© Algebra and Discrete Mathematics
Volume **34** (2022). Number 2, pp. 317–325
DOI:10.12958/adm1624 RESEARCH ARTICLE

On a stronger notion of connectedness in c-spaces

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Communicated by I. V. Protasov

Abstract. In this article, a stronger form of connectedness called Y-connectedness in c-spaces is introduced and some of its properties are studied. Using the notion of touching, some conditions under which union of Y-connected sub c-spaces of a c-space become Y-connected is also discussed.

Introduction

Axiomatization of the notion of connected sets by Reinhard Börger in 1983 is a great achievement in the study of connectedness. It is known that topological connectedness was used in the study of connectedness in continuous figures whereas graph theoretical connectedness was used in the study of discrete figures. But these two concepts are not mutually compatible as there are topological structures whose connectedness cannot be induced from a graph and viceversa [2, 10]. Compatibility is a must for an applied mathematician to work with as study of continuous figures is achieved by discretizing them. In fact, the proposed theory of Börger, known as the theory of c-spaces, unified various abstract notions of connectedness present in different branches and find many applications in various

The author is thankful to Late (Prof.) P.T. Ramachandran, Adjunct Professor of Mathematics, Kannur University for his guidance to do this work. The financial support from the U.G.C, Govt. of India is greatly acknowledged. The author wish to extend his sincere thanks to the unknown referees, whose comments and suggestions improved this article to a great extent.

2020 MSC: 05C40, 54A05, 54D05.

Key words and phrases: c-space, Y-connected c-space, t-closure, touching, connectivity spaces.

branches like Digital Topology, Signal Processing, Pattern Recognition and many more [4, 5, 10, 13, 14].

1. Preliminaries

All terminologies in this section are standard and are taken from [4, 8, 11, 12] unless otherwise mentioned. A nonempty set X together with a collection $\mathcal{C}$ of subsets of X which satisfies the following axioms is called a c-space.

(i) $\varnothing \in \mathcal{C}$ and $\{x\} \in \mathcal{C}$ for every $x \in X$.

(ii) If $\{C_i : i \in I\}$ is a non empty collection of subsets in $\mathcal{C}$ with $\bigcap_{i\in I} C_i \neq \varnothing$, then $\bigcup_{i\in I} C_i \in \mathcal{C}$.

The collection $\mathcal{C}$ of subsets X which satisfies the above axioms is called a *c-structure* [8] or a *connectivity class* [4, 5, 14] of X. Elements of a c-structure are called *connected sets*.

One trivial example is *indiscrete c-space* where *indiscrete c-structure* is given by $\mathcal{I}_X = \mathcal{P}(X)$, the power set of X. Another example is *discrete c-space*, where discrete c-structure is given by $\mathcal{D}_X = \{\varnothing\} \cup \{\{x\} : x \in X\}$. Unless otherwise specified, the c-space (X, C_X) is represented by X. For example, considering $\mathbb{R}$ as a c-space, connected sets are precisely intervals in $\mathbb{R}$. A c-space Y is said to be a *sub c-space* of the c-space X if $Y \subseteq X$ and $\mathcal{C}_Y = \{A \in \mathcal{C}_X : A \subseteq Y\}$. A c-space is said to be C_1 if it contains no two element connected sets.

A point $x \in X$ is said to *touch* a set $A \subset X$ if there is a nonempty subset $C \subseteq A$ such that $\{x\} \cup C$ is connected. The set of all points touching the set A is denoted by $t(A)$. Further, the subsets A and B of X are said to touch if there is a point $x \in A \cup B$ which touches both A and B. A is said to be *t-closed* if it contains all of its touching points. The t-closure of a set A is defined to be the smallest t-closed set containing A and is denoted by $\bar{A}$. We may note that t-closure is not a Kuratowski closure operator. By convention, $t^2(A) = t(t(A))$, $t^3(A) = t(t^2(A))$ and so on.

A function $f : X \to Y$ is called a *c-continuous* function if it maps connected sets of X to connected sets of Y. Let $\{X_i : i \in I\}$ be a family of c-spaces and let $X = \prod X_i$. Then $\mathcal{C} = \{A \subset X : \pi_i(A) \in \mathcal{C}_{X_i} \text{ for every } i\}$, where π_is are the projection functions defined on the set X, is a c-structure on X and X with this c-structure is called the product space of $\{X_i : i \in I\}$. Obviously it is the largest c-structure on X which make each π_i c-continuous. To make the concept more clear, an example of connected and disconnected sets from $\mathbb{R}^2$ is given in Figure 1.

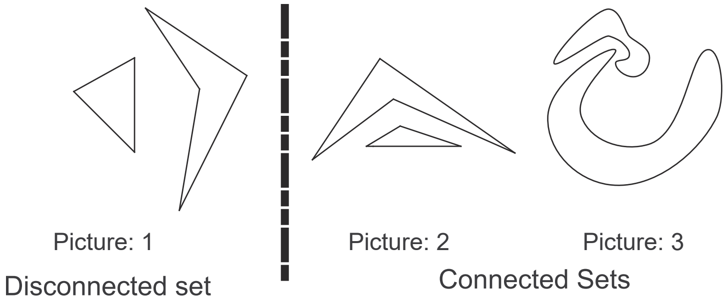

FIGURE 1.

A function $f : X \to Y$ is said to be a quotient map if $\mathcal{C}_Y$ is the smallest c-structure on Y which make f c-continuous. Then $\mathcal{C}_Y = \langle\{f(C) : C \in \mathcal{C}_X\}\rangle$, the smallest c-structure on Y which contains all $f(C)$s. In otherwords, it is the c-structure generted by the collection $\{f(C) : C \in \mathcal{C}_X\}$. In this case we say that Y is a quotient space of X. Many properties of quotient space can be seen in [12]. Further, if $(X, \mathcal{C}_X)$ be the sum of the the family of c-spaces $\{(X_i, \mathcal{C}_{X_i}) : i \in I\}$, then $\mathcal{C}_X = \bigcup_{i\in I}\{C \times \{i\} : C \in \mathcal{C}_{X_i}\}$.

Let X be any set and α be any cardinal with $\alpha \leqslant |X|$. Then a c-structure $\mathcal{C}$ on X is said to be *α-generated* if there is a sub collection $\mathcal{B} \subseteq \{A \in \mathcal{C} : |A| \leqslant \alpha\}$ such that $\mathcal{C} = \langle\mathcal{B}\rangle$. A c-space is said to be α-generated if its c-structure is α-generated [9]. 2-generated c-spaces are of special interest to us [8, 9].

2. On a stronger notion of connectedness in c-spaces

A topological space is connected if it cannot be written as the union of two disjoint nonempty open sets. Alternately, a topological space X is connected if and only if any continuous map f from X to the two element discrete topological space $\{0, 1\}$ is a constant (Page 164 of [7]). In his work [3], Dai Bo introduced the idea of Z-connectedness by replacing the two element discrete space $\{0, 1\}$ by a T_1 topological space Z.

Analogously, Joseph Muscat and David Buhagiar [8] observed that a c-space X is connected if and only if any c-continuous function f from X to the two element discrete c-space $\{0, 1\}$ is a constant. In our work on stronger notion of connectedness, we replace the two element discrete c-space $\{0, 1\}$ by a C_1 c-space. It can be noted that the associated c-space of a T_1 topological space is C_1.

Definition 1. Let Y be any C_1 c-space with more than one element. A c-space X is said to be Y-connected if every c-continuous function $f : X \to Y$ is constant.

Remark 1. The restriction that Y should be a C_1 c-space is mandatory. For otherwise, Y contains a two element connected set, say $\{a, b\}$. Let $f : X \to \{a, b\}$ be any non constant function. Then f is c-continuous as every subset of $\{a, b\}$ is connected in Y.

Now we list some examples of Y-connected spaces.

1) Let Y be the Brunnian closure of an infinite discrete c-space.

a. Consider the c-space $(X, \mathcal{C}_X)$ where $X = \{1, 2, 3\}$ and $\mathcal{C}_X = \mathcal{D}_X \cup \{\{1, 2\}, X\}$. Let $f : X \to Y$ be any c-continuous map. Since X is a finite connected set, we must have $f(X) = \{a\}$ for some $a \in Y$, so that f is a constant map and hence X is Y-connected c-space.

b. Consider the c-space $(X, \mathcal{C}_X)$ where $X = \{1, 2, 3, 4\}$ and $\mathcal{C}_X = \mathcal{D}_X \cup \{\{1, 2\}, \{3, 4\}, \{1, 2, 3\}, X\}$. Similar arguments as above shows that the c-space X is Y-connected.

2) Let Y be any C_1 c-space with more than one point. Then we can note that all 2-generated connected c-spaces are Y-connected. Using this we can construct infinite c-spaces which are Y-connected. For example, the c-space $(X, \mathcal{C}_X)$ where $X = \mathbb{N}$ and $\mathcal{C}_X = \langle\{n, n + 1\} : n \in \mathbb{N}\rangle$.

Proposition 1 ([8]). *A c-space X is connected if and only if any c-continuous function f from X to the two element discrete c-space $\{0, 1\}$ is a constant.*

Proposition 2. *Every Y-connected space is connected.*

Proof. Let X be any Y-connected space. Consider the two element discrete c-space $D = \{0, 1\}$. Let $f : X \to D$ be any c-continuous function.

As D is a discrete c-space, any function $g : D \to Y$ is c-continuous. Since composition of two c-continuous functions is again c-continuous, $g \circ f : X \to Y$ is c-continuous. Since X is Y-connected, $g \circ f$ is a constant map for every g. This implies that f is a constant map. Then by Proposition 1, X is a connected c-space. □

Remark 2. We may note that converse of the above proposition is not true. For example, any surjective function $f : X \to Y$ with corresponding c-structures $\mathcal{C}_X = \mathcal{D}_X \cup \{X\}$ and $\mathcal{C}_Y = \mathcal{D}_Y \cup \{Y\}$ is c-continuous and will serve our purpose.

Combining the Proposition 2 with the above remark, it follows that Y-connectedness is a stronger notion of connectedness in c-spaces.

Remark 3. Further we may note that if X is a finite c-space, then X is connected if only if X is Y-connected.

By Proposition 2, we know that a Y-connected c-space is always connected. Conversely let X be any finite connected c-space. Let $f : X \to Y$ be any c-continuous function. Since $|f(X)|$ is finite, we have $f(X) = \{a\}$ for some $a \in Y$. Hence f is a constant function. This implies that X is Y-connected.

The above remark can be generalized to any c-space X with $|X| < |Y|$.

Proposition 3 ([8])**.** *Let X and Y be c-spaces and $f : X \to Y$ be a c-continuous function. Let $x \in X$ and $A \subset X$ such that x touches A. Then $f(x)$ touches $f(A)$.*

The following proposition easily follows from the above result.

Proposition 4. *Let X and Y be c-spaces and $f : X \to Y$ be a c-continuous function. Let A, B be subsets of X such that A touches B. Then $f(A)$ touches $f(B)$.*

In the next theorem we investigate some conditions under which union of Y-connected spaces become Y-connected.

Theorem 1. *Let X be a c-space and $\{X_\alpha\}_{\alpha\in I}$ be a family of Y-connected sub c-spaces of X. Then*

1) *If $\bigcap_{\alpha\in I} X_\alpha \neq \varnothing$, then $\bigcup_{\alpha\in I} X_\alpha$ is Y-connected.*
2) *If X_α touches X_β for all α and β in I, then $\bigcup_{\alpha\in I} X_\alpha$ is Y-connected.*
3) *If $I = \mathbb{N}$ and X_n touches X_{n+1} for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} X_n$ is Y-connected.*
4) *Let $\{X_\alpha\}_{\alpha\in I}$ be a directed family of Y-connected sub c-spaces of X, direction being the set inclusion. (That is, for each pair of elements $t_1, t_2 \in I$, there exists $t_3 \in I$ such that $X_{t_1} \subset X_{t_3}$ and $X_{t_2} \subset X_{t_3}$.) Then $\bigcup_{\alpha\in I} X_\alpha$ is Y-connected.*

Proof. Since each X_α is Y-connected, by Proposition 2, each X_α is connected.

1) Consider a c-continuous function $f : \bigcup_{\alpha\in I} X_\alpha \to Y$. Since each X_α is Y-connected, the image $f(X_\alpha)$ coinsides with a singleton $\{f(p)\}$. Hence $f(\bigcup_{\alpha\in I} X_\alpha) = \bigcup_{\alpha\in I} f(X_\alpha) = \{f(p)\}$, a singleton and hence $\bigcup_{\alpha\in I} X_\alpha$ is Y-connected.

2) Let $f : \bigcup_{\alpha\in I} X_\alpha \to Y$ be a nonconstant c-continuous function. Then there exist $x, y \in \bigcup_{\alpha\in I} X_\alpha$ with $x \neq y$ such that $f(x) = k_1$ and $f(y) = k_2$ with $k_1 \neq k_2$. Let $x \in X_\alpha$ and $y \in X_\beta$ for some $\alpha, \beta \in I$.

Obviously, $f \circ i_\alpha : X_\alpha \to Y$ is a c-continuous function, where $i_\alpha : X_\alpha \to \bigcup_{\alpha\in I} X_\alpha$ is the inclusion function. X_α being Y-connected, $f \circ i_\alpha$ is constant for every α. In particular, we have $(f \circ i_\alpha)(X_\alpha) = \{k_1\}$. That is, $f(X_\alpha) = \{k_1\}$. Similarly $f(X_\beta) = \{k_2\}$.

Since X_α touches X_β, by Proposition 4, $f(X_\alpha)$ touches $f(X_\beta)$. That is, $\{k_1\}$ touches $\{k_2\}$. Consequently the set $\{k_1, k_2\}$ is connected in Y. This is possible only if $k_1 = k_2$, a contradiction. Thus the only possible c-continuous functions $f : \bigcup_{\alpha\in I} X_\alpha \to Y$ are constant functions and hence the result.

3) First let us prove that $\bigcup_{i=1}^{n} X_i$ is Y-connected. Using the Principle of Induction, it is enough to prove the case for $n = 2$.

Let $n = 2$. Since X_1 touches X_2, X_2 touches X_1. Then by case (2) above, $X_1 \cup X_2$ is Y-connected. Hence our claim.

Let $C_n = \bigcup_{i=1}^{n} X_i$. Then C_n is Y-connected for each $n \geqslant 1$. Since $\bigcap_{n\in\mathbb{N}} C_n = X_1 \neq \varnothing$, by case 1) above, $\bigcup_{n=1}^{\infty} X_n$ is Y-connected.

4) Let $f : \bigcup_{\alpha\in I} X_\alpha \to Y$ be a nonconstant c-continuous function. Then there exist $x, y \in \bigcup_{\alpha\in I} X_\alpha$ such that $f(x) = k_1$, $f(y) = k_2$ with $x \neq y$ and $k_1 \neq k_2$, where $k_1, k_2 \in Y$.

Let $x \in X_\alpha$ and $y \in X_\beta$ for some α, $\beta \in I$. I being a directed set, there exists $\gamma \in I$ such that $X_\alpha \subset X_\gamma$ and $X_\beta \subset X_\gamma$. Since $f\restriction_{X_\gamma}: X_\gamma \to Y$ is c-continuous and since X_γ is Y-connected, $f\restriction_{X_\gamma}$ is a constant. Let $f(X_\gamma) = \{k_3\}$, for some $k_3 \in Y$. Then obviously $k_1 = k_2 = k_3$, a contradiction. Hence f must be a constant function. □

3. More on Y-connected c-spaces

In this section, results relating to product, quotient, sum and t-closure of Y-connected c-spaces are discussed. As c-spaces can contain nonconnected c-subspaces, it follows that Y-connectedness is not a hereditary property. Further, since sum of two connected spaces is not connected, it follows that sum of Y-connected c-spaces is not Y-connected.

Theorem 2. *Finite product of Y-connected c-spaces is Y-connected.*

Proof. Let X_1 and X_2 be two Y-connected c-spaces. It is enough to prove that $X_1 \times X_2$ is Y-connected.

Consider the product space $X_1 \times X_2$. Fix a point (a, b) in $X_1 \times X_2$.

For $x \in X_1$, let $T_x = (X_1 \times \{b\}) \cup (\{x\} \times X_2)$.

Now obviously $X_1 \times \{b\}$ is a Y-connected space c-isomorphic with X_1 and $\{x\} \times X_2$ is a Y-connected space c-isomorphic with X_2. Then by Theorem 1, T_x is Y-connected, being the union of two Y-connected c-spaces that have a point (x, b) in common.

Now

$$X_1 \times X_2 = \bigcup_{x \in X_1} T_x$$

Then by Theorem 1, $X_1 \times X_2$ is Y-connected being the union of Y-connected c-spaces which have a point (a, b) in common. □

The above theorem can be extended to the artbitrary product, which is as follows.

Theorem 3. *The product $X = \prod_{\alpha \in A} X_\alpha$ of Y-connected spaces is Y-connected.*

Proof. Assuming that X is not Y-connected, we can find a c-continuous map $f : X \to Y$ such that $f(a) \neq f(b)$ for some points $a, b \in X$. By Proposition 2, every space X_α is connected (being Y-connected) so is its isomorphic copy $\dot{X}_\alpha = \{x \in X : \{\beta \in A : x(\beta) \neq a(\beta)\} \subseteq \{\alpha\}\}$ in X. By the definition of a c-structure, the union $V = \bigcup_{\alpha \in A} \dot{X}_\alpha$ is a connected subset of X. Since each X_α is Y-connected, so is its isomorphic copy $\dot{X}_\alpha$, which implies that $f(X_\alpha) = \{f(a)\}$.

Now consider the subset $C = V \cup \{b\}$ of X and observe that for every $\alpha \in A$ its projection $\pi_\alpha(C) = \pi_\alpha(V) = X_\alpha$ is connected. Then C is a connected subset of X by the definition of the c-structure on $X = \prod_{\alpha \in A} X_\alpha$. Since f is c-continuous, the image $f(C) = f(V) \cup \{f(b)\} = \{f(a), f(b)\}$ is a connected subset of Y. Since Y contains no connected two element subsets, $f(b) = f(a)$, which contradicts the choice of a, b. □

Theorem 4. *Let $f : X \to Z$ be a surjective c-continuous function. Then if X is Y-connected, so is Z.*

Proof. Given any c-continuous map $g : Z \to Y$, observe that the composition $g \circ f : X \to Y$ is c-continuous and hence a constant as X is Y-connected. Then $g(Z) = g \circ f(X)$ is a singleton, which means that Z is Y-connected. □

Corollary 1. *Let $\{X_\alpha : \alpha \in I\}$ be a family of nonempty Y-connected c-spaces. If $\prod_{\alpha \in I} X_\alpha$ is Y-connected, then each X_α is Y-connected.*

Corollary 2. *Quotient space of a Y-connected space is Y-connected.*

Two preceding theorems imply the following characterization.

Theorem 5. *The product $\prod_{\alpha\in A} X_\alpha$ of a family $(X_\alpha)_{\alpha\in A}$ of c-spaces is Y-connected if and only if for every $\alpha \in A$, the c-space X_α is Y-connected.*

Theorem 6. *If A is a Y-connected sub c-space of a c-space X, then $\overline{A}$ is also Y-connected.*

Proof. Let X be the given c-space. Well-order X. Let $\leqslant$ be a well-order on X. Let A be a Y-connected sub c-space of X and $f : \overline{A} \to Y$ be any c-continuous function. f being c-continuous on A and since A is Y-connected, f is constant on A. Let $f(x) = a$ for every $x \in A$. Let $K = \{x \in \overline{A} : f(x) = a\}$. We claim that $K = \overline{A}$.

We know that $\overline{A} = t^\gamma(A)$ where γ is the least ordinal such that $t^\gamma(A) = t^{\gamma+1}(A)$. Let $x \in \overline{A}$. If $x \in A$, our claim trivially follows. So let $x \notin A$. Then there exists an ordinal λ such that $x \in t^\lambda(A)$ and $x \notin t^\alpha(A)$ for every $\alpha < \lambda$.

Case I. Let λ is not a limit ordinal. Since $x \in t^\lambda(A)$, x is a touching point of $t^{\lambda-1}(A)$. Then there exits a subset C of $t^{\lambda-1}(A)$ such that $K_1 = C \cup \{x\}$ is connected in X. Clearly $K_1 \subseteq \overline{A}$.

Assume that $y < x$ implies $y \in K$ for every $y \in X$. As $t^{\lambda-1}(A) \subsetneq t^\lambda(A)$, we have $y < x$ for every $y \in C$. Then by our assumption $y \in K$ and hence $f(y) = a$. Now

$$f(K_1) = f(C) \cup \{f(x)\} = \{a, f(x)\}$$

with f being c-continuous, and since Y is a C_1 c-space, $\{a, f(x)\}$ is connected if and only if $f(x) = a$. Hence $x \in K$. Thus by Principle of Transfinite Induction $K = \overline{A}$.

Case II. Let λ be a limit ordinal. Then

$$t^\lambda(A) = \bigcup_{\alpha<\lambda} t^\alpha(A)$$

Since $x \in t^\lambda(A)$, $x \in t^\alpha(A)$ for some $\alpha < \lambda$. By Case I above, we have $K = \overline{A}$.

From above cases, it follows that $f(x) = a$ for every $x \in \overline{A}$. That is, f is a constant function on $\overline{A}$ and hence $\overline{A}$ is Y-connected. □

Remark 4. Let A be a Y-connected sub c-space of a c-space X and B be a sub c-space of X such that $A \subset B \subset \overline{A}$. Then B need not be Y-connected.

Consider the c-space $X = \{1, 2, 3\}$ endowed with the c-structure $\mathcal{C}_X = \mathcal{D}_X \cup \{\{1, 2\}, \{2, 3\}, X\}$. Observe that for the sub c-space $A = \{1\}$, we have $t(A) = \{1, 2\}$ and $\overline{A} = t(t(A)) = X$. On the other hand, the sub c-space $B = \{1, 3\}$ of X is not Y-connected as it is not connected.

Conclusion

In the present study, a stronger notion of connecteness is produced by replacing the discrete c-space $\{0, 1\}$ by a C_1 c-space Y with more than one point. One can think about achieving stronger connecteness in alternate ways and can try to correlate these various notions. In a similar way, we can also think about the concept of weaker form of connectedness too.

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Received by the editors: 29.05.2020
and in final form 11.11.2021.