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An outer measure on a commutative ring

Dariusz Dudzik and Marcin Skrzyński

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Abstract. We show how to produce a reasonable outer measure on a commutative ring from a given measure on a family of prime ideals of this ring. We provide a few examples and prove several properties of such outer measures.

Introduction

Throughout the present paper, R is a nonzero commutative ring with identity. We denote by $\operatorname{Spec}(R)$ the family of all the prime ideals of R. (Notice that, by definition, every prime ideal is proper).

It is well known [1] that topological properties of $\operatorname{Spec}(R)$ equipped with the Zariski topology reflect algebraic properties of R. But are there useful relationships between algebraic or geometric properties of R and measures on $\operatorname{Spec}(R)$? This question seems to be quite interesting and not worked out in the specialist literature. The present paper provides some basic remarks concerning the question and, hopefully, is a starting point for further study.

In the paper, we will show that an arbitrary measure on a suitable subfamily of $\operatorname{Spec}(R)$ induces an outer measure on R with good multiplicative properties. We will also discuss a few elementary examples of such outer measures.

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By "measure" we mean a "non-negative σ-additive measure". We denote by 2^X the power set of a set X. We define

$$|X| = \begin{cases} \text{the cardinality of } X, & \text{if } X \text{ is finite,} \\ +\infty, & \text{otherwise.} \end{cases}$$

By $R^\times$ we denote the set of invertible elements of R. Notice that $\wp \cap R^\times = \varnothing$ whenever $\wp \in \operatorname{Spec}(R)$. We define $\operatorname{Max}(R)$ to be the family of all the maximal ideals of R. One can prove that $\operatorname{Max}(R) \subseteq \operatorname{Spec}(R)$ and $\bigcup \operatorname{Max}(R) = R \setminus R^\times$.

We refer to [1] for more information about commutative rings and to [2] for elements of measure theory.

1. Construction

We will use the definition of outer measure taken from [2].

Definition 1. We say that $\mu^* : 2^X \longrightarrow [0, +\infty]$ is an outer measure on a set X, if the following conditions are satisfied:

(1) $\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu^*(B_n)$ for every sequence $\{B_n\}_{n=1}^{\infty}$ of subsets of X and every $A \subseteq \bigcup_{n=1}^{\infty} B_n$,

(2) $\mu^*(\varnothing) = 0$.

Let $\mathcal{P} \subseteq \operatorname{Spec}(R)$ be such that $\bigcup \mathcal{P} = R \setminus R^\times$, and let $\mathfrak{M}$ be a σ-algebra of subsets of $\mathcal{P}$. For a set $A \subseteq R$ we define

$$\Omega(A) = \left\{ \mathcal{S} \in \mathfrak{M} : \bigcup \mathcal{S} \supseteq A \setminus R^\times \right\}.$$

Proposition 1. *Suppose that $\mu : \mathfrak{M} \longrightarrow [0, +\infty]$ is a measure. Then the function $\mu^* : 2^R \longrightarrow [0, +\infty]$ defined by*

$$\mu^*(A) = \inf_{\mathcal{S} \in \Omega(A)} \mu(\mathcal{S})$$

is an outer measure on R. (This outer measure will be referred to as the outer measure induced by μ).

Proof. It is obvious that $\mu^*(\varnothing) = 0$. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of subsets of R and let ε be an arbitrary positive real number. Observe that

$$\forall\, n \in \mathbb{N} \setminus \{0\}\, \exists\, \mathcal{S}_n \in \Omega(B_n) : \mu(\mathcal{S}_n) \leqslant \mu^*(B_n) + \frac{\varepsilon}{2^n}.$$

If $A \subseteq \bigcup_{n=1}^{\infty} B_n$, then $\bigcup_{n=1}^{\infty} \mathcal{S}_n \in \Omega(A)$ and hence

$$\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu(\mathcal{S}_n) \leqslant \varepsilon + \sum_{n=1}^{\infty} \mu^*(B_n).$$

Since ε is arbitrary, the above inequalities yield $\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu^*(B_n)$. □

The outer measure induced by a measure on a family of prime ideals is a slight modification of a well known measure-theoretical construction. In the next section we give examples that illustrate and motivate this modification.

2. Examples

We denote by (a) the principal ideal generated by an element $a \in R$. Consider a further example of a "covering by prime ideals".

Example 1. We assume that R is a unique factorization domain and define $\mathcal{P}_{\text{irr}}(R) = \{(0)\} \cup \{(a) : a \in R, a \text{ is irreducible}\}$. Observe that $\mathcal{P}_{\text{irr}}(R) \subseteq \text{Spec}(R)$ and $\bigcup \mathcal{P}_{\text{irr}}(R) = R \setminus R^{\times}$. Moreover, if $n \in \mathbb{N} \setminus \{0, 1\}$ and $R = \mathbb{C}[x_1, \ldots, x_n]$, then $\mathcal{P}_{\text{irr}}(R) \cap \text{Max}(R) = \varnothing$.

Recall that for every ideal I of the ring of integers there exists exactly one $m \in \mathbb{N} \cup \{0\}$ such that $I = (m)$. Notice also that $\text{Max}(\mathbb{Z}) = \{(p) : p \in \mathbb{P}\}$, where $\mathbb{P}$ stands for the set of prime numbers.

Proposition 2. *Let $\mu^* : 2^{\mathbb{Z}} \longrightarrow [0, +\infty]$ be the outer measure induced by the counting measure on $\text{Max}(\mathbb{Z})$, and let $A \subseteq \mathbb{Z}$ be such that $A \setminus \{-1, 1\} \neq \varnothing$. Then*

(i) $\mu^*(\{-1, 1\}) = 0$,

(ii) $\mu^*(A) = 1$ *if and only if*

$$\exists\, d \in \mathbb{N} \setminus \{0, 1\}\, \forall\, k \in A \setminus \{-1, 1\} : d \mid k$$

(in particular, $\mu^*(A) = 1$ whenever A is a singleton or a proper ideal of $\mathbb{Z}$),

(iii) $\mu^*(A) \leqslant |A|$.

Moreover, in the case where $A \cap \{-1, 1\} = \varnothing$ and A is a finite set, $\mu^*(A) = |A|$ if and only if the elements of A are pairwise relatively prime.

Proof. Since $\{-1,1\} = \mathbb{Z}^\times$, we have $\varnothing \in \Omega(\{-1,1\})$. Equality (i) follows.

By the above characterization of $\mathrm{Max}(\mathbb{Z})$ and the definition of counting measure, $\mu^*(A) = 1$ if and only if $A \setminus \{-1,1\} \subseteq (p_1)$ for a prime number p_1. The latter condition means precisely that

$$\exists\, p_1 \in \mathbb{P}\,\forall\, k \in A \setminus \{-1,1\} : \, p_1 \mid k.$$

Finally, if $d \in \mathbb{N} \setminus \{0,1\}$, $k \in \mathbb{Z}$ and $d \mid k$, then k is divisible by every prime factor of d. Property (ii) follows.

Property (iii) is an immediate consequence of the definition of outer measure and the fact that $\mu^*(\{k\}) \leqslant 1$ for all $k \in \mathbb{Z}$.

Assume that $A \cap \{-1,1\} = \varnothing$ and A is a finite set. Let us define $\ell = |A|$. Observe that $\mu^*(A) \neq |A|$ if and only if

$$\exists\, \mathcal{S} \in \Omega(A) : \, |\mathcal{S}| \leqslant \ell - 1.$$

Since the cardinality of A is greater than the cardinality of $\mathcal{S}$, the latter condition holds true if and only if

$$\exists\, s,t \in A\, \exists\, p_2 \in \mathbb{P} : \begin{cases} s \neq t, \\ s,t \in (p_2), \end{cases}$$

and this means precisely that there exist two distinct elements of A which are not relatively prime. □

Let $n \in \mathbb{N} \setminus \{0\}$. Consider a σ-algebra $\mathfrak{N}$ of subsets of $\mathbb{C}^n$, a measure $\lambda : \mathfrak{N} \longrightarrow [0,+\infty]$, and the map

$$\Phi : \mathbb{C}^n \ni z \longmapsto \{f \in \mathbb{C}[x_1,\dots,x_n] : \, f(z) = 0\} \in \mathrm{Max}(\mathbb{C}[x_1,\dots,x_n]).$$

The family $\mathfrak{M} = \{\mathcal{S} \subseteq \mathrm{Max}(\mathbb{C}[x_1,\dots,x_n]) : \, \Phi^{-1}(\mathcal{S}) \in \mathfrak{N}\}$ is a σ-algebra of subsets of $\mathrm{Max}(\mathbb{C}[x_1,\dots,x_n])$. The function $\eta : \mathfrak{M} \ni \mathcal{S} \mapsto \lambda(\Phi^{-1}(\mathcal{S})) \in [0,+\infty]$ is a measure.

Let us define $U = \mathbb{C}[x_1,\dots,x_n]^\times$. (Obviously, $U = \mathbb{C} \setminus \{0\}$).

Proposition 3. *If* $\eta^* : 2^{\mathbb{C}[x_1,\dots,x_n]} \longrightarrow [0,+\infty]$ *is the outer measure induced by* η *and* $A \subseteq \mathbb{C}[x_1,\dots,x_n]$ *is such that* $A \setminus U \neq \{0\}$*, then*

$$\eta^*(A) = \inf\{\lambda(Z) : \, Z \in \mathfrak{N},\, Z \cap f^{-1}(0) \neq \varnothing \textit{ for every } f \in A \setminus \mathbb{C}\}.$$

Proof. If $A \subseteq U$, then $\{Z \in \mathfrak{N} : \, Z \cap f^{-1}(0) \neq \varnothing$ for every $f \in A \setminus \mathbb{C}\} = \mathfrak{N}$, and hence

$$\inf\{\lambda(Z) : \, Z \in \mathfrak{N},\, Z \cap f^{-1}(0) \neq \varnothing \text{ for every } f \in A \setminus \mathbb{C}\} = 0 = \eta^*(A).$$

By Hilbert's Nullstellensatz, the map Φ is bijective. Consequently, $\mathfrak{M} = \{\Phi(Z) : Z \in \mathfrak{N}\}$. Suppose that $A \setminus \mathbb{C} \neq \varnothing$. Then for any $Z \in \mathfrak{N}$ the following equivalences hold true:

$$\Phi(Z) \in \Omega(A) \iff (\forall\, f \in A \setminus U\, \exists\, \wp \in \Phi(Z) :\, f \in \wp) \iff$$

$$(\forall\, f \in A \setminus \mathbb{C}\, \exists\, z \in Z :\, f(z) = 0) \iff \Big(\forall\, f \in A \setminus \mathbb{C} :\, Z \cap f^{-1}(0) \neq \varnothing\Big).$$

(The second equivalence holds because 0 belongs to every ideal). Therefore,

$$\eta^*(A) = \inf_{\mathcal{S} \in \Omega(A)} \eta(\mathcal{S}) =$$

$$= \inf\{\lambda(\Phi^{-1}(\Phi(Z))) :\, Z \in \mathfrak{N},\, Z \cap f^{-1}(0) \neq \varnothing \text{ for every } f \in A \setminus \mathbb{C}\},$$

which completes the proof. □

Example 2. Let $\eta^* : 2^{\mathbb{C}[x,y]} \longrightarrow [0, +\infty]$ be the outer measure induced by the counting measure on $\mathrm{Max}(\mathbb{C}[x, y])$. Consider the set $E = \{f, g, h, k\} \subset \mathbb{C}[x, y]$, where

$$f(x, y) = x^2 - y + 1,\ g(x, y) = y^2,\ h(x, y) = xy - 1,\ k(x, y) = xy + 1.$$

Since $f^{-1}(0) \cap g^{-1}(0) \cap h^{-1}(0) \cap k^{-1}(0) = \varnothing$ and $f^{-1}(0) \cap g^{-1}(0) \neq \varnothing$, we have $\eta^*(E) \in \{2, 3\}$. Observe that $\{f, g\}$, $\{f, h\}$ and $\{f, k\}$ are the only two-element subsets of E which have a common zero. Consequently, no three-element subset of E has a common zero. It follows, therefore, that $\eta^*(E) = 3$.

Notice that in the example above, if I is a proper ideal of $\mathbb{C}[x, y]$, then $I \subseteq \wp$ for an ideal $\wp \in \mathrm{Max}(\mathbb{C}[x, y])$ and hence $\eta^*(I) = 1$.

Let K be a nonempty compact subset of $\mathbb{R}^n$ and let $\mathcal{C}(K, \mathbb{R})$ stand for the ring of all the continuous functions $f : K \longrightarrow \mathbb{R}$. Recall that $\mathcal{C}(K, \mathbb{R})^\times = \{f \in \mathcal{C}(K, \mathbb{R}) :\, f(x) \neq 0 \text{ for all } x \in K\}$. The map

$$\Psi : K \ni x \longmapsto \{f \in \mathcal{C}(K, \mathbb{R}) :\, f(x) = 0\} \in \mathrm{Max}(\mathcal{C}(K, \mathbb{R}))$$

is well known to be a bijection [3]. Consequently, if $\mathfrak{B}$ is a σ-algebra of subsets of K and $\xi : \mathfrak{B} \longrightarrow [0, +\infty]$ is a measure, then $\mathfrak{M} = \{\Psi(Z) :\, Z \in \mathfrak{B}\}$ is a σ-algebra of subsets of $\mathrm{Max}(\mathcal{C}(K, \mathbb{R}))$ and

$$\eta : \mathfrak{M} \ni \mathcal{S} \mapsto \xi(\Psi^{-1}(\mathcal{S})) \in [0, +\infty]$$

is a measure. The obvious counterpart of Proposition 3 remains true.

Example 3. Let $\eta^* : 2^{\mathcal{C}(K,\mathbb{R})} \longrightarrow [0,+\infty]$ be the outer measure induced by η. We will denote by W the set of all the polynomial functions $f : K \longrightarrow \mathbb{R}$. Since

$$\forall\, x \in K\, \exists\, f \in W :\, f^{-1}(0) = \{x\},$$

we have $\eta^*(W) = \eta^*(\mathcal{C}(K,\mathbb{R})) = \xi(K)$.

Now, suppose that K is the Euclidean closed unit ball and ξ is the n-dimensional Lebesgue measure. If E stands for the set of all the radially symmetric functions belonging to $\mathcal{C}(K,\mathbb{R})$ and L is the straight line segment that joins the origin to a boundary point of K, then

$$\forall\, f \in E \setminus \mathcal{C}(K,\mathbb{R})^{\times} :\, L \cap f^{-1}(0) \neq \varnothing.$$

Consequently, $\eta^*(E) = \xi(L) = 0$ whenever $n \geqslant 2$. It is easy to see that if $n = 1$, then $\eta^*(E) = 1$.

3. General properties

In the theorem below (it is the main result of the paper) we use the notations and assumptions of Proposition 1. For $n \in \mathbb{N} \setminus \{0\}$ and $A_1, \dots, A_n \subseteq R$ we define $A_1 \dots A_n = \{a_1 \dots a_n :\, a_1 \in A_1, \dots, a_n \in A_n\}$. Moreover, if $A \subseteq R$, then $A^n = \{a^n :\, a \in A\}$ and $A^{\bullet n} = \underbrace{A \dots A}_{n}$.

Theorem 1. *Let $A, B \subseteq R$ and let C be a nonempty subset of $R^{\times}$. Then*
(i) $\mu^*(R^{\times}) = 0$,
(ii) $\mu^*(A) = \mu^*(A \setminus R^{\times})$,
(iii) $\mu^*(\{0\}) = \min\{\mu^*(E) :\, E \subseteq R,\, E \setminus R^{\times} \neq \varnothing\}$,
(iv) $\forall\, n \in \mathbb{N} \setminus \{0\} :\, \mu^*(A^n) = \mu^*(A^{\bullet n}) = \mu^*(A)$,
(v) $\mu^*(AC) = \mu^*(A)$,
(vi) $\mu^*(AB) \geqslant \max\{\mu^*(A), \mu^*(B)\}$ *whenever* $A \cap R^{\times} \neq \varnothing$ *and* $B \cap R^{\times} \neq \varnothing$,
(vii) $\mu^*(AB) \leqslant \mu^*(A) + \mu^*(B)$,
(viii) $\mu^*(AB) = \mu^*(A)$ *whenever* $A \cap R^{\times} = \varnothing$ *and* $B \cap R^{\times} \neq \varnothing$,
(ix) $\mu^*(AB) = \min\{\mu^*(A), \mu^*(B)\}$ *whenever* $A \cap R^{\times} = \varnothing$ *and* $B \cap R^{\times} = \varnothing$.

Proof. Properties (i) and (ii) are obvious.

Property (iii) follows from the facts that $0 \notin R^{\times}$ and 0 belongs to every ideal of R.

Fix a positive integer n. Let $a_1, \dots, a_n \in R$. The product $a_1 \dots a_n$ is not invertible if and only if there exists an index $i \in \{1, \dots, n\}$ such that

a_i is not invertible. Similarly, $a_1 \dots a_n \in \wp$ for an ideal $\wp \in \mathrm{Spec}(R)$ if and only if there exists an index $i \in \{1, \dots, n\}$ such that $a_i \in \wp$. Therefore, $\Omega(A^n) = \Omega(A^{\bullet n}) = \Omega(A)$. Property (iv) follows.

Let $a \in R$ and $c \in R^\times$. Observe that $ac \notin R^\times$ if and only if $a \notin R^\times$. Moreover,

$$\forall \wp \in \mathrm{Spec}(R) : \; ac \in \wp \Leftrightarrow a \in \wp.$$

Consequently, $\Omega(AC) = \Omega(A)$.

Suppose that $C_1 = A \cap R^\times \neq \varnothing$ and $C_2 = B \cap R^\times \neq \varnothing$. Since $AC_2 \cup BC_1 \subseteq AB$, we have $\max\{\mu^*(AC_2), \mu^*(BC_1)\} \leqslant \mu^*(AB)$. Property (v) yields $\mu^*(AC_2) = \mu^*(A)$ and $\mu^*(BC_1) = \mu^*(B)$. This completes the proof of (vi).

Let $\mathcal{S} \in \Omega(A)$ and $\mathcal{T} \in \Omega(B)$. Suppose that $ab \notin R^\times$ for some $a \in A$ and $b \in B$. Then $a \notin R^\times$ or $b \notin R^\times$. By the definition of ideal, we get therefore

$$ab \in \bigcup \mathcal{S} \cup \bigcup \mathcal{T}.$$

Consequently, $\mathcal{S} \cup \mathcal{T} \in \Omega(AB)$ and hence $\mu^*(AB) \leqslant \mu(\mathcal{S}) + \mu(\mathcal{T})$. Since $\mathcal{S}$ and $\mathcal{T}$ are arbitrarily chosen, it follows that $\mu^*(AB) \leqslant \mu^*(A) + \mu^*(B)$.

Assume that $A \cap R^\times = \varnothing$ and $C_2 = B \cap R^\times \neq \varnothing$. Then, by the definition of ideal, $\Omega(A) \subseteq \Omega(AB)$ which implies that $\mu^*(AB) \leqslant \mu^*(A)$. On the other hand, by (v), we have $\mu^*(A) = \mu^*(AC_2) \leqslant \mu^*(AB)$. Therefore, $\mu^*(AB) = \mu^*(A)$.

Finally, assume that $A \cap R^\times = \varnothing$ and $B \cap R^\times = \varnothing$. Then $\mu^*(AB) \leqslant \min\{\mu^*(A), \mu^*(B)\}$ (cf. the proof of property (viii)). Suppose now that $\mu^*(AB) < \min\{\mu^*(A), \mu^*(B)\}$. Then

$$\exists \mathcal{U} \in \Omega(AB) : \begin{cases} \mu^*(\bigcup \mathcal{U}) < \mu^*(A), \\ \mu^*(\bigcup \mathcal{U}) < \mu^*(B). \end{cases}$$

(Notice that $\mu^*(\bigcup \mathcal{U}) \leqslant \mu(\mathcal{U})$). Consequently,

$$\mu^*(A \setminus \bigcup \mathcal{U}) \geqslant \mu^*(A) - \mu^*(A \cap \bigcup \mathcal{U}) \geqslant \mu^*(A) - \mu^*(\bigcup \mathcal{U}) > 0$$

and, in the same way, $\mu^*(B \setminus \bigcup \mathcal{U}) > 0$. Since $AB \cap R^\times = \varnothing$ and therefore $AB \subseteq \bigcup \mathcal{U}$, we get

$$\exists\, a \in A \,\exists\, b \in B \,\exists\, \wp \in \mathcal{U} \subseteq \mathrm{Spec}(R) : \begin{cases} ab \in \wp, \\ a \notin \wp, \; b \notin \wp, \end{cases}$$

a contradiction. Property (ix) follows. $\square$

We will conclude the paper with an example illustrating the behavior of $\mu^*(AB)$ in the case where A and B both contain invertible elements.

Example 4. Let $\mu^* : 2^{\mathbb{Z}} \longrightarrow [0, +\infty]$ be the outer measure induced by the counting measure on $\mathrm{Max}(\mathbb{Z})$. If $A = \{1, 2, 3\}$, $B_1 = \{1, 2, 3, 5\}$, $B_2 = \{1, 2, 5, 7\}$ and $B_3 = \{1, 5, 7, 11\}$, then $\mu^*(A) = 2$, $\mu^*(B_1) = \mu^*(B_2) = \mu^*(B_3) = 3$, $\mu^*(AB_1) = 3$, $\mu^*(AB_2) = 4$ and $\mu^*(AB_3) = 5$.

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Contact information

D. Dudzik
Institute of Mathematics,
Pedagogical University of Cracow,
ul. Podchorążych 2,
30-084 Kraków, Poland
E-Mail(s): dariusz.dudzik@gmail.com

M. Skrzyński
Institute of Mathematics,
Cracow University of Technology,
ul. Warszawska 24,
31-155 Kraków, Poland
E-Mail(s): pfskrzyn@cyf-kr.edu.pl

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