigma\,[M/C]\subseteq\sigma[M]$. Let $L\in\sigma\,[M/C]$ such that $L\in\mathbb{F}_{\tau^C}$. As $L\in\sigma[M]$, then by Remark 2.3 $L\in\mathbb{F}_\tau$. So by hypothesis $\mathrm{Ann}^N_M\,(\mathfrak{N}_\tau)\neq 0$. Hence there exists $0\neq L'\subseteq L$ such that $(\mathfrak{N}_\tau)_M\,L'=0$. We claim that $(\mathfrak{N}_{\tau^C}\,(M/C))_{M/C}\,L'=0$. In fact as $C\subseteq\mathfrak{N}_\tau$, then $C\subseteq P$ for all $P\in\mathrm{Spec}_\tau(M)$. So by Remark 2.3 we have that $\mathfrak{N}_{\tau^C}\,(M/C)=\frac{\mathfrak{N}_\tau}{C}$. So $(\mathfrak{N}_{\tau^C}\,(M/C))_{M/C}\,L'=\left(\frac{\mathfrak{N}_\tau}{C}\right)_{M/C}L'$. Let $f:\frac{M}{C}\to L'$ be a morphism, then $f\circ\pi\in\mathrm{Hom}_R\,(M,L')$, where $\pi:M\to M/C$ is the canonical epimorphism.

Thus $(f \circ \pi)(\mathfrak{N}_\tau) = 0$. Hence $f\left(\frac{\mathfrak{N}_\tau}{C}\right) = 0$. Therefore $\left(\frac{\mathfrak{N}_\tau}{C}\right)_{M/C} L' = 0$. So $\left(\mathfrak{N}_{\tau^C}(M/C)\right)_{M/C} L' = 0$. Thus $\operatorname{Ann}^L_{M/C}\left(\mathfrak{N}_{\tau^C}(M/C)\right) \neq 0$.

Lemma 2.14. *Let M be a finitely generated module and projective in $\sigma[M]$. If I and N are fully invariant submodules of M such that $0 \neq I \subseteq N$ and $I_M I = 0$, then there exists a non zero fully invariant submodule A of M such that A is maximal with respect $A_M A = 0$ and $A \subseteq N$.*

Proof. Let $F = \{0 \neq I \subseteq_{FI} M \mid I_M I = 0 \text{ and } I \subseteq N\}$. By hypothesis $F \neq \varnothing$. Let $\{I_i\}$ be a chain in F. We claim that $(\cup I_i)_M (\cup I_i) = 0$. In fact if $(\cup I_i)_M (\cup I_i) \neq 0$, then there exists $x \in \cup I_i$ and $f : M \to \cup I_i$ a morphism such that $f(x) \neq 0$. Hence there exist j and $j\prime$ such that $x \in I_j$ and $f(x) \in I_{j\prime}$. On the other hand, as M is finitely generated module, then there exits j'' such that $f(M) \subseteq I_{j''}$. Let $r = \max\{j,\ j',\ j''\}$. So $x \in I_r$ and $f : M \to I_r$. Thus $(I_r)_M I_r \neq 0$, which is a contradiction. Therefore $(\cup I_i)_M (\cup I_i) = 0$. Moreover it is clear that $\cup I_i$ is a fully invariant submodule of M and $\cup I_i \subseteq N$. Therefore every chain $\{I_i\}$ in F is bounded. By Zorn's lemma we have the result.

Note that the Lemma 2.14 is not true in general. Consider the following example.

Example 2.15. We consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ where p is a prime number. Since $\mathbb{Z}_{p^n\,M}\mathbb{Z}_{p^m} = 0$ for all n and m. In particular, $\mathbb{Z}_{p^n\,M}\mathbb{Z}_{p^n} = 0$ for all positive integer n. So M does not have maximal submodule N of M, such that $N_M N = 0$. Moreover M is an artinian module and, so M has τ-Krull dimension for all $\tau \in M$-tors. Notice that in this case M is not a finitely generated module.

Theorem 2.16. *Let $M \in R$-Mod be a progenerator finitely generated in $\sigma[M]$ and $\tau \in M$-tors with $\operatorname{Spec}_\tau(M) \neq \varnothing$. If $\tau \neq \chi$ and M has τ-Krull dimension then the following conditions are equivalent.*

i) ($\mathfrak{A}$) *For any fully invariant submodules A and B of M with $B \subseteq \mathfrak{N}_\tau$, there exists an integer $r > 0$ such that $(B^r)_M \overline{A} \subseteq \overline{B_M A}$.*

($\mathfrak{B}$) $\operatorname{Ann}^N_M(\mathfrak{N}_\tau) \neq 0$ *for any nonzero $N \in \sigma[M]$ such that $N \in \mathbb{F}_\tau$.*

ii) a) The τ-purification of any τ-nilpotent fully invariant submodule of M is τ-nilpotent.

b) If $C \subseteq \mathfrak{N}_\tau$ is a τ-pure fully invariant submodule of M, then $\mathfrak{N}_{\tau^C}(M/C) = \frac{\mathfrak{N}_\tau}{C}$ does not contain a nonzero τ^C-idempotent fully invariant submodule of M/C.

iii) $\mathfrak{N}_\tau$ *is τ-nilpotent.*

Proof. $i) \Rightarrow ii)$ By Proposition 2.12 $ii)$ $a)$ follows from $(\mathfrak{A})$. Now, by Proposition 2.12 $i)$ and Lemma 2.13 we have that conditions $(\mathfrak{A})$ and $(\mathfrak{B})$ are inherited by all module M/C where C is a τ-pure fully invariant submodule of M and $C \subseteq \mathfrak{N}_\tau$. As $t_\tau(M)$ is a fully invariant submodule of M and $t_\tau(M) \subseteq \mathfrak{N}_\tau$, then we may assume that M is τ-torsionfree, and we have to show that if $I \subseteq \mathfrak{N}_\tau$ is a τ-idempotent fully invariant submodule of M, then $I = 0$. Suppose otherwise, then by the condition $(\mathfrak{B})$ we have that $0 \neq \mathrm{Ann}_M^{\overline{I}}(\mathfrak{N}_\tau) \subseteq \mathrm{Ann}_M^{\overline{I}}(I)$. Let $X = \mathrm{Ann}_M^{\overline{I}}(I)$. As I is a fully invariant submodule of M, then $\overline{I}$ is a fully invariant submodule of M. Moreover by Proposition 1.14. we have that X is a fully invariant submodule of M. So by condition $(\mathfrak{A})$ we know that $(I^r)_M\overline{X} \subseteq \overline{I_M X} = \overline{0}$ for some integer $r > 0$. Since $M \in \mathbb{F}_\tau$, then $(I^r)_M\overline{X} = 0$. As I is τ-idempotent, then $\frac{I}{I_M I} \in \mathbb{T}_\tau$. So by Corollary 1.11 we have that $\frac{I}{I^r} \in \mathbb{T}_\tau$. By Lemma 1.10 $I_M\overline{X} = \frac{I_M\overline{X}}{(I^r)_M\overline{X}} \in \mathbb{T}_\tau$. Hence $I_M\overline{X} \subseteq t_\tau(M) = 0$. Since $X \subseteq \overline{I}$, then $\overline{X} \subseteq \overline{I}$. So $X = \mathrm{Ann}_M^{\overline{I}}(I) = \overline{X}$. We claim that $\overline{X} = \overline{I}$. In fact, we suppose that $\overline{X} \subsetneq \overline{I}$, then $\frac{\overline{I}}{\overline{X}} \in \mathbb{F}_\tau$. So by hypothesis $\mathrm{Ann}_M^{\frac{\overline{I}}{\overline{X}}}(\mathfrak{N}_\tau) \neq 0$. Since $I \subseteq \mathfrak{N}_\tau$, then $0 \neq \mathrm{Ann}_M^{\frac{\overline{I}}{\overline{X}}}(\mathfrak{N}_\tau) \subseteq \mathrm{Ann}_M^{\frac{\overline{I}}{\overline{X}}}(I)$. So we put $\mathrm{Ann}_M^{\frac{\overline{I}}{\overline{X}}}(I) = \frac{Y}{\overline{X}}$. Hence $I_M\frac{Y}{\overline{X}} = 0$. Now, by [3, Proposition 5.5] we have that $I_M Y \subseteq \overline{X}$. By [5, Proposition 1.3] $I_M(I_M Y) \subseteq I_M\overline{X}$. So $\left(I^2\right)_M Y \subseteq I_M\overline{X} = 0$. Moreover by Lemma 1.10 $I_M Y = \frac{I_M Y}{\left(I^2\right)_M Y} \in \mathbb{T}_\tau$. Thus $I_M Y \subseteq t_\tau(M) = 0$. Hence $Y \subseteq \mathrm{Ann}_M^{\overline{I}}(I)$. Therefore $\mathrm{Ann}_M^{\overline{I}}(I) = \overline{X} \subsetneq Y \subseteq \mathrm{Ann}_M^{\overline{I}}(I)$ which is a contradiction. So $\overline{X} = \overline{I}$. Thus $I_M I \subseteq I_M\overline{I} = I_M\overline{X} = 0$. So $I = \frac{I}{I_M I} \in \mathbb{T}_\tau$. But $I \subseteq M \in \mathbb{F}_\tau$. Thus $I = 0$.

$ii) \Rightarrow iii)$ As $\mathrm{Spec}_\tau(M) \neq \varnothing$ and $\tau \neq \chi$, then $\mathfrak{N}_\tau \subsetneq M$. Let $C = t_\tau(M)$. Since $C = t_\tau(M) \subseteq \mathfrak{N}_\tau$, then by Remark 2.3 we have that $\mathfrak{N}_{\tau^C}(M/C) = \frac{\mathfrak{N}_\tau}{C}$. If $\mathfrak{N}_{\tau^C}(M/C)$ is τ^C-nilpotent, then $\left[\frac{\mathfrak{N}_\tau}{C}\right]^n \in \mathbb{T}_{\tau^C}$ for some integer $n > 0$. Hence by Corollary 2.11. we have that $\left[\frac{\mathfrak{N}_\tau}{C}\right]^n = \frac{(\mathfrak{N}_\tau)^n + C}{C} = \frac{(\mathfrak{N}_\tau)^n + t_\tau(M)}{t_\tau(M)} \in \mathbb{T}_{\tau^C}$. By Remark 2.3 $\frac{(\mathfrak{N}_\tau)^n + t_\tau(M)}{t_\tau(M)} \in \mathbb{T}_\tau$. So $(\mathfrak{N}_\tau)^n \in \mathbb{T}_\tau$. Thus $\mathfrak{N}_\tau$ is τ-nilpotent. Thus, we may assume that $M \in \mathbb{F}_\tau$. If $\mathfrak{N}_\tau = 0$, there is noting to prove; so we assume that $\mathfrak{N}_\tau \neq 0$. By Lemma 2.9 $\mathfrak{N}_\tau$ contains a fully invariant submodule $0 \neq I$ of M such that $I_M I = 0$. On the other hand as M is finitely generated, then by Lemma 2.4 we choose a fully invariant submodule A of M maximal with respect to $0 \neq A \subseteq \mathfrak{N}_\tau$ and $A_M A = 0$. By $(ii)(a)$ we have that $\overline{A}$ is τ-nilpotent. As A is a fully invariant submodule of M, then $\overline{A}$ is fully invariant submodule of M. Moreover $\overline{A} \subseteq \mathfrak{N}_\tau$. Hence by Remark 2.3 we have that $\frac{M}{\overline{A}} \in \mathbb{F}_{\tau^{\overline{A}}}$.

By (ii) (b) we have that $\mathfrak{N}_{\tau^{\overline{A}}}\left(\frac{M}{\overline{A}}\right) = \frac{\mathfrak{N}_\tau}{\overline{A}}$. Hence if $\mathfrak{N}_{\tau^{\overline{A}}}\left(\frac{M}{\overline{A}}\right) = 0$, then $\mathfrak{N}_\tau = \overline{A}$. Thus $\mathfrak{N}_\tau$ is τ-nilpotent So we may suppose that $\mathfrak{N}_{\tau^{\overline{A}}}\left(\frac{M}{\overline{A}}\right) \neq 0$. Repeating the argument for $\frac{M}{\overline{A}}$ there exists $\frac{A_2}{\overline{A}}$ maximal with respect $0 \neq \frac{A_2}{\overline{A}} \subseteq \mathfrak{N}_{\tau^{\overline{A}}}\left(\frac{M}{\overline{A}}\right)$ and $\left(\frac{A_2}{\overline{A}}\right)_{\frac{M}{\overline{A}}}\left(\frac{A_2}{\overline{A}}\right) = \frac{\overline{A}}{\overline{A}}$. By Lemma 2.10 we have that $\frac{(A_2)_M(A_2)+\overline{A}}{\overline{A}} = \frac{\overline{A}}{\overline{A}}$. Thus $(A_2)_M\,(A_2) \subseteq \overline{A}$. So we obtain a chain $0 = A_0 = C_0 \subsetneq A_1 \subseteq C_1 \subsetneq A_2 \subseteq C_2 \subsetneq \cdots \subsetneq A_j \subseteq C_j \subsetneq A_{j+1} \subseteq C_{j+1} \subsetneq \cdots \subseteq \mathfrak{N}_\tau$ where $A_1 = A$, $C_j = \overline{A_j}$, and each fully invariant submodule A_{j+1} of M is chosen maximal with respect to $C_j \subsetneq A_{j+1} \subseteq \mathfrak{N}_\tau$ and $(A_{j+1})_M\,(A_{j+1}) \subseteq C_j$. Note that C_j is τ-nilpotent for all j.

Suppose that the above chain is infinite, and set

$$I = \overline{\bigcup_{j=1}^{\infty} C_j} \subseteq \mathfrak{N}_\tau$$

By hypothesis (ii) (b), I is not τ-idempotent. We claim that there exists $n, j \in \mathbb{N}$ such that $I^n \subseteq \overline{\overline{I^{n+1}} + C_j}$. In fact, suppose that $I^n \nsubseteq \overline{\overline{I^{n+1}} + C_j}$ for all $n, j \in \mathbb{N}$. We proceed to select a subsequence $\{B_n\}$ of $\{C_n\}$ such that $\overline{I^n} \cap B_{n+1} \nsubseteq \overline{\overline{I^{n+1}} + B_n}$ for all $n \geqslant 1$. Choose $B_1 = C_0 = 0$. Since I is not τ-idempotent, then $\overline{I_M I} \subsetneq \overline{I}$. Thus. $I = \overline{I} \nsubseteq \overline{I_M I} = \overline{I^2}$. Hence there exists $C_j \nsubseteq \overline{I^2}$ for some $j > 0$. We choose j minimal with respect to this property and set $B_2 = C_j$. Thus

$$\overline{I} \cap B_2 = I \cap C_j = C_j \nsubseteq \overline{I^2} = \overline{\overline{I^2} + 0} = \overline{\overline{I^2} + B_1}$$

Now suppose that $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_n$ have been chosen such that

$$\overline{I^{n-1}} \cap B_n \nsubseteq \overline{\overline{I^n} + B_{n-1}}$$

Now, if $\overline{I^n} \cap C_j \subseteq \overline{\overline{I^{n+1}} + B_n}$ for all j, then we have that

$$I^n = I^n \cap I \subseteq \overline{I^n} \cap \overline{\bigcup_{j=1}^{\infty} C_j} = \overline{\overline{I^n} \cap \left(\bigcup_{j=1}^{\infty} C_j\right)} \subseteq \overline{\overline{I^{n+1}} + B_n}$$

As $B_n = C_{j\prime}$ for some $j\prime$, then $I^n \subseteq \overline{\overline{I^{n+1}} + C_{j\prime}}$ which is impossible, since we suppose that $I^n \nsubseteq \overline{\overline{I^{n+1}} + C_j}$ for all n, j. Therefore $\overline{I^n} \cap C_j \nsubseteq \overline{\overline{I^{n+1}} + B_n}$ for some j. So we may take j minimal, and $B_{n+1} = C_j$.

Moreover we have that $B_n \subsetneq B_{n+1}$. So by [13, Lemma 3.2.8] we have that I does not have τ-Krull dimension which is a contradiction since M has τ-Krull dimension. Therefore $I^n \subseteq \overline{\overline{I^{n+1}} + C_j}$ for some n, j. On the other hand $\overline{I^{n+1}} + C_j \subseteq \overline{I^{n+1} + C_j}$. Hence $\overline{\overline{I^{n+1}} + C_j} \subseteq \overline{I^{n+1} + C_j}$. Moreover it is clear that $\overline{I^{n+1} + C_j} \subseteq \overline{\overline{I^{n+1}} + C_j}$. Thus $\overline{\overline{I^{n+1}} + C_j} = \overline{I^{n+1} + C_j}$. Thus $\frac{I^n+C_j}{I^{n+1}+C_j} \subseteq \frac{\overline{\overline{I^{n+1}}+C_j}}{I^{n+1}+C_j} = \frac{\overline{I^{n+1}+C_j}}{I^{n+1}+C_j} \in \mathbb{T}_\tau$. By Lemma 1.10 and Lemma 2.10 we have that $\frac{(I^n+C_j)/C_j}{(I^{2n}+C_j)/C_j} \in \mathbb{T}_{\tau^{C_j}}$. Now by Lemma 2.10 we have that

$$\frac{\frac{I^n+C_j}{C_j}}{\left(\frac{I^n+C_j}{C_j}\right)_{M/C_j}\left(\frac{I^n+C_j}{C_j}\right)} = \frac{\frac{I^n+C_j}{C_j}}{\frac{(I^n)_M(I^n)+C_j}{C_j}} = \frac{\left(I^n+C_j\right)/C_j}{\left(I^{2n}+C_j\right)/C_j} \in \mathbb{T}_{\tau^{C_j}}.$$

So the fully invariant submodule $\frac{I^n+C_j}{C_j}$ of $\frac{M}{C_j}$ is τ^{C_j}-idempotent. Now by $(ii)(b)$ we have that $\frac{I^n+C_j}{C_j} = 0$. Hence $I^n \subseteq C_j$. If $n = 1$, then $I \subseteq C_j$. So $I = C_j$. If $n > 1$, then $\left(I^{n-1} + A_{j+1}\right)_M \left(I^{n-1} + A_{j+1}\right) = I^{2n-2} + \left(I^{n-1}\right)_M A_{j+1} + \left(A_{j+1}\right)_M I^{n-1} + \left(A_{j+1}\right)^2$. Since A_{j+1} was chosen maximal with the property $\left(A_{j+1}\right)_M \left(A_{j+1}\right) \subseteq C_j$, then $\left(I^{n-1} + A_{j+1}\right)_M \left(I^{n-1} + A_{j+1}\right) = \left(I^{n-1} + A_{j+1}\right)^2 \subseteq I^n + C_j \subseteq C_j$. Hence $I^{n-1} + A_{j+1} = A_{j+1}$. So $I^{n-1} \subseteq A_{j+1} \subseteq C_{j+1}$. Repeating this argument, we obtain $I \subseteq C_k$ with $k = j + n - 1$. But $I \subseteq C_k \subsetneq A_{k+1} \subseteq I$, which is a contradiction. Therefore, the chain $\{C_n\}$ must be finite. Consequently $C_n = \mathfrak{N}_\tau$ for some n, and $\mathfrak{N}_\tau$ is τ-nilpotent.

$iii) \Longrightarrow i)$ As $\mathfrak{N}_\tau$ is τ-nilpotent, then exists $k > 0$ such that $\mathfrak{N}_\tau^k \subseteq t_\tau(M)$, then it follows for two fully invariant submodules A and B of M with $B \subseteq \mathfrak{N}_\tau$ that $\left(B^k\right)_M \overline{A} \subseteq \left(\mathfrak{N}_\tau^k\right)_M \overline{A} \subseteq t_\tau(M)_M \overline{A} \subseteq t_\tau(M) \subseteq \overline{B_M A}$. So $(\mathfrak{A})$ holds. Now, let $N \in \sigma[M]$ with $N \in \mathbb{F}_\tau$, then $t_\tau(M)_M N = 0$. By [5, Proposition 1.3] we have that $\left(\mathfrak{N}_\tau^k\right)_M N \subseteq t_\tau(M)_M N = 0$. If $\mathfrak{N}_\tau^i$ is the smallest power of $\mathfrak{N}_\tau$ such that $\left(\mathfrak{N}_\tau^i\right)_M N = 0$, then $\left(\mathfrak{N}_\tau^{i-1}\right)_M N \neq 0$ where $\mathfrak{N}_\tau^0 = M$. As M is generator in $\sigma[M]$, then $\left(\mathfrak{N}_\tau^0\right)_M N = M_M N \neq 0$. Hence $0 \neq \left(\mathfrak{N}_\tau^{i-1}\right)_M N \subseteq \mathrm{Ann}_M^N\left(\mathfrak{N}_\tau\right)$. So $(\mathfrak{B})$ holds.

Notice that in Theorem 2.16 the hypothesis M is finitely generated is used only in the proof of $ii) \Rightarrow iii)$

Note that the Theorem 2.16 is not true in general. Consider the following example.

Example 2.17. We consider the example 1.12 of [5]. In that example, R is the trivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $M = E\left(S\right)$ where S is the only one simple R-module and $E\left(S\right)$ denotes the injective hull of S. Also note that M is a finite module. Hence M is an artinian module. Moreover the authors showed that $\mathrm{Spec}(M) = \varnothing$. Thus $\mathfrak{N}_\tau = M$ for all $\tau \in M$-tors. So $\mathfrak{N}_\tau$ is not τ-nilpotent for all $\chi \neq \tau \in M$-tors.

3. FIS-invariant torsion theories

In this section we use the concept of FIS-invariant torsion theory $\tau \in M$-tors given in [6]. We will show that the conditions *ii*) *a*) and *ii*) *b*) of the Theorem 2.16 are satisfied when M is a progenerator in $\sigma[M]$ and $\tau \in M$-tors is a FIS-invariant torsion theory. Furthermore if M is a finitely generated module and M has τ-Krull dimension, then $\mathfrak{N}_\tau$ is τ-nilpotent.

The following definition was given in [6, Definition 2.1].

Definition 3.1. *Let $M \in R$-Mod. A hereditary torsion theory $\tau \in M$-tors is K-invariant for a fully invariant submodule K of M if the module $\frac{K}{K_M D}$ is τ-torsion for any τ-dense submodule D of M. If τ is K-invariant for every fully invariant submodule K of M, then τ is called FIS-invariant (Fully Invariant Submodule invariant torsion theory).*

We observe that ξ and χ are FIS-invariant torsion theories.

Proposition 3.2. *Let $M \in R$-Mod be projective in $\sigma[M]$ and let R be a commutative ring. If M is a multiplication module, then τ is FIS-invariant for all $\tau \in M$-tors.*

Proof. As M is a multiplication module, then M is a duo module (every submodule N of M is fully invariant). We claim that M generates all its submodules. In fact, let N be a submodule of M, then $N = IM$ for some ideal I of R. Since R is commutative ring, then for each $r \in I$ we may define the morphism $f_r : M \to M$ as $f_r(m) = rm$. So $f_r(M) = rM \subseteq IM = N$. Thus $f_r : M \to N$ for all $r \in I$. Hence we have that $N = IM = \sum_{r\in I} f_r(M) \subseteq M_M N \subseteq N$. So $M_M N = N$. Thus M generates N.

On the other hand. As M is a multiplication module. If N and L are submodules of M, then there exist ideals I and J of R such that $N = IM$ and $L = JM$. So $N_M L = \sum_{f:M\to L} f(N) = \sum_{f:M\to L} f(IM) = I\sum_{f:M\to L} f(M) = I(M_M L) = IL = I(JM) = (IJ)M = (JI)M = J(IM) = L_M N$. Therefore $N_M L = L_M N$.

Now let $\tau \in M$-tors and K, D be submodules of M such that D is τ-dense in M. As $\frac{M}{D} \in \mathbb{T}_\tau$, then by Lemma 1.10 we have that $\frac{M_M K}{D_M K} \in \mathbb{T}_\tau$.

Since $M_M K = K$ and $D_M K = K_M D$, then $\frac{M_M K}{D_M K} = \frac{K}{K_M D} \in \mathbb{T}_\tau$. So τ is FIS-invariant.

Notice that if R is a von Neumann regular ring and M is a multiplication R-module, such that M generates all its submodules, then τ is FIS-invariant for all $\tau \in M$-tors. The proof is similar as the proof of Proposition 3.2.

Proposition 3.3. *Let $M \in R$-Mod be projective in $\sigma[M]$ and let $\tau \in M$-tors. If K is a fully invariant submodule of M, then the following conditions are equivalent:*

i) τ is K-invariant

ii) If $K_M N' = 0$ for a τ-dense submodule N' of a τ-torsionfree $N \in \sigma[M]$, then $K_M N = 0$.

Proof. *i)* $\Longrightarrow$ *ii)* Let $N \in \sigma[M]$ with $N \in \mathbb{F}_\tau$ and let N' be a τ-dense submodule of N such that $K_M N' = 0$. We know that $K_M N = \sum_{f:M\to N} f(K)$. Let $f : M \to N$ be a morphism. So we can consider the morphism $\pi \circ f : M \to N/N'$ where $\pi : N \to N/N'$ is the canonical projection. We put $\ker(\pi \circ f) = D$. Thus $\pi(f(D)) = 0$. So $f(D) \subseteq N'$. By [5, Proposition 1.3] we have that $K_M [f(D)] = 0$. We claim that $f(K_M D) = 0$. In fact let $h : M \to D$ be a morphism. As $f \circ h : M \to f(D)$, then $(f \circ h)(K) = 0$. Hence $f(K_M D) = 0$. Now we consider the restriction morphism $f_{|K} : K \to N$. As K is a fully invariant submodule of M, then $K_M D \subseteq K$. Thus $f_{|K}(K_M D) = f(K_M D) = 0$. So we have the morphism $\widehat{f_{|K}} : \frac{K}{K_M D} \to N$ such that $\widehat{f_{|K}}(x + K_M D) = f_{|K}(x) = f(x)$. On the other hand as $\ker(\pi \circ f) = D$, then $M/D \in \mathbb{T}_\tau$. Thus D is τ-dense in M. Since τ is K-invariant, then $\frac{K}{K_M D} \in \mathbb{T}_\tau$. Since $N \in \mathbb{F}_\tau$, then $\widehat{f_{|K}}\left(\frac{K}{K_M D}\right) = 0$. Therefore $0 = \widehat{f_{|K}}(x + K_M D) = f(x)$ for all $x \in K$.

ii) $\Longrightarrow$ *i)* Let D be a τ-dense submodule of M, then $\frac{D+\overline{K_M D}}{\overline{K_M D}}$ is τ-dense submodule of $\frac{M}{\overline{K_M D}}$. Thus $\frac{M}{D+\overline{K_M D}} \in \mathbb{T}_\tau$. As $K_M D \subseteq D \cap \overline{K_M D}$, then by [3, Proposition 5.5] we have that $K_M\left(\frac{D}{D\cap\overline{K_M D}}\right) = 0$. Since $\frac{D}{D\cap\overline{K_M D}} \cong \frac{D+\overline{K_M D}}{\overline{K_M D}}$, then $K_M\left(\frac{D+\overline{K_M D}}{\overline{K_M D}}\right) = 0$. Since $\frac{M}{\overline{K_M D}} \in \mathbb{F}_\tau$, then by hypothesis $K_M\left(\frac{M}{\overline{K_M D}}\right) = 0$. By [3, Proposition 5.5] we have that $K_M M \subseteq \overline{K_M D}$. As K is a fully invariant submodule of M, then $K_M M = K$. So $K \subseteq \overline{K_M D}$. Thus $\frac{K}{K_M D} \subseteq \frac{\overline{K_M D}}{K_M D} \in \mathbb{T}_\tau$. Therefore τ is K-invariant.

Lemma 3.4. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\tau \in M$-tors. If τ is FIS-invariant and $N \in \sigma[M]$ is a τ-cocritical module, then* $\mathrm{Ann}_M(N)$ *is a τ-pure prime submodule of M.*

Proof. Let K and L be fully invariant submodules of M such that $K_M L \subseteq \mathrm{Ann}_M(N)$, then $(K_M L)_M N = 0$. Suppose that $L \nsubseteq \mathrm{Ann}_M(N)$. Thus $L_M N \neq 0$. If $L_M N = N$, then by [3, Proposition 5.6] we have that $0 = K_M(L_M N) = K_M N$. Thus $K \subseteq \mathrm{Ann}_M(N)$. If $0 \neq L_M N \subsetneq N$, then $\frac{N}{L_M N} \in \mathbb{T}_\tau$. So $L_M N$ is a τ-dense submodule of N. Now by Proposition 3.2 we have that $K_M N = 0$. So $K \subseteq \mathrm{Ann}_M(N)$.

Note that if τ is FIS-invariant in the Theorem 2.16, then by Lemma 3.4 $\mathrm{Spec}_\tau(M) \neq \varnothing$ since M has τ-Krull dimension. (thus, there exist τ-cocritical modules).

Lemma 3.4 is not true in general. Consider the following example.

Example 3.5. We consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ where p is a prime number. We know that $\mathbb{Z}_p$ is a ξ-cocritical module. On the other hand, $\mathrm{Ann}_M(\mathbb{Z}_p) = \mathbb{Z}_{p^\infty}$, but $\mathbb{Z}_{p^\infty}$ is not a prime submodule of M.

Proposition 3.6. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\tau \in M$-tors be FIS-invariant. If $M \in \mathbb{F}_\tau$ and M has τ-Krull dimension, then $\mathfrak{N}_\tau$ does not contain τ-idempotent fully invariant submodules $0 \neq N$ of M.*

Proof. Suppose that I is a τ-idempotent fully invariant submodule of M such that $I \subseteq \mathfrak{N}_\tau$. We claim that $I_M N = 0$ for all $N \in F_\tau$. In fact, we choose an ordinal α which is minimal with respect to the property that there exists a τ-torsion free module N with k_τ-dim$(N) = \alpha$ and $I_M N \neq 0$. Assume, that N is α-τ-critical, then k_τ-dim$(N/L) < \alpha$ for all τ-pure submodules $0 \neq L$ of N. Hence $I_M\left(\frac{N}{L}\right) = 0$. By [3, Proposition 5.5] we have that $I_M N \subseteq L$. So $\overline{I_M N} \subseteq \overline{L} = L$ for all τ-pure submodules $0 \neq L$ of N. Thus $0 \neq \overline{I_M N}$ is the unique non zero minimal τ-pure submodule of N. On the other hand, if N' is a nonzero τ-pure submodule of $\overline{I_M N}$, then N' is τ-pure in N. Thus $N' = \overline{I_M N}$. Hence $\frac{\overline{I_M N}}{L} \in \mathbb{T}_\tau$ for all $0 \neq L$ submodules of $\overline{I_M N}$. Therefore $\overline{I_M N}$ is 0-τ-critical (τ-cocritical). Since nonzero submodules of α-τ-critical modules are α-τ-critical, then $\alpha = 0$. Hence N is 0-τ-critical. So N is a τ-cocritical module. By Lemma 3.4, $\mathrm{Ann}_M(N)$ is a prime τ-pure submodule of M. Thus $\mathfrak{N}_\tau \subseteq \mathrm{Ann}_M(N)$. As $I \subseteq \mathfrak{N}_\tau$, then $I_M N = 0$ this is a contradiction. Thus N cannot be τ-critical. Notice that I annihilates any τ-critical submodule of N. By [12, Proposition 3.2.4] we have that there exists $0 \neq L$ a τ-critical submodule of N. Therefore k_τ-dim$(L) \leqslant k_\tau$-dim$(N) = \alpha$. If

k_τ-dim$(L) < \alpha$, then $I_M L = 0$. If k_τ-dim$(L) = \alpha$, then $I_M L = 0$. Hence $L \subseteq \mathrm{Ann}_M^N(I) \neq 0$. We put $H = \mathrm{Ann}_M^N(I)$. So $I_M H = 0$. Thus $H \subsetneq N$. Now if $t_\tau(N/H) = H'/H$, then by Proposition 3.3 $I_M H' = 0$. Hence $H' \subseteq \mathrm{Ann}_M^N(I) = H$. Thus $t_\tau(N/H) = 0$. So $N/H \in \mathbb{F}_\tau$. So there exists a τ-critical submodule $0 \neq T/H$ of N/H. By the preceding paragraph $0 \neq T/H \subseteq \mathrm{Ann}_M^{N/H}(I)$. Hence $I_M(T/H) = 0$. So by [3, Proposition 5.5] we have that $I_M T \subseteq H$. By [3, Proposition 5.6] we have that $(I_M I)_M T = I_M(I_M T) \subseteq I_M H = 0$. As I is τ-idempotent, then $\frac{I}{I_M I} \in \mathbb{T}_\tau$. So by Lemma 1.10 $I_M T = \frac{I_M T}{(I_M I)_M T} \in \mathbb{T}_\tau$. Since $I_M T \subseteq H \subseteq N \in \mathbb{F}_\tau$, then $I_M T = 0$. Hence $H \subsetneq T \subseteq \mathrm{Ann}_M^N(I) = H$ this is a contradiction. Therefore $I_M N = 0$, for $N \in F_\tau$. Thus $I = I_M M = 0$.

Proposition 3.7. *Let $M \in R$-Mod be projective in $\sigma[M]$, let $\tau \in M$-tors and K a fully invariant submodule of M. If τ is K-invariant, then $\overline{K}_M \overline{N} \subseteq \overline{K_M N}$for all $N \in \sigma[M]$.*

Proof. It is clear that $K_M N \subseteq \overline{K_M N} \cap N$. So by [3, Proposition 5.5] we have that $K_M\left(\frac{N}{\overline{K_M N} \cap N}\right) = 0$. Hence $K_M\left(\frac{N + \overline{K_M N}}{\overline{K_M N}}\right) = 0$. On the other hand $\frac{N+\overline{K_M N}}{\overline{K_M N}} \subseteq \frac{\overline{N}}{\overline{K_M N}} \in \mathbb{F}_\tau$ and $\left(\frac{\overline{N}}{\overline{K_M N}}\right) \Big/ \left(\frac{N+\overline{K_M N}}{\overline{K_M N}}\right) \cong \frac{\overline{N}}{N+\overline{K_M N}} \in \mathbb{T}_\tau$. As τ is K-invariant, then by Proposition 3.3 $K_M\left(\frac{\overline{N}}{\overline{K_M N}}\right) = 0$. We claim that $\overline{K}_M\left(\frac{\overline{N}}{\overline{K_M N}}\right) = 0$. In fact, if $f : M \to \frac{\overline{N}}{\overline{K_M N}}$ is a morphism, then $f(K) = 0$. Hence we can define the morphism $\widehat{f} : \frac{M}{K} \to \frac{\overline{N}}{\overline{K_M N}}$ such that $\widehat{f}(x + K) = f(x)$. Since $\frac{\overline{K}}{K} \in \mathbb{T}_\tau$ and $\frac{\overline{N}}{\overline{K_M N}} \in \mathbb{F}_\tau$, then $\widehat{f}\left(\frac{\overline{K}}{K}\right) = 0$. So $0 = \widehat{f}(x + K) = f(x)$ for all $x \in \overline{K}$. Hence $f\left(\overline{K}\right) = 0$. Thus $\overline{K}_M\left(\frac{\overline{N}}{\overline{K_M N}}\right) = 0$. By [3, Proposition 5.5] we have that $\overline{K}_M \overline{N} \subseteq \overline{K_M N}$.

Notice that if τ is FIS-invariant in Proposition 3.7 we have that $\overline{K}_M \overline{N} \subseteq \overline{K_M N}$ for all fully invariant submodules K of M and for all $N \in \sigma[M]$.

Lemma 3.8. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\tau \in M$-tors. If τ is FIS-invariant and I is a fully invariant submodule of M, then τ^I is FIS-invariant.*

Proof. Let K/I be a fully invariant submodule of M/I and D/I a τ^I-dense submodule of M/I. By Lemma 2.10, $\left(\frac{K}{I}\right)_{M/I}\left(\frac{D}{I}\right) = \frac{K_M D + I}{I}$. Hence $\frac{K/I}{\left(\frac{K}{I}\right)_{M/I}\left(\frac{D}{I}\right)} = \frac{K/I}{\frac{K_M D+I}{I}} \cong \frac{K}{K_M D+I}$. Since K/I is a fully invariant submodule of M/I, then K is a fully invariant submodule of M and by Remark 2.3 D is τ-dense in M. Hence $\frac{K}{K_M D} \in \mathbb{T}_\tau$. Therefore $\frac{K}{K_M D+I} \in \mathbb{T}_\tau$. Thus by Remark 2.3 $\frac{K/I}{\left(\frac{K}{I}\right)_{M/I}\left(\frac{D}{I}\right)}$ is τ^I-torsion. So τ^I is FIS-invariant.

Corollary 3.9. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\tau \in M$-tors. Suppose that $M \in \mathbb{F}_\tau$ and M has τ-Krull dimension. If τ is FIS-invariant, then the following conditions hold:*

i) The τ-purification of any τ-nilpotent fully invariant submodule of M is τ-nilpotent.

ii) If $I \subseteq \mathfrak{N}_\tau$ is a τ-pure fully invariant submodule of M, then $\mathfrak{N}_{\tau^I}\left(\frac{M}{I}\right) = \frac{\mathfrak{N}_\tau}{I}$ does not contain nonzero τ^I-idempotent fully invariant submodules of $\frac{M}{I}$.

Proof. i) Let I be a τ-nilpotent fully invariant submodule of M, then $I^n \subseteq t_\tau(M)$ for some positive integer n. Now, by Proposition 3.7 $\left(\overline{I}\right)^n \subseteq \overline{I^n} \subseteq \overline{t_\tau(M)} = t_\tau(M)$. So $\overline{I}$ is τ-nilpotent.

ii) By [17, Proposition 18.2 (4)] we have that $\frac{M}{I}$ is projective in $\sigma\left[\frac{M}{I}\right]$. So we apply Lemma 3.8 and Proposition 3.6 to get the result.

Theorem 3.10. *Let $M \in R$-Mod be a finitely generated module, progenerator in $\sigma[M]$ and $\tau \in M$-tors FIS invariant. If $\tau \neq \chi$ and M has τ-Krull dimension, then $\mathfrak{N}_\tau$ is τ-nilpotent.*

Proof. Apply Corollary 3.9 and Theorem 2.16 to get the result.

Notice that the Example 2.17 shows that the Theorem 3.10 is not true in general.

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