© Algebra and Discrete Mathematics
Volume **33** (2022). Number 2, pp. 118–127
DOI:10.12958/adm1612 RESEARCH ARTICLE

An identity on automorphisms of Lie ideals in prime rings*

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Communicated by A. P. Petravchuk

Abstract. In the present paper it is shown that a prime ring R with center Z satisfies s_4, the standard identity in four variables if R admits a non-identity automorphism σ such that $[u, v] - u^m[u^\sigma, u]^n u^\sigma \in Z$ for all u in some noncentral ideal L of R, whenever $\mathrm{char}(R) > n + m$ or $\mathrm{char}(R) = 0$, where n and m are fixed positive integer.

Introduction

Throughout this article, R is a prime ring with center Z. For given $x, y \in R$, the Lie commutator of x, y is denoted by $[x, y]$ and defied by $[x, y] = xy - yx$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$. The standard identity s_4 in four variables is defined as follows:

$$s_4 = \sum (-1)^\tau X_{\tau(1)} X_{\tau(2)} X_{\tau(3)} X_{\tau(4)}$$

where $(-1)^\tau$ is the sign of a permutation τ of the symmetric group of degree 4.

The theory of commuting and centralizing maps on (semi-)prime rings was motivated by the result of Posner [20] and was developed by

*This research is supported by the National Board of Higher Mathematics (NBHM), India, Grant No. 02011/16/2020 NBHM (R. P.) R & D II/7786.

2020 MSC: 16N60, 16W20, 16R50.

Key words and phrases: prime ring, automorphisms; maximal right ring of quotients, generalized polynomial identity.

Brešar [4–6]. Posner's second theorem sates that if there exists a nonzero centralizing derivation on a prime ring R, then R is commutative. Mayne [18] obtained an analogous result for automorphisms of prime rings. Many people have extended Posner's result in various ways and obtained many powerful results. In [16], Lee and Lee generalized Posner's result by showing that if $\text{char}(R) \neq 2$ and $[d(x), x] \in Z$ for all x in a noncentral Lie ideal of R, then R is commutative. In [15], Lanski proved that if $[d(x), x]_n = 0$ for all x in a noncommutative Lie ideal of R, then $\text{char}(R) = 2$ and $R \subseteq M_2(\mathbb{F})$ for $\mathbb{F}$ a field. A similar extension for Lie ideals in automorphism case was obtained by Mayne [19].

In [7], Carini and De Filippis studied the power-centralizing derivations on noncentral Lie ideals of prime rings. They proved that, if $\text{char}(R) \neq 2$ and $[d(x), x]^n \in Z$ for all x in a noncentral Lie ideal of R, then R satisfies s_4, the standard identity in four variables. Recently, Wang [22], obtained similar result for automorphisms of prime rings. To be more specific, Wang discussed the following: Let R be a prime ring with center Z, L a noncentral Lie ideal of R and σ a nontrivial automorphism of R such that $[u^\sigma, u]^n \in Z$ for all $u \in L$. If either $\text{Char}(R) > n$ or $\text{char}(R) = 0$, then R satisfies s_4.

On other hand, the representative work of Herstein should be mention at least. Herstein [12], proved that if there exists a nonzero derivation d on a prime ring R such that the map $x \mapsto d(x)$ is commuting on R, then R may be noncommutative. That is, the following relation $[d(x), x]d(x) + d(x)[d(x), x] = 0$ for all $x \in R$ does not imply that $d = 0$. Motivated by the above result Cheng [10] proved the following, which can be considered as an extension of Posner's second theorem: if R is a 2-torsion free noncommutative prime ring and d be a derivation of R such that $[d(x), x]d(x) = 0$ for all $x \in R$, then $d = 0$.

The property $x^n = x$ has been among the favourites of many ring theorists over the last many decades since Jacobson [14] first studied the commutativity of rings satisfying this condition in order to generalize the classical Wedderburn theorem [23]. This result was further generalized by Sercoid and MacHale [21] who proved that commutativity of an arbitrary ring R (not necessarily prime) follows even if the above condition is weakened as $(xy)^n = xy$ for all $x, y \in R$ and integer $n = n(x, y) > 1$. Further, Bell and Ligh [3] obtained direct sum decomposition of ring satisfying the property $xy = (xy)^2 f(x, y)$, where $f(X, Y) \in \mathbb{Z}\langle X, Y\rangle$, the ring of polynomial in two non-commuting indeterminates. Later, Ashraf [1] established a decomposition theorem for ring satisfying $yx = x^m f(xy)x^n$ or $xy = x^m f(xy)x^n$ where m, n are non-negative integers and $f(X) \in$

$X^2\mathbb{Z}[X]$, which in turn allows us to determine the commutativity of R. Now in this perspective and inspired by Wang [22] and Cheng [10] works, one can consider the following related ring property:

Let $m \geqslant 0$, $n \geqslant 0$ be fixed integers and L a Lie ideal of prime ring R which admits an automorphism σ such that $[u, v] - u^m[u^\sigma, u]^n u^\sigma \in Z$.

In the present paper, it is shown that if R admits an automorphismσ satisfy the above condition, if $\text{char}(R) > n + m$ or $\text{char}(R) = 0$, then R satisfies s_4, the standard identity in four variables.

1. Preliminaries

For the sake of completeness we shall touch upon a few preliminary notions required for the exposition of the main theorem. Some of these notions are classical and we present them briefly, R will be prime ring with center Z and maximal right ring of quotients $Q = Q_{mr}(R)$. Note that Q is also a prime ring and the center C of Q, which is called the extended centroid of R, is a field. Moreover, $Z \subseteq C$ (for more explanation we refer to [2]). It is well known that any automorphism of R can be uniquely extended to an automorphism of Q. An automorphism σ of R is called Q-inner if there exists an invertible element $g \in Q$ such that $x^\sigma = gxg^{-1}$ for all $x \in R$. Otherwise, σ is called Q-outer. We denote by G the group of all automorphisms of R and by A_i the group consisting of all Q-inner automorphisms of R. Recall that a subset $\mathfrak{A}$ of G is said to be independent (modulo A_i) if for any $a_1, a_2 \in \mathfrak{A}$, $a_1 a_2^{-1} \in A_i$ implies $a_1 = a_2$. For instance, if a is an outer automorphism of R, then 1 and a are independent (modulo A_i). We present some well-known facts that will be used in the sequel.

Fact 1. It is well known that any automorphisms of R can be extended to Q.

Fact 2. Let R be a prime ring and I a two-sided ideal of R. Then I, R, and Q satisfy the same generalized polynomial identities with coefficients in Q (see [8]).

Fact 3. Suppose that R is a prime ring and $\mathfrak{A}$ an independent subset of G modulo A_i. Let $\phi = \chi(x_i^{a_j}) = 0$ be a generalized identity with automorphisms of R reduced with respect to $\mathfrak{A}$. If for all $x_i \in X$, $a_j \in \mathfrak{A}$, the $x_i^{a_j}$-word degree of $\phi = \chi(x_i^{a_j})$ is strictly less than $\text{char}(R)$ when $\text{char}(R) \neq 0$, then $\chi(z_{ij}) = 0$ is also a generalized polynomial identity of R (see [9, Theorem 3]).

Fact 4. Recall that, in case $\mathrm{char}(R) = 0$, an automorphism σ of Q is called *Frobenius* if $(x)^\sigma = x$ for all $x \in C$. Moreover, in case $\mathrm{char}(R) = p \geqslant 2$, an automorphism σ is *Frobenius* if there exists a fixed integer t such that $(x)^\sigma = x^{p^t}$ for all $x \in C$. In [9, Theorem 2] Chuang proves that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and $\sigma \in \mathrm{Aut}(R)$ an automorphism of R which is not Frobenius, then R also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.

Fact 5. Let R be a prime ring and L a noncentral Lie ideal of R. If $\mathrm{char}(R) \neq 2$, then there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If $\mathrm{char}(R) = 2$ and $\dim_c RC > 4$, then there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus if either $\mathrm{char}(R) \neq 2$ or $\dim_C RC > 4$, then we may conclude that there exists a nonzero ideal I of R such that $[I, I] \subseteq L$.

Fact 6. Let R be a prime ring with extended centroid C. Then the following conditions are equivalent:

(i) $\dim_C RC \leqslant 4$.
(ii) R satisfies S_4, the standard identity in four variables.
(iii) R is commutative or R embeds in $M_2(\mathbb{F})$, where $\mathbb{F}$ is a field.
(iv) R is algebraic of bounded degree 2 over C.
(v) R satisfies $[[x^2, y], [x, y]]$.

2. The results in prime rings

We begin with the following results which are imperative to establish of our main theorem.

Theorem 1. *Let R be a prime ring and σ a non-identity automorphism of R such that $[u, v] - u^m[u^\sigma, u]^n u^\sigma = 0$ for all u, v in a noncentral Lie ideal L of R, where n, m are fixed positive integer. If either* $\mathrm{char}(R) > n + m$ *or* $\mathrm{char}(R) = 0$, *then R satisfies s_4, the standard identity in four variables.*

Proof. We assume that $\dim_C RC > 4$. In view of Fact 5, there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Using our hypothesis, we find that

$$[[x, y], [z, w]] - [x, y]^m[[[x, y]^\sigma, [x, y]]^n[x, y]^\sigma = 0 \text{ for all } x, y \in I. \tag{1}$$

Firstly, if σ is Q-inner, then there exists an invertible element $q \in Q$ such that $x^\sigma = qxq^{-1}$ for all $x \in R$. By [8, Theorem 2],

$$[[x, y], [z, w]] - [x, y]^m[q[x, y]q^{-1}, [x, y]]^n q[x, y]q^{-1} = 0$$

is also an identity for RC. By Martindale's theorem in [17], RC is a primitive ring with nonzero socle. Since RC is primitive, there exist a vector space $\mathcal{V}$ over a division ring $\mathcal{D}$ such that RC is a dense ring of $\mathcal{D}$-linear transformations over $\mathcal{V}$. We divide the proof into two steps:

Step 1. Our aim is to show that for any $v \in \mathcal{V}$, v and vq are linearly $\mathcal{D}$-dependent. If v and vq are linearly $\mathcal{D}$-independent for some $v \in \mathcal{V}$, then we consider the following cases:

If $vq^{-1} \notin Span_{\mathcal{D}}\{v, vq\}$, then the set $\{v, vq, vq^{-1}\}$ is linearly $\mathcal{D}$-independent. By the density of RC there exist $x_0, y_0 \in RC$ such that

$$\begin{array}{lllll} vx_0 = v, & vqx_0 = 0, & vz_0 = vq & vqz = v & vq^{-1}x_0 = -vq \\ vy_0 = v, & vqy_0 = -v, & vw_0 = v & vqw = 0 & vq^{-1}y_0 = 0. \end{array}$$

We can easily see that

$$0 = v([[x, y], [z, w]] - [x, y]^m[q[x_0, y_0]q^{-1}, [x_0, y_0]]^n q[x_0, y_0]q^{-1}) = v \neq 0,$$

a contradiction.

On the other hand if $vq^{-1} \in Span_{\mathcal{D}}\{v, vq\}$, then $vq^{-1} = v\alpha + vq\beta$ for some $\alpha, \beta \in \mathcal{D}$. In view of the density of RC, there exist $x_0, y_0, z_0, w_0 \in RC$ such that

$$\begin{array}{llll} vx_0 = v, & vqx_0 = 0 & vz_0 = qv & vqz_0 = v \\ vy_0 = v, & vqy_0 = v & vw_0 = v & vqw_0 = 0. \end{array}$$

Hence we find that

$$0 = v([q[x_0, y_0]q^{-1}, [x_0, y_0]]^n q[x_0, y_0]q^{-1}) = \gamma v \neq 0$$

for some $\gamma \in \mathcal{D}$, again a contradiction.

Step 2. We have that v and qv are $\mathcal{D}$-dependent for every $v \in \mathcal{V}$. For each $v \in \mathcal{V}$, we write $vq = v\lambda_v$ where $\lambda_v \in \mathcal{D}$. Fix $0 \neq u \in \mathcal{V}$. Let $0 \neq v \in \mathcal{V}$ and write $vq = v\lambda_v$. Suppose first that v and u are $\mathcal{D}$-independent. Then $(u + v)\lambda_{u+v} = (u + v)q = uq + vq = u\lambda_u + v\lambda_v$. So $u(\lambda_{u+v} - \lambda_u) = v(\lambda_v - \lambda_{u+v})$, and hence $\lambda_{u+v} = \lambda_u = \lambda_v$. Suppose next that u and v are $\mathcal{D}$-dependent. Indeed, for any $w \in \mathcal{V}$, w and u are $\mathcal{D}$-independent, and then, by the proof above, we have $\lambda_w = \lambda_v$. Clearly, w and v are $\mathcal{D}$-independent. So $\lambda_w = \lambda_v$, implying that $\lambda_u = \lambda_v$. Thus λ_v is the independent choice of $v \in \mathcal{V}$. Consequently, $vq = v\lambda$ for all $v \in \mathcal{V}$, where $\lambda = \lambda_v$. By standard argument we see that $q \in C$, a contradiction. Thus $\dim_C RC \leqslant 4$, and by Fact 6, R satisfies s_4, the standard identity in four variables.

Next we assume that σ is not Q-inner, then by Chuang [10, Main Theorem], R satisfies $[[x,y],[z,w]]-[x,y]^m[[x,y]^\sigma,[x,y]]^n[x,y]^\sigma=0$. Since either $\mathrm{char}(R)>n$ or $\mathrm{char}(R)=0$, it follows from Fact 3 that $[[x,y],[z,w]]-[x,y]^m[[w_1,z_1],[x,y]]^n[w_1,z_1]=0$ for all $x,y,z,w\in R$. Note that this is a polynomial identity and thus there exists a field $\mathbb{F}$ such that $R\subseteq M_k(\mathbb{F})$, the ring of $k\times k$ matrices over a field $\mathbb{F}$, where $k\geqslant 1$. Moreover, R and $M_k(\mathbb{F})$ satisfy the same polynomial identity [15, Lemma 1], that is $[[x,y],[z,w]]-[x,y]^m[[w_1,z_1],[x,y]]^n[w_1,z_1]=0$ for all $x,y,w,z,z_1,w_1\in M_k(\mathbb{F})$. But by choosing $x=e_{11}$, $y=e_{21}$, $w=e_{12}$, $z=e_{12}$, $w_1=e_{11}$, $z_1=e_{12}$ we get

$$\begin{aligned} 0 &= [[e_{11},e_{21}],[e_{12},e_{12}]]-[e_{11},e_{21}]^m[[e_{11},e_{12}],[e_{11},e_{21}]]^n[e_{11},e_{12}] \\ &= (-1)^n e_{12}, \end{aligned}$$

a contradiction. This completes the proof. □

Let $\mathcal{V}_\mathcal{D}$ be a right vector space over a division ring $\mathcal{D}$. We denote $\mathrm{End}(\mathcal{V}_\mathcal{D})$ the ring of $\mathcal{D}$-linear transformations on $\mathcal{V}_\mathcal{D}$. A map $T:\mathcal{V}\to\mathcal{V}$ is called a semilinear transformation if T is additive and there is an automorphism ζ of $\mathcal{D}$ such that $T(v\gamma)=(Tv)\zeta(\gamma)$ for all $v\in\mathcal{V}$ and $\gamma\in\mathcal{D}$. Moreover, by a theorem of Jacobson [13, Isomorphism Theorem, p.79], there exists an invertible semilinear transformation $T:\mathcal{V}\to\mathcal{V}$ such that $\sigma(x)=TxT^{-1}$ for all $x\in\mathrm{End}(\mathcal{V}_\mathcal{D})$, where σ is an automorphism of $\mathrm{End}(\mathcal{V}_\mathcal{D})$.

Lemma 1. *Let σ be an automorphism of $\mathrm{End}(\mathcal{V}_\mathcal{D})$ such that for every $x,y,z,w,z_1\in\mathrm{End}(\mathcal{V}_\mathcal{D})$,*

$$[[x,y],[z,w]]-[x,y]^m[[[x,y]^\sigma,[x,y]]^n[x,y]^\sigma,z_1]=0,$$

where n,m are fixed positive integer. If $\dim(\mathcal{V}_\mathcal{D})\geqslant 2$, then σ is identity map of $\mathrm{End}(\mathcal{V}_\mathcal{D})$.

Proof. By a theorem of Jacobson [13, Isomorphism Theorem, p.79], there exists an invertible semilinear transformation $T:\mathcal{V}\to\mathcal{V}$ such that $\sigma(x)=TxT^{-1}$ for all $x\in\mathrm{End}(\mathcal{V}_\mathcal{D})$, where σ is an automorphism of $\mathrm{End}(\mathcal{V}_\mathcal{D})$. In particular, there exists an automorphism ζ of $\mathcal{D}$ such that $T(v\gamma)=(Tv)\zeta(\gamma)$ for all $v\in\mathcal{V}$ and $\gamma\in\mathcal{D}$. Using our hypothesis, we find that $0=[[x,y],[z,w]]-[x,y]^m[[[x,y]^\sigma,[x,y]]^n[x,y]^\sigma,z_1]=[[x,y],[z,w]]-[x,y]^m[[T[x,y]T^{-1},[x,y]]^nT[x,y]^{-1},z_1]$ for all $x,y,z,w,z_1\in\mathrm{End}(\mathcal{V}_\mathcal{D})$. We divide our proof into the following cases:

There exists $v \in \mathcal{V}$ such that v and $T^{-1}v$ are $\mathcal{D}$-independent. Suppose first that $\{v, vT, vT^{-1}\}$ is $\mathcal{D}$-independent. Let $x, y, z \in \mathrm{End}(\mathcal{V}_\mathcal{D})$ such that

$$\begin{array}{lll} xv = Tv, & xT^{-1}v = -v, & xTv = 0 \\ yv = Tv, & yT^{-1}v = 0, & yTv = v \\ wv = Tv & z_1T^{-1}v = 0 & zTv = Tv \\ zv = 0 & w_1T^{-1} = 0 & wTv = 0 \\ z_1v = 0, & z_1T^{-1}v = v, & z_1Tv = -v. \end{array}$$

Then $[x, y]v = 0$, $[x, y]T^{-1}v = v$, $[x, y]Tv = Tv$, $[z, w]v = Tv$ and hence

$$0 = ([[[x, y], [z, w]] - [x, y]^m[T[x, y]T^{-1}, [x, y]]^nT[x, y]T^{-1}, z])v = v,$$

a contradiction.

Suppose next that $\{v, Tv, T^{-1}v\}$ is $\mathcal{D}$-dependent. Then there exist $\mu, \chi \in \mathcal{D}$ such that $Tv = v\mu + T^{-1}v\chi$. Moreover, we claim that $\chi \neq 0$. Indeed, if $\chi = 0$, then $Tv = v\mu$ and $v = T^{-1}v\mu$, a contradiction. Let $x, y, z, w, z_1 \in \mathrm{End}(\mathcal{V}_\mathcal{D})$ such that

$$\begin{array}{lll} xv = Tv, & xT^{-1}v = -v, & z_1v = 0 \\ yv = Tv, & yT^{-1}v = 0, & z_1T^{-1}v = -v \\ zv = 0 & wv = Tv & zT^{-1}v = v. \end{array}$$

We can easily see that

$$0 = ([[[[x, y], [z, w]] - [x, y]^m[T[x, y]T^{-1}, [x, y]]^nT[x, y]T^{-1}, z])v = \eta v,$$

for some $\eta \in \mathcal{D}$, a contradiction.

We have that v and $T^{-1}v$ are $\mathcal{D}$-dependent for every $v \in \mathcal{V}$. For each $v \in \mathcal{V}$, we write $T^{-1}v = v\alpha_v$ where $\alpha_v \in \mathcal{D}$. Fix $0 \neq u \in \mathcal{V}$. Let $0 \neq v \in \mathcal{V}$ and write $T^{-1}v = v\alpha_v$. Suppose first that v and u are $\mathcal{D}$-independent. Then $(u + v)\alpha_{u+v} = (u + v)q = uq + vq = u\alpha_u + v\alpha_v$. So $u(\alpha_{u+v} - \alpha_u) = v(\alpha_v - \alpha_{u+v})$, and hence $\alpha_{u+v} = \alpha_u = \alpha_v$. Suppose next that u and v are $\mathcal{D}$-dependent. Since $\dim(\mathcal{V}_\mathcal{D}) \geqslant 2$, there exists $w \in \mathcal{V}$ such that w and u are $\mathcal{D}$-independent, and then, by the proof above, we have $\alpha_w = \alpha_v$. Clearly, w and v are $\mathcal{D}$-independent. So $\alpha_w = \alpha_v$, implying that $\alpha_u = \alpha_v$. Thus α_v is independent of the choice of $v \in \mathcal{V}$. Consequently, $T^{-1}v = v\alpha$ for all $v \in \mathcal{V}$, where $\alpha = \alpha_v$. Now we have $\sigma(x)v = T(x(v\alpha)) = T((xv)\alpha) = xv$ for all $x \in \mathrm{End}(\mathcal{V}_\mathcal{D})$ and $v \in \mathcal{V}$. In particular, $(\sigma(x) - x)V = 0$ for all $x \in \mathrm{End}(\mathcal{V}_\mathcal{D})$. Thus $\sigma(x) = x$ for all $x \in \mathrm{End}(\mathcal{V}_\mathcal{D})$. This implies σ is the identity map of $\mathrm{End}(\mathcal{V}_\mathcal{D})$, proving the lemma. □

Using both of these lemmas, we are ready to prove our main theorem.

Theorem 2. *Let R be a prime ring with center Z which admits a non-identity automorphism σ such that $[u, v] - u^m[u^\sigma, u]^n u^\sigma \in Z$ for all u in a noncentral ideal L of R, where n, m are fixed positive integer. If* $\mathrm{char}(R) > n + m$ *or* $\mathrm{char}(R) = 0$*, then R satisfies s_4, the standard identity in four variables.*

Proof. We assume that $\dim_C RC > 4$. Then by Fact 5, there exists a nonzero ideal I of R such that $0 \neq [I, I] \subseteq L$. By assumption, we get

$$[[x, y], [z, w]] - [x, y]^m[[x, y]^\sigma, [x, y]]^n[x, y]^\sigma \in Z \text{ for all } x, y \in I. \tag{2}$$

Suppose σ is Q-inner automorphism, there exists an invertible element $g \in Q$ such that $x^\sigma = gxg^{-1}$ for all $x \in R$. Then I satisfies

$$[[x, y], [z, w]] - [x, y]^m[[x, y]^\sigma, [x, y]]^n[x, y]^\sigma \in Z. \tag{3}$$

By a Theorem of Chuang[8], I and Q satisfy the same generalized polynomial identities. Thus Q satisfied

$$[[x, y], [z, w]] - [x, y]^m[[x, y]^\sigma, [x, y]]^n[x, y]^\sigma \in C. \tag{4}$$

Since $g \notin C$, therefore

$$\phi(t) = [[[x, y], [z, w]] - [x, y]^m[[x, y]^\sigma, [x, y]]^n[x, y]^\sigma, z_1]$$

for all $x, y, z, w, z_1 \in Q$ is a nontrivial generalized polynomial identity on Q. Denote by F the algebraic closure of C if C is infinite and set $F = C$ for C finite. Then $Q \otimes_C F$ is a prime ring with extended centroid F [11, Theorem 3.5]. Clearly $Q \cong Q \otimes_C C \subseteq Q \otimes_C F$. So we may regards Q as a subring $Q \otimes_C F$ and hence $\phi(t)$ is also a nontrivial generalized polynomial identity of $Q \otimes_C F$. Let $\mathcal{Q} = Q_{mr}(Q \otimes_C F)$, the maximal right ring of quotients of $Q \otimes_C F$. By [2, Theorem 6.4.4], $\phi(t)$ is also a nontrivial generalized polynomial identity on $\mathcal{Q}$. By Martindale's theorem [17], $\mathcal{Q} \cong \mathrm{End}(\mathcal{V}_\mathcal{D})$, where $\mathcal{V}$ is a vector space over a division ring $\mathcal{D}$ and $\mathcal{D}$ is finite dimension over its center F. Recall that F is either algebraically closed or finite. From the finite dimensionality of D over F, it follows that $\mathcal{D} = F$. Hence $\mathcal{Q} \cong \mathrm{End}(\mathcal{V}_F)$. By Lemma 1, we get a contradiction.

We now assume that σ is Q-outer automorphism, due to Chuang [8, Main Theorem], I and Q satisfies the same polynomial identity and hence R as well. Therefore R satisfies

$$[[[x, y], [z, w]] - [x, y]^m[[x, y]^\sigma, [x, y]]^n[x, y]^\sigma, z_1] = 0.$$

Since either $\mathrm{char}(R) > n+m$ or $\mathrm{char}(R) = 0$, it follows from Lemma 1 that

$$[[[x,y],[z,w]] - [x,y]^m[[s,t],[x,y]]^n[s,t],z] = 0$$

for all $x, y, s, t, z, w, z_1 \in R$. Note that this is a polynomial identity and thus there exists a field $\mathbb{F}$ such that $R \subseteq M_k(\mathbb{F})$, the ring of $k \times k$ matrices over a field $\mathbb{F}$, where $k > 1$. Moreover, R and $M_k(\mathbb{F})$ satisfy the same polynomial identity [15, Lemma 1], that is

$$[[[x,y],[z,w]] - [x,y]^m[[s,t],[x,y]]^n[s,t],z] = 0$$

for all $x, y, s, t, z \in M_k(\mathbb{F})$. Let e_{ij} be a matrix unit with 1 in the (i,j)-entry and zero elsewhere. Since $\dim_C RC > 4$, we see that $k > 2$. By choosing $x = e_{11}$, $y = e_{21}$,$z = e_{12}$, $w = e_{12}$, $s = e_{11}$, $t = e_{12}$, $z_1 = e_{31}$ we get

$$\begin{aligned} 0 &= [[[x,y],[z,w]] - [x,y]^m[[s,t],[x,y]]^n[s,t],z] \\ &= [[[e_{11},e_{21},[e_{12},e_{12}]] - [e_{11},e_{21}]^m[[e_{11},e_{12}],[e_{11},e_{21}]]^n[e_{11},e_{12}],e_{31}] \\ &= (-1)^{n+1}e_{31}, \end{aligned}$$

a contradiction. Thus $\dim_C RC \leqslant 4$. In View of Fact 6, we get required result. With this the proof is complete. □

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Received by the editors: 15.05.2020
and in final form 24.08.2022.