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Factorization of elements in noncommutative rings, I

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Abstract. We extend the classical theory of factorization in noncommutative integral domains to the more general classes of right saturated rings and right cyclically complete rings. Our attention is focused, in particular, on the factorizations of right regular elements into left irreducible elements. We study the connections among such factorizations, right similar elements, cyclically presented modules of Euler characteristic 0 and their series of submodules. Finally, we consider factorizations as a product of idempotents.

1. Introduction

The study of factorizations has always given strong impulses to algebra in its history. Modern commutative algebra was practically born in 1847, when Gabriel Lamé announced at the Paris Académie des Sciences his solution to Fermat's Problem [18]. In his proof, he claimed that the ring $\mathbb{Z}[\zeta_p]$ is a UFD for every prime integer p, where ζ_p denotes a primitive p-th root of the unity. He didn't know that Ernst Kummer had proved four years before that $\mathbb{Z}[\zeta_{23}]$ is not a UFD. This mistake made apparent the necessity of a rigorous study of the subrings of $\mathbb{C}$ that are UFDs, that is, the necessity of a rigorous foundation of commutative algebra.

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Richard Dedekind, Gauss's last student, said to his collaborators around 1855 that the goal of number theory was to do for the general ring of integers of an algebraic number field what Kummer had done for the particular case of $\mathbb{Z}[\zeta_p]$. Dedekind completely succeeded in his programme in 1871, and one of his main result was that each proper ideal of the ring of integers of an algebraic number field can be factored in an essentially unique way as the product of prime ideals.

In the classical noncommutative setting, the study of factorizations of elements into irreducible elements in a noncommutative integral domain and the theory of noncommutative UFDs [5], started by Asano and Jacobson [16, pp. 33–36] for noncommutative PIDs, lead P. M. Cohn to the discovery of the theory of free ideal rings and the study of factorization in rings of noncommutative polynomials (see [6]).

The Auslander-Reiten theory and its almost split sequences were born studying the factorizations of morphisms into irreducible morphisms [1,3]. (Recall that a morphism h between indecomposable modules is *irreducible* if it is not invertible and, in every factorization $h = \beta\alpha$ of h, either α is left invertible or β is right invertible. Irreducible morphisms are the arrows of the Auslander-Reiten quiver.)

It is therefore natural to wonder if Cohn's factorization theory can be also extended to the case of any noncommutative ring R, not necessarily an integral domain. This is what we begin to do in this paper.

Over an arbitrary ring R, a great simplification takes place when we limit ourselves to the study of right regular elements $a \in R$ (i.e., such that $a \neq 0$ and a is not a left zerodivisor) and we restrict our study to the factorizations of the right regular element a as a product of right regular elements. In this paper we deal with this case, postponing the general case of a not necessarily a right regular element to a further paper.

The study of factorizations of elements in noncommutative rings can proceed in a number of different directions. Our attention is focused, in particular, on the factorizations of right regular elements into left irreducible elements. We study the connections among such factorizations, right similar elements, cyclically presented modules of Euler characteristic 0 and their series of submodules. We finally consider factorizations as a product of idempotents. Our main example is the case of factorization of elements in the ring $M_n(k)$ of $n \times n$ matrices with entries in a ring k. When k is a division ring, the right regular elements of the ring $M_n(k)$ are the invertible matrices. Thus the study of factorizations of regular elements in the ring $M_n(k)$ is the noncommutative analogue of the study of factorizations of invertible elements in a division ring k, for which all factorizations are trivial. As far as singular matrices are concerned, every such matrix is a product of idempotent matrices [17].

In the study of factorization in a noncommutative domain R, it is a natural step to replace an element $a \in R$ by the cyclically presented right R-module R/aR. When R is commutative, R/aR is isomorphic to R/bR if and only if a and b are associated. Also, when R is a domain, the right R-modules R/aR and R/bR are isomorphic if and only if the left R-modules R/Ra and R/Rb are isomorphic, so that the condition is right-left symmetric. We generalize this point of view to the case of an element a of an arbitrary ring R, considering, instead of a left factor $b \in R$ of a, the cyclic right R-module Rb/Ra.

In the continuation of this paper, we will essentially follow Cohn's "lattice method" [6, Section 5]. The factorizations of an element a of a ring R will be described by the partially ordered set of all principal right ideals of R between aR and R_R itself.

For any subset X of a ring R, the *left annihilator* $\operatorname{l.ann}_R(X)$ is the set of all $r \in R$ such that $rx = 0$ for every $x \in X$. Similarly for the *right annihilator* $\operatorname{r.ann}_R(X)$. We denote by $U(R)$ the group of all invertible elements of a ring R, that is, all the elements $c \in R$ for which there exists $d \in R$ with $cd = dc = 1$.

2. Right regular elements, left irreducible elements

Let R be an associative ring with an identity $1 \neq 0$. We consider factorizations of an element $a \in R$. If $xy = 1$, that is, if $x \in R$ is right invertible and $y \in R$ is left invertible, then a has always the trivial factorizations $a = (ax)y$ and $a = x(ya)$. If a, b are elements of R, we say that a is a *left divisor of* b (in R), and write $a|_l b$, if there exists an element $x \in R$ with $ax = b$. Similarly for right divisors, in which case we use the symbol $|_r$. Right invertible elements are left divisors of all elements of R and left invertible elements are right divisors of all elements of R. An element $u \in R$ is right invertible if and only if $u|_l 1$.

For any ring R, the relation $|_l$ is a preorder on R, that is, a relation on R that is reflexive and transitive. If $a, b \in R$, we will say that a and b are *left associates*, and write $a \sim_l b$, if $a|_l b$ and $b|_l a$. Clearly, $\sim_l$ is an equivalence relation on the set R and the preorder $|_l$ on R induces a partial order on the quotient set $R/{\sim_l}$. The equivalence class $[1]$ of $1 \in R$ in $R/{\sim_l}$ is the least element of $R/{\sim_l}$ and consists of all right invertible elements of R. The equivalence class $[0]$ of $0 \in R$ in $R/{\sim_l}$ is the greatest element of $R/{\sim_l}$ and consists only of 0.

One has that $a|_l b$ if and only $bR \subseteq aR$, that is, if and only if the principal right ideal generated by b is contained in the principal right ideal generated by a. Thus the quotient set $R/{\sim_l}$ is in one-to-one correspondence with the set $\mathcal{L}_p(R_R)$ of all principal right ideals of R. If $R/{\sim_l}$ is partially ordered by the partial order induced by $|_l$ as above and $\mathcal{L}_p(R_R)$ is partially ordered by set inclusion $\subseteq$, then the one-to-one correspondence $R/{\sim_l} \to \mathcal{L}_p(R_R)$ turns out to be an anti-isomorphism of partially ordered sets. In particular, two elements of R are left associates if and only if they generate the same principal right ideal of R.

Every element $a \in R$ always has, among its left divisors, all right invertible elements of R and all left associates with a. These are called the *trivial left divisors* of a. If these are all the left divisors of a, $a \neq 0$ and a is not right invertible, then a is said to be a *left irreducible* element of R.

Lemma 1. *The following conditions are equivalent for an element $a \in R$:*

(1) *a is a left irreducible element.*

(2) *The right ideal aR is nonzero and is a maximal element in the set $\mathcal{L}_p(R_R) \setminus \{0, R\}$ of all nonzero proper principal right ideals of R.*

Recall that an element a of a ring R is a *left zerodivisor* if it is nonzero and there exists $b \in R$, $b \neq 0$ such that $ab = 0$, and is *right regular* if it is $\neq 0$ and is not a left zerodivisor. Thus $a \in R$ is right regular if and only if $ax = 0$ implies $x = 0$ for every $x \in R$. For any element $a \in R$, left multiplication by a is a right R-module homomorphism $\lambda_a \colon R_R \to R_R$, which is a monomorphism if and only if a is right regular. Notice that a right ideal I of a ring R is isomorphic to R_R if and only if it is a principal right ideal of R generated by a right regular element of R. Similarly, we define right zerodivisors and left regular elements. An element is a *zerodivisor* if it is either a right zerodivisor or a left zerodivisor. An element is *regular* if it is both right regular and left regular.

Lemma 2. *A right regular element is right invertible if and only if it is invertible.*

Proof. Let $a \in R$ be a right regular right invertible element. Then there exists $b \in R$ such that $ab = 1$. Then $0 = (ab - 1)a = a(ba - 1)$. Now a right regular implies $ba = 1$, so a is also left invertible. □

Lemma 3. *Let R be a ring and S the set of all right regular elements of R. Then:*

(1) *If $a, b \in S$, then $ab \in S$.*
(2) *If $a, b \in R$ and $ab \in S$, then $b \in S$.*
(3) *Suppose that every principal right ideal of R generated by a right regular element of R is essential in R_R. Then $a, b \in R$ and $ab \in S$ imply that $a \in S$ and $b \in S$.*
(4) *If $a \in S$ and $I \subseteq J$ are right ideals of R, then J/I and aJ/aI are isomorphic right R-modules.*

Proof. The proofs of (1) and (2) are elementary.

For (3), assume that every principal right ideal of R generated by a right regular element of R is essential in R_R, and that $a, b \in R$ and $ab \in S$. Then $b \in S$ by (2). Let us show that $\mathrm{r.ann}_R(a) \cap bR = 0$. If $bx \in R$ and $bx \in \mathrm{r.ann}_R(a)$, then $abx = 0$, so that $x = 0$. It follows that $\mathrm{r.ann}_R(a) \cap bR = 0$. Since bR is essential in R_R, we get that $\mathrm{r.ann}_R(a) = 0$. Equivalently, $a \in S$.

Part (4) follows immediately from the fact that left multiplication by $a \in S$ is a right R-module isomorphism $R_R \to aR$. □

We will say that a ring R is a *right saturated* ring if every left divisor of a right regular element is right regular, that is, if for every $a, b \in R$, $ab \in S$ implies $a \in S$.

Examples 1. (1) Every (not necessarily commutative) integral domain is a right saturated ring. Every commutative ring is a (right) saturated ring.

(2) By Lemma 3(3), if R is a ring and every principal right ideal of R generated by a right regular element of R is essential in R_R, then R is a right saturated ring.

(3) Recall that a ring R is *directly finite* (or *Dedekind finite*) if every right invertible element is invertible (equivalently, if every left invertible element is invertible). Every right saturated ring is directly finite.

(4) Let R be a ring and suppose that the right R-module R_R has finite Goldie dimension. Then every principal right ideal of R generated by a right regular element of R is essential in R_R [21, Lemma II.2.3], so that R is a right saturated ring (Example (2)).

(5) If R is a right nonsingular ring and every principal right ideal of R generated by a right regular element of R is essential in R_R, then every right regular element of R is left regular. In fact, let R be a right nonsingular ring and suppose that every principal right ideal of R generated by a right regular element of R is essential in R_R. Let $a \in R$ be a right regular element. Suppose that $xa = 0$ for some $x \in R$. Then left multiplication

$\lambda_x\colon R_R \to R_R$ by x induces a right R-module morphism $R/aR \to R_R$. Now aR is essential, so R/aR is singular [14, Proposition 1.20(b)], and R_R is nonsingular. Thus the right R-module morphism $R/aR \to R_R$ is the zero morphism. Hence $x = 0$, so that a is left regular.

The preorder $|_l$ on R induces a preorder on the set S of all right regular elements of R. The set S contains all left invertible elements.

Lemma 4. *Let a, b be two right regular elements of R. Then $a \sim_l b$ if and only if $b = au$ for some invertible element $u \in R$.*

Proof. If $a \sim_l b$, there exist $u, v \in R$ such that $au = b$ and $bv = a$. Thus $a(1 - uv) = 0$. Since a is right regular, we get that u is right invertible and v is left invertible. By symmetry, since b is also right regular, we get that v is right invertible and u is left invertible. Thus u is invertible. The converse is clear. □

For the case where a, b are not right regular elements, we need a further definition. Let a be an element of a ring R and I a subgroup of the additive group R. We say that u is *right invertible modulo I* if there exists $v \in R$ such that $uv - 1 \in I$. Similarly, u is *left invertible modulo I* if there exists $v \in R$ such that $vu - 1 \in I$.

Notice that if u is right invertible modulo I and I is a right ideal, then all the elements of the coset $u + I$ are also right invertible modulo I.

Lemma 5. *Let a, b be elements of a ring R. Then $a \sim_l b$ if and only if $b = au$ for some element $u \in R$ right invertible modulo* $\mathrm{r.ann}_R(a)$.

Proof. If $a \sim_l b$, there exist $u, v \in R$ such that $au = b$ and $bv = a$. Thus $a(1 - uv) = 0$, so that u is right invertible modulo $\mathrm{r.ann}_R(a)$. Conversely, suppose $b = au$ for element $u \in R$ right invertible modulo $\mathrm{r.ann}_R(a)$. Then $bR = auR \subseteq aR$ and there exists $v \in R$ with $a(1 - uv) = 0$. Thus $aR = auvR \subseteq auR = bR$, so $aR = bR$ and $a \sim_l b$. □

Lemma 6. *Let R be a right saturated ring. The following conditions are equivalent for a right regular element $a \in R$ that is not right invertible:*

(1) *a is left irreducible.*

(2) *For every factorization $a = bc$ of the element a $(b, c \in R)$, either b is invertible or c is invertible.*

Proof. (1) $\Longrightarrow$ (2) If a is right regular left irreducible and $a = bc$ is a factorization of a, then b is either right invertible or left associate with a. Moreover, b and c are right regular (Lemma 3(3)). If b is right invertible, then it is invertible by Lemma 2. If b is left associate with a, then there exists an invertible element $d \in R$ such that $b = ad$ (Lemma 4), so that $a = bc = adc$, hence $1 = dc$. Thus c is invertible.

(2) $\implies$ (1) Since a is right regular, we must have $a \neq 0$. Suppose that (2) holds. In order to prove that a is left irreducible, we must show that every left divisor b of a is trivial. Now, if b is a left divisor of a, there is a factorization $a = bc$. By (2), either b is invertible, hence it is a trivial divisor of a, or c is invertible, in which case b and a are left associates. $\square$

The preorder $|_l$ on the set S of all right regular elements induces a partial order on the quotient set $S/\!\sim_l$. The equivalence class $[1_R]$ of 1_R in $S/\!\sim_l$ is the least element of $S/\!\sim_l$. By Lemma 4, the equivalence class $[1_R]$ of 1_R in $S/\!\sim_l$ consists of all invertible elements of R. Thus we can consider the subset $S^* := S \setminus [1_R]$ consisting of all right regular elements of R that are not invertible in R.

The maximal elements in $S^*/\!\sim_l$ are exactly the equivalence classes modulo $\sim_l$ of the left irreducible elements of R modulo $\sim_l$. Two elements $a, b \in S^*$ are equivalent modulo $\sim_l$ if and only if $a = bu$ for some invertible element $u \in R$ (Lemma 4).

3. Cyclically presented modules, right similar elements

We now pass from right regular elements of a ring R to cyclically presented right modules over R. The proof of the following lemma is elementary.

Lemma 7. *The following conditions are equivalent for a right R-module M_R:*

(1) *There exists a right regular element $a \in R$ such that $M_R \cong R_R/aR$.*

(2) *There exists a short exact sequence of the form $0 \to R_R \to R_R \to M_R \to 0$.*

We call the modules satisfying the equivalent conditions of Lemma 7 *cyclically presented modules of Euler characteristic* 0. When R is an IBN ring, so that the Euler characteristic of a module with a finite free resolution is well defined, the modules satisfying the conditions of Lemma 7 are exactly the cyclic modules with a finite free resolution of length 1 of Euler characteristic $\chi(M_R)$ equal to 0.

Obviously, cyclically presented modules of Euler characteristic 0 are cyclic modules. Take any other epimorphism $R_R \to M_R$ of such a module

M_R, that is, fix any other generator of M_R. Then the kernel of the epimorphism is the annihilator of the new generator of M_R, and is a right ideal I of R.

Lemma 8. *Let R be a ring, $a \in R$ a right regular element, and I a right ideal of R such that $R/aR \cong R/I$. Then:*

(1) $R_R \oplus R_R \cong R_R \oplus I$.

(2) *I is a projective right ideal of R that can be generated by two elements.*

(3) *I is isomorphic to the kernel of an epimorphism $R_R \oplus R_R \to R_R$.*

(4) *There exist elements $b, c \in R$ such that $bR = cR$ and I_R is isomorphic to the submodule of $R_R \oplus R_R$ consisting of all pairs $(x, y) \in R_R \oplus R_R$ such that $bx = cy$.*

(5) *There exists $b \in R$ such that $I = (aR : b) := \{ x \in R \mid bx \in aR \}$ and $aR + bR = R$. Moreover, $r.ann_R(b) \subseteq I$, and the two right R-modules $I/r.ann_R(b)$ and $aR \cap bR$ are isomorphic. Conversely, if $a, b \in R$, a is right regular and $aR + bR = R$, then $R/(aR : b)$ is a cyclically presented right R-module of Euler characteristic* 0.

(Statement (5) appears in Cohn [7, Proposition 3.2.1]).

Proof. From $R/aR \cong R/I$, it follows that $R_R \oplus R_R \cong R_R \oplus I$ by Schanuel's Lemma [1, p. 214]. This proves (1). Statements (2) and (3) follow immediately from (1).

(4) By (3), I is isomorphic to the kernel of an epimorphism $R_R \oplus R_R \to R_R$. Every such an epimorphism is of the form $(x, y) \mapsto bx + cy$ with $bR + cR = R$. We can substitute $-c$ for c, getting that every epimorphism $R_R \oplus R_R \to R_R$ is of the form $(x, y) \mapsto bx - cy$, where b, c are elements of R and $bR + cR = R$. The kernel of this epimorphism consists of all pairs $(x, y) \in R_R \oplus R_R$ such that $bx = cy$.

(5) Let $\varphi \colon R/I \to R/aR$ be an isomorphism and assume that $\varphi(1 + I) = b + aR$. Surjectivity of φ gives $aR + bR = R$. Injectivity gives that $I = \{ x \in R \mid bx \in aR \}$. This proves the first part of (5). For the second part, notice that $r.ann_R(b) \subseteq (aR : b) = I$. Finally, from $I = \{ x \in R \mid bx \in aR \}$, it follows that the mapping $I \to aR \cap bR$, $x \to bx$, is a well defined epimorphism with kernel $r.ann_R(b)$, so that $I/r.ann_R(b) \cong aR \cap bR$.

For the converse, let a, b be elements of R, with a right regular and $aR + bR = R$. Then left multiplication by b induces an isomorphism $R/(aR : b) \to R/aR$. □

The reason why we have introduced cyclically presented R-modules in the study of factorizations of elements of R is the following. Suppose that $a = a_1a_2\dots a_n$ is a factorization in R, where $a, a_1, \dots, a_n \in R$. Then $aR = a_1a_2\dots a_nR \subseteq a_1a_2\dots a_{n-1}R \subseteq \dots \subseteq a_1R \subseteq R$ is a series of principal right ideals of R, so that $aR/aR = a_1a_2\dots a_nR/aR \subseteq a_1a_2\dots a_{n-1}R/aR \subseteq \dots \subseteq a_1R/aR \subseteq R/aR$ is a series of submodules of the cyclically presented right R-module R/aR. Similarly, $Ra/Ra = Ra_1a_2\dots a_n/Ra \subseteq Ra_2a_3\dots a_n/Ra \subseteq \dots \subseteq Ra_n/Ra \subseteq R/Ra$ is a series of submodules of the cyclically presented left R-module R/Ra.

We would like the submodules $a_1a_2\dots a_iR/aR$ in these series and the factor modules

$$(a_1a_2\dots a_{i-1}R/aR)/(a_1a_2\dots a_iR/aR) \cong a_1a_2\dots a_{i-1}R/a_1a_2\dots a_iR$$

to be cyclically presented modules as well. Unluckily, the situation is the following:

Lemma 9. *Let x, y be elements of a ring R. Then the right R-module xR/xyR is cyclically presented if and only if there exist $z, w \in R$ such that $xyR + xwR = xR$ and, for every $t \in R$, one has $xwt \in xyR$ if and only if $t \in zR$.*

Proof. The right R-module xR/xyR is cyclically presented if and only if there exists $z \in R$ with $R/zR \cong xR/xyR$, i.e., if and only if there exist $z, w \in R$ such that left multiplication $\lambda_{xw}\colon R_R \to xR/xyR$ is an epimorphism with kernel zR. Now it is easy to conclude. □

In the previous lemma, the situation is much simpler when x is right regular. In this case, it suffices to take $z := y$ and $w := 1$, and $xR/xyR \cong R/yR$ always turns out to be cyclically presented.

Therefore, we will suppose that all the elements $a_1, a_2, \dots, a_n \in R$ are right regular, in which case $a = a_1a_2\dots a_n$ is also right regular (Lemma 3(1)). Under this hypothesis, the right R-modules $aR/aR = a_1a_2\dots a_nR/aR \subseteq a_1a_2\dots a_{n-1}R/aR \subseteq \dots \subseteq a_1R/aR \subset R/aR$ of the series are cyclically presented of Euler characteristic 0, and so are the factors $a_1a_2\dots a_{i-1}R/a_1a_2\dots a_iR$ of the series.

Two elements a, b of an arbitrary ring R are said to be *right similar* if the cyclically presented right R-modules $R/aR, R/bR$ are isomorphic. If two elements a, b of R are left associates, then they generate the same principal right ideals aR, bR of R, hence they are clearly right similar.

4. Factorizations of right regular elements and right cyclically complete rings

Let R be a ring and $U(R)$ its group of units, that is, the group of all elements with a (two-sided) inverse. The direct product $U(R)^{n-1}$ of $n-1$ copies of $U(R)$ acts on the set of factorizations $F_n(a)$ of length $n \geqslant 1$ of an element a into right regular elements. Here a is an element of R, the set of factorizations of length n of a into right regular elements is the set of all n-tuples $F_n(a) := \{\, (a_1, a_2, \dots, a_n) \mid a_i \in S,\ a_1 a_2 \dots a_n = a \,\}$. In particular, $F_n(a)$ is empty if a is not right regular. The direct product $U(R)^{n-1}$ acts on the set $F_n(a)$ via $(u_1, u_2, \dots, u_{n-1}) \cdot (a_1, a_2, \dots, a_n) = (a_1 u_1, u_1^{-1} a_2 u_2, u_2^{-1} a_3 u_3, \dots, u_{n-1}^{-1} a_n)$. We say that two factorizations of a of length n, m respectively, are *equivalent* if $n = m$ and the two factorizations are in the same orbit under the action of $U(R)^{n-1}$.

In Section 3, we have associated with every factorization $(a_1, \dots, a_n)$ of length n of an element $a \in R$ into right regular elements, the series

$$aR/aR = a_1 \dots a_n R/aR \subseteq a_1 \dots a_{n-1} R/aR \subseteq \dots \subseteq a_1 R/aR \subseteq R/aR$$

of cyclically presented submodules of Euler characteristic 0 of the cyclically presented right R-module R/aR. In this series, the factors of the series are cyclically presented modules of Euler characteristic 0 as well.

Proposition 1. *Let a be a right regular element of a ring R. Two factorizations*

$$(a_1, a_2, \dots, a_n) \quad \text{and} \quad (b_1, b_2, \dots, b_m)$$

of a into right regular elements $a_1, \dots a_n, b_1, \dots, b_m$ are equivalent if and only if their associated series of submodules of the cyclically presented right R-module R/aR are equal.

(Two series $0 = A_0 \leqslant A_1 \leqslant A_2 \leqslant \dots \leqslant A_n = R/aR$, $0 = B_0 \leqslant B_1 \leqslant B_2 \leqslant \dots \leqslant B_m = R/aR$ of R/aR are equal if $n = m$ and $A_i = B_i$ for every $i = 0, 1, 2, \dots, n$.)

Proof. The two series

$$aR/aR = a_1 \dots a_n R/aR \subseteq a_1 \dots a_{n-1} R/aR \subseteq \dots \subseteq a_1 R/aR \subseteq R/aR$$

and

$$aR/aR = b_1 \dots b_m R/aR \subseteq b_1 \dots b_{n-1} R/aR \subseteq \dots \subseteq b_1 R/aR \subseteq R/aR$$

are equal if and only if $n = m$ and $a_1a_2 \dots a_iR = b_1b_2 \dots b_iR$ for every $i = 1, 2, \dots, n-1$, that is, if and only if $a_1a_2 \dots a_i \sim_l b_1b_2 \dots b_i$ for every $i = 1, 2, \dots, n-1$.

Suppose $a_1a_2 \dots a_i \sim_l b_1b_2 \dots b_i$ for every $i = 1, 2, \dots, n-1$. Then there exist invertible elements $u_1, u_2, \dots, u_{n-1} \in R$ such that $b_1b_2 \dots b_i = a_1a_2 \dots a_iu_i$ (Lemma 4). Set $u_0 = u_n = 1$. Let us prove by induction on i that $b_i = u_{i-1}^{-1}a_iu_i$ for every $i = 1, 2, \dots, n$. The case $i = 1$ is obvious. Suppose that (*) $b_i = u_{i-1}^{-1}a_iu_i$ for every $i = 1, 2, \dots, j$ with $j < n$. Let us show that $b_{j+1} = u_j^{-1}a_{j+1}u_{j+1}$. Replacing in $b_1b_2 \dots b_{j+1} = a_1a_2 \dots a_{j+1}u_{j+1}$ the equalities (*), which we assume to hold, we get that

$$u_0^{-1}a_1u_1u_1^{-1}a_2u_2 \dots u_{j-1}^{-1}a_ju_jb_{j+1} = a_1a_2 \dots a_{j+1}u_{j+1},$$

so that $u_jb_{j+1} = a_{j+1}u_{j+1}$. It follows that $b_{j+1} = u_j^{-1}a_{j+1}u_{j+1}$, as desired. This concludes the proof by induction. It is now clear that the two factorizations $(a_1, a_2, \dots, a_n)$, $(b_1, b_2, \dots, b_n)$ are in the same orbit.

Conversely, suppose that the factorizations $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$ are equivalent. Then there exist invertible elements $u_1, u_2, \dots, u_{n-1} \in R$ such that $b_i = u_{i-1}^{-1}a_iu_i$ for every $i = 1, 2, \dots, n$, where $u_0 = u_n = 1$. It follows that $b_1b_2 \dots b_iR = u_0^{-1}a_1u_1u_1^{-1}a_2u_2 \dots u_{i-1}^{-1}a_iu_iR = a_1a_2 \dots a_iR$, and the two series of cyclically presented submodules of R/aR coincide. $\square$

Remark 1. We have defined two factorizations of a into right regular elements to be equivalent if one differs from the other by insertion of units. Now suppose that the ring R is not directly finite, that is, there exist noninvertible elements $c, d \in R$ with $cd = 1$. Suppose that $(a_1, a_2, \dots, a_n)$ is a factorization of an element $a \in R$ into right regular elements. Then $a = a_1a_2 \dots a_n = (a_1c)(da_2)a_3 \dots a_n$, but in this second factorization the element a_1c is not right regular, otherwise c would be right regular (Lemma 3(2)). But c is right invertible, hence it would be invertible by Lemma 2, contradiction. Thus, it is natural to define two factorizations of a into right regular elements to be equivalent if one differs from the other by insertion of units, because insertion by elements invertible only on one side is not sufficient.

In the following, it will be often convenient to restrict our attention to the rings for which the projective right ideals I of R, studied in Lemma 8, are all necessarily principal right ideals generated by a right regular element. We will call them right cyclically complete rings. Thus a ring R is *right cyclically complete* if for every $a, b \in R$ with a right regular and $aR + bR = R$, the right ideal $(aR : b)$ is a principal right ideal generated by a right regular element. Equivalently, a ring R is right cyclically complete if and only if, for every right regular element $a \in R$, the right annihilator of any generator of R/aR is a principal right ideal generated by a right regular element.

Examples 2. (1) If R is a projective free (that is, a ring for which every projective ideal is free) IBN ring, then R is right cyclically complete. In fact, if $a, b \in R$, a is right regular and $aR + bR = R$, then $R/(aR : b)$ is a cyclically presented right R-module of Euler characteristic 0 (Lemma 8), so that $R/(aR : b) \cong R/rR$ for some right regular element $r \in R$. By Schanuel's Lemma, $R \oplus (aR : b) \cong R \oplus rR \cong R_R^2$, so that $(aR : b)$ is a finitely generated projective right ideal. But R is projective free, hence $(aR : b) \cong R_R^n$ for some nonnegative integer n. The isomorphism $R \oplus (aR : b) \cong R_R^2$ and R IBN imply that $(aR : b) \cong R_R$. Therefore $(aR : b)$ is a principal right ideal generated by a right regular element.

(2) Assume that R_R cancels from direct sums, that is, if A_R, B_R are arbitrary right R-modules and $R_R \oplus A_R \cong R_R \oplus B_R$, then $A_R \cong B_R$. Then R is right cyclically complete. In fact, we can argue as in Example (1), as follows. If $a, b \in R$, a is right regular and $aR + bR = R$, then $R/(aR : b) \cong R/rR$ for some right regular element $r \in R$, so that $R \oplus (aR : b) \cong R_R^2$. But R_R cancels from direct sums, so that $(aR : b) \cong R_R$ is a principal right ideal generated by a right regular element.

(3) If R is any ring of stable range 1, then R is right cyclically complete. In fact, if R is a ring of stable range 1, then R_R cancels from direct sums ([10, Theorem 2] or [11, Theorem 4.5]), and we conclude by Example (2).

(4) If R is any semilocal ring, then R is right cyclically complete. In fact, semilocal rings are of stable range 1 ([4] or [11, Theorem 4.4]). Thus this Example (4) is a special case of Example (3). In particular, local rings are right cyclically complete.

(5) Commutative Dedekind rings are right cyclically complete rings. To see this, we can argue as in Examples (1) and (2). Let R be a Dedekind ring and let a, b be elements of R, with a right regular and $aR + bR = R$. Then $R \oplus (aR : b) \cong R_R^2$. By [20, Lemma 6.18], we get that $(aR : b) \cong R_R$ is a principal right ideal generated by a right regular element.

(6) Right Bézout domains, that is, the (not necessarily commutative) integral domains in which every finitely generated right ideal is principal, are right cyclically complete rings. To prove this, let R be a right Bézout domain and a, b be elements of R, with a right regular and $aR + bR = R$. Then, as in the previous examples, $R \oplus (aR : b) \cong R_R^2$, so that $(aR : b)$ is a finitely generated right ideal. But R is right Bézout, so that $(aR : b)$ is a principal right ideal of R. If $(aR : b) = 0$, then $R_R \cong R_R^2$, so that

R has a nontrivial idempotent. But R is an integral domain, and this is a contradiction, which proves that $(aR : b)$ is a nonzero principal right ideal of R. As R is an integral domain, the nonzero principal right ideal $(aR : b)$ of R is generated by a right regular element.

(7) Unit-regular rings are of stable range 1, and are therefore right cyclically complete (Example (3)).

(8) Every commutative ring is right cyclically complete. In fact, if $a, b \in R$, a is right regular and $aR + bR = R$, then $R/(aR : b) \cong R/aR$, so that $R/(aR : b)$ and R/aR have the same annihilators, that is, $(aR : b) = aR$. The same proof shows that every right duo ring is right cyclically complete.

(9) We will now give an example of a ring R that is not right cyclically complete. Let V_k be a vector space of countable dimension over a field k, and let $v_0, v_1, v_2, \dots$ be a basis of V_k. Set $R := \mathrm{End}(V_k)$. Notice that an element $a \in R$ is a right regular element of R if and only if it is an injective endomorphism of V_k. For instance, the element $a \in R$ such that $a\colon v_i \mapsto v_{2i}$ for every $i \geqslant 0$ is a right regular element of R. Apply the exact functor $\mathrm{Hom}({}_RV_k, -)\colon \text{Mod-}k \to \text{Mod-}R$ to the split exact sequence $0 \to V_k \xrightarrow{a} V_k \to V_k \to 0$, getting a split exact sequence $0 \to R_R \xrightarrow{\lambda_a} R_R \to R_R \to 0$. Then $R/aR \cong R_R$. To show that R is not right cyclically complete, it suffices to prove that R_R has a generator whose right annihilator is not a principal right ideal generated by a right regular element. As a generator of R_R, consider the element $x \in R$ such that $x\colon v_i \mapsto v_{i-1}$ for every $i \geqslant 1$ and $x\colon v_0 \mapsto 0$. The endomorphism x of V_k is surjective, hence split surjective, i.e., right invertible, so that it is a generator of R_R. Its right annihilator is $\mathrm{r.\,ann}_R(x) = \{\, y \in R \mid xy = 0 \,\} = \{\, y \in R \mid \mathrm{im}(y) \subseteq \ker(x) = v_0k \,\} = \mathrm{Hom}(V_k, v_0k)$. This is a projective cyclic right ideal of R, which contains no right regular elements of R, because no element of $\mathrm{Hom}(V_k, v_0k)$ is an injective mapping. This proves that R is not a right cyclically complete ring.

Lemma 10. *A ring R is right cyclically complete if and only if, for every right ideal I of R with R/I a cyclically presented module of Euler characteristic 0, I is a principal right ideal generated by a right regular element.*

Lemma 11. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of right modules over a right cyclically complete ring R. If A and C are cyclically presented modules of Euler characteristic 0 and B is cyclic, then B is a cyclically presented module of Euler characteristic 0.*

Proof. Since B is cyclic, we have that $B \cong R/I$ for some right ideal I of R. Then there exists a right ideal J of R with $I \subseteq J$, $C \cong R/J$ and $A \cong J/I$. Now $C \cong R/J$ is a cyclically presented module of Euler characteristic 0 and R is cyclically complete, so that $J = rR$ is the principal right ideal generated by a right regular element r of R. Thus left multiplication $\lambda_r \colon R_R \to R_R$ by r is a monomorphism, which induces an isomorphism $\lambda_r \colon R_R \to rR = J$. If K is the inverse image of $I \subseteq J$ via this isomorphism, then K is a right ideal of R and $rK = I$. Then $A \cong J/I = rR/rK \cong R/K$. But A is a cyclically presented module of Euler characteristic 0 and R is cyclically complete, so $K = sR$ is the principal right ideal generated by a right regular element s of R. Then $I = rK = rsR$, so that $B \cong R/I = R/rsR$ is a cyclically presented module of Euler characteristic 0. □

The following Proposition is motivated by [7, Proposition 0.6.1].

Proposition 2. *Let R be a right cyclically complete ring. Let*

$$0 = M_0 \subseteq M_1 \subseteq . \cdots \subseteq M_n = M_R \tag{1}$$

be a finite series of submodules of a right R-module M_R. Assume that all the factors M/M_i $(i = 0, 1, 2, \ldots, n)$ are cyclically presented R-modules of Euler characteristic 0. Let $a \in R$ be any right regular element such that $M \cong R_R/aR$. Then there exists a factorization $a = a_1 a_2 \ldots a_n$ of a with $a_1, a_2, \ldots, a_n \in R$ right regular elements, such that $M_j/M_i \cong R/a_{n-j+1}a_{n-j+2} \ldots a_{n-i-1}a_{n-i-2}R$ for every $0 \leqslant i < j \leqslant n$. In particular, all the modules M_j/M_i $(0 \leqslant i < j \leqslant n)$ are cyclically presented R-module M_R of Euler characteristic 0.

Proof. Let $a \in R$ be a right regular element such that $M \cong R_R/aR$, so that there exists an epimorphism $\varphi\colon R_R \to M_R$ with $\ker \varphi = aR$. By the Correspondence Theorem applied to the epimorphism φ, there is a one-to-one correspondence between the set $\mathcal{L}$ of all right ideals of R containing aR and the set $\mathcal{L}'$ of all submodules of M_R. The correspondence is given by $I_R \mapsto \varphi(I_R)$ for every $I_R \in \mathcal{L}$ and its inverse is given by $N_R \mapsto \varphi^{-1}(N_R)$ for every $N_R \in \mathcal{L}'$. Thus if $K_i := \varphi^{-1}(M_i)$, then the finite series of right ideals of R corresponding to the series (1) is the series $aR = \ker \varphi = K_0 \subseteq K_1 \subseteq . \cdots \subseteq K_n = R_R$. If $\varphi_i := \varphi|_{K_i} \colon K_i \to M_i$ denotes the restriction of φ, then φ_i is an epimorphism with kernel aR, so that $M_i \cong K_i/aR$ for every $i = 0, 1, 2, \ldots, n$. If $0 \leqslant i \leqslant j \leqslant n$, the composite mapping of $\varphi_j \colon K_j \to M_j$ and the canonical projection

$M_j \to M_j/M_i$ is an epimorphism $K_j \to M_j/M_i$ with kernel K_i, so that

$$K_j/K_i \cong M_j/M_i. \tag{2}$$

For $j = n$, we get in particular that $R_R/K_i \cong M/M_i$ for every $i = 0, 1, \dots, n$. Thus, the modules R_R/K_i are cyclically presented of Euler characteristic 0, so that the right ideals K_i are of our type, so that they are isomorphic to R_R because R is right cyclically complete. Thus we have right regular elements $c_0 := a, c_1, c_2 \dots, c_{n-1}, c_n := 1$ of R such that $K_i = c_i R$ for every $i = 0, 1, \dots, n$. Now $K_0 \subseteq K_1 \subseteq \dots \subseteq K_n = R_R$, so that $c_i = c_{i+1} b_{i+1}$ for suitable elements $b_1, b_2 \dots, b_n \in R$. These elements b_i are right regular by Lemma 3(2). Then, for $0 \leqslant i < j \leqslant n$, we get that $c_i = c_{i+1} b_{i+1} = c_{i+2} b_{i+2} b_{i+1} = c_{i+3} b_{i+3} b_{i+2} b_{i+1} = \dots = c_j b_j b_{j-1} \dots b_{i+2} b_{i+1}$. In particular, for $i = 0$ and $j = n$, we have that $a = c_0 = c_n b_n b_{n-1} \dots b_2 b_1 = b_n b_{n-1} \dots b_2 b_1$.

Finally, from (2), we get that

$$\begin{aligned} M_j/M_i \cong K_j/K_i = c_j R/c_i R &= c_j R/c_j b_j b_{j-1} \dots b_{i+2} b_{i+1} R \\ &\cong R/b_j b_{j-1} \dots b_{i+2} b_{i+1} R \end{aligned}$$

by Lemma 3(4). Now change the notation, orderly substituting the sequence $a_1, a_2, \dots, a_n$ for the sequence $b_n, b_{n-1}, \dots, b_1$. □

5. Correspondence between factorizations and series of submodules

Let R be a ring and S be the set of all right regular elements of R. Let $\mathcal{S} := \dot{\bigcup}_{n \geqslant 1} S^n$ be the disjoint union of the sets S^n of all n-tuples of elements of S, i.e., the cartesian product of n copies of S. The set $\mathcal{S}$ can be seen as the set of all factorizations of finite length into right regular elements. Let $\mathcal{M}_R$ be the class of all series of finite length of cyclically presented right R-modules of Euler characteristic 0:
$\mathcal{M}_R := \{ (M_1, M_2, \dots, M_n) \mid n \geqslant 1,\ 0 = M_0 \leqslant M_1 \leqslant M_2 \leqslant \dots \leqslant M_n$, and the modules M_i and M_i/M_{i-1} are cyclically presented right R-modules of Euler characteristic 0 for every $i = 1, 2, \dots, n\}$.
Two series $(M_1, M_2, \dots, M_n), (M'_1, M'_2, \dots, M'_m) \in \mathcal{M}_R$ of finite length are *isomorphic* if $n = m$ and there is a right R-module isomorphism $\varphi\colon M_n \to M'_m$ such that $\varphi(M_i) = M'_i$ for every $i = 1, 2, \dots, n-1$. In this case, we will write $(M_1, M_2, \dots, M_n) \cong (M'_1, M'_2, \dots, M'_m)$, so that $\cong$

turns out to be an equivalence relation on the class $\mathcal{M}_R$, and the quotient class $\mathcal{M}_R/\cong$ has a set of representatives modulo $\cong$.

In the first paragraph of Section 4, we have considered, for any $a \in R$, the set $F_n(a) := \{ (a_1, a_2, \dots, a_n) \mid a_i \in S,\ a_1 a_2 \dots a_n = a \}$ of factorizations of length n of a into right regular elements. For every fixed right regular element $a \in R$, let $\mathcal{S}(a)$ be the disjoint union of the sets $F_n(a)$, $n \geqslant 1$, so that $\mathcal{S}(a)$ is a subset of $\mathcal{S}$. Similarly, we can consider the set

$$\mathcal{M}_R(R/aR) := \{ (M_1, M_2, \dots, M_n) \in \mathcal{M}_R \mid M_n = R/aR \}.$$

It is the set of all finite series of submodules of R/aR. There is a mapping

$$f \colon \mathcal{S}(a) \to M_R(R/aR)$$

defined by

$$f(a_1, a_2, \dots, a_n) = (a_1 a_2 \dots a_{n-1} R/aR, \dots, a_1 R/aR, R/aR)$$

for every $(a_1, a_2, \dots, a_n) \in \mathcal{S}(a)$.

Proposition 3. *Let a be a right regular element of a ring R.*

(1) *If R is right cyclically complete, then the mapping $f \colon \mathcal{S}(a) \to M_R(R/aR)$ is surjective.*

(2) *Two factorizations in $\mathcal{S}(a)$ are mapped to the same element of $M_R(R/aR)$ via f if and only if they are equivalent factorizations of a.*

Proof. (1) Let $(M_1, M_2, \dots, M_n)$ be an element of $\mathcal{M}_R(R/aR)$. Then $M := M_n = R/aR$, and all the modules M/M_i are cyclically presented modules of Euler characteristic 0 by Lemma 11. Thus we can apply Proposition 2. Following its proof, we can take as $\varphi \colon R_R \to M$ the canonical projection, so that $M_i = K_i/aR$, $K_i = c_i R$ for suitable right regular elements c_i, with $c_i = c_{i+1} a_{n-i}$, where $a = a_1 a_2 \dots a_n$ and $a_1, \dots, a_n \in R$ are right regular elements of R. Thus $f(a_1, \dots, a_n) = (M_1, \dots, M_n)$.

(2) is Proposition 1. □

If a and b are right similar right regular elements of a right cyclically complete ring R, there is an isomorphism $f \colon R/aR \to R/bR$, which is defined by left multiplication by an element $c \in R$. This bijection $f = \lambda_c$ induces a bijection $\mathcal{M}_R(R/aR) \to \mathcal{M}_R(R/bR)$, in which every series of submodules of R/aR is mapped to an isomorphic series of submodules of R/bR. Since R is right cyclically complete, this bijection $\mathcal{M}_R(R/aR) \to$

$\mathcal{M}_R(R/bR)$ induces a bijection $\mathcal{S}(a)/\cong\ \to \mathcal{S}(b)/\cong$ (Proposition 3), where, for two factorizations $(a_1, a_2, \dots, a_n)$ and $(a'_1, a'_2, \dots, a'_n)$ in $\mathcal{S}(a)$, we write $(a_1, a_2, \dots, a_n) \cong (a'_1, a'_2, \dots, a'_n)$ if the two factorizations are equivalent according to the definition given in the first paragraph of Section 4. Fix $(a_1, a_2, \dots, a_n) \in \mathcal{S}(a)$. The factorization $a = a_1 a_2 \dots a_n$ of a corresponds to the series of submodules

$$0 = \frac{aR}{aR} \leqslant \frac{a_1 \dots a_{n-1}R}{aR} \leqslant \frac{a_1 \dots a_{n-2}R}{aR} \leqslant \dots \leqslant \frac{a_1 R}{aR} \leqslant \frac{R}{aR}$$

of R/aR, which is mapped isomorphically by λ_c to the series

$$0 = \frac{caR + bR}{bR} \leqslant \frac{ca_1 \dots a_{n-1}R + bR}{bR} \leqslant \frac{ca_1 \dots a_{n-2}R + bR}{bR} \leqslant \dots$$
$$\dots \leqslant \frac{ca_1 R + bR}{bR} \leqslant \frac{cR + bR}{bR} = \frac{R}{bR}$$

of submodules of R/bR. As far as the factors are concerned, it follows that $R/(ca_1 \dots a_i R + bR) \cong R/a_1 \dots a_i R$. Set $d_0 := 1$ and $d_n := b$. For $i = 1, \dots n-1$, let d_i be a right regular element such that $ca_1 \dots a_i R + bR = d_i R$. Such a d_i exists because R is right cyclically complete. Then $a_1 \dots a_i R \subseteq a_1 \dots a_{i-1} R$, so that $0 \neq d_i R \subseteq d_{i-1} R$. Hence there exists $b_i \in R$ with $d_i = d_{i-1} b_i$, and b_i is right regular, $i = 1, 2, \dots, n$. Then $b = d_n = d_{n-1} b_n = d_{n-2} b_{n-1} b_n = d_{n-3} b_{n-2} b_{n-1} b_n = \dots = d_0 b_1 b_2 \dots b_n = b_1 b_2 \dots b_n$ is the factorization of b corresponding to the factorization $a = a_1 a_2 \dots a_n$ of a, up to equivalence of factorizations.

The set $\mathcal{S}(a)$ can obviously be partially ordered by the refinement relation $\leqslant$. The partially ordered set $(\mathcal{S}(a), \leqslant)$ has no maximal elements, because

$$(a_1, a_2, \dots, a_n) \leqslant (a_1, a_2, \dots, a_i, 1, a_{i+1}, \dots, a_n).$$

If we want to relate maximal elements of $\mathcal{S}(a)$ with respect to the refinement relation $\leqslant$ and factorizations of a as a product of left irreducible right invertible elements, we must exclude from the factorizations into right regular elements left invertible factors (recall that left invertible elements are always right regular). Thus, let T be the set of all elements of R that are right regular but not left invertible, so that $T \subseteq S$. Set $\mathcal{T}(a) := \{\, (a_1, a_2, \dots, a_n) \mid n \geqslant 1,\ a_i \in T,\ a_1 a_2 \dots a_n = a \,\} \subseteq \mathcal{S}(a)$. The set $\mathcal{T}(a)$ is also partially ordered by the refinement relation $\leqslant$. We leave to the reader the proof of the following easy lemma.

Lemma 12. *Let a be an element of a right saturated ring. An element $(a_1, a_2, \ldots, a_n) \in \mathcal{T}(a)$ is a maximal element of $\mathcal{T}(a)$ with respect to the refinement relation $\leqslant$ if and only if a_i is left irreducible for every $i = 1, 2, \ldots, n$.*

In particular, if R is a right saturated ring and $a \in R$, then $\mathcal{T}(a)$ has a maximal element if and only if a has a factorization as a product of finitely many right regular left irreducible elements that are not left invertible. Such a factorization is very rarely unique up to equivalence, that is, $\mathcal{T}(a)$ modulo equivalence has very rarely a unique maximal element. For instance, if R is a commutative ring, $(a_1, \ldots, a_n)$ is a maximal element of $\mathcal{T}(a)$ and $\sigma \in S_n$ is a permutation, then $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ is a maximal element of $\mathcal{T}(a)$ as well.

Recall that a nonzero element a of an integral domain R is *rigid* [8, p. 43] if $a = bc = b'c'$ implies $b \in b'R$ or $b' \in bR$. More generally, this definition can be extended to any element a of any ring R. Thus we say that an element a of a ring R is *right rigid* if the principal right ideals between aR and R form a chain under inclusion. When the ring R is a right cyclically complete ring and $a \in R$ is right regular, then the principal right ideals between aR and R form a *finite* chain under inclusion if and only if a has a unique factorization into right regular left irreducible elements up to equivalence of factorizations.

6. Idempotents

Particularly interesting, in study of factorizations in rings with zero-divisors and the uniqueness of such factorizations, is the case of idempotents. In fact, consider the example of matrices. In [9], J. A. Erdos showed that singular matrices over commutative fields factorize as a product of idempotent matrices. This result was later extended to matrices over euclidean rings and division rings [17]. Fountain [13] considered the case of commutative Hermite rings, using techniques of semigroup theory. Inspired by this paper, Ruitenburg [19] studied the case of noncommutative Hermite rings. Fountain and Ruitenberg determined a clear connection between product decompositions of singular matrices into idempotents and product decompositions of invertible matrices into elementary ones.

If an element $x \in R$ is a product $x = e_1 \ldots e_n$ of idempotents $e_i \in R$, then x is annihilated both by left multiplication by $1 - e_1$ and by right multiplication by $1 - e_n$ [12, Section 2], so that, whenever $x \in R$ is a product of finitely many idempotents, we must necessarily have that either $x = 1$, or both $\operatorname{l.ann}(x) \neq 0$ and $\operatorname{r.ann}(x) \neq 0$.

If $e \in R$ is an idempotent and we consider the factorizations $e = xy$ of e as a product of two elements $x, y \in R$, then e always has the trivial factorizations $e = u(ve)$, $e = (eu)v$ and $e = (eu)(e)$, where $u, v \in R$ are any two elements with $uv = 1$. The composition factors of these factorizations $e = xy$, i.e., the composition factors R/xR and xR/xyR of the series $eR = xyR \subseteq xR \subseteq R$ are 0 and R/eR in all these three types of factorizations. We will say that e is an *irreducible idempotent* if these are the only factorizations of e as a product of two elements x and y in R and $e \neq 1$. For example:

Proposition 4. *Let k be a division ring, V_k a right vector space over k of finite dimension $n \geqslant 2$, $R := \mathrm{End}(V_k)$ its endomorphism ring and $\varphi \in R$ an idempotent endomorphism. Then φ is is an irreducible idempotent of R if and only if* $\dim(\ker \varphi) = 1$.

Proof. If $\dim(\ker \varphi) = 0$, then φ is the identity, so that it is not an irreducible idempotent.

If $\dim(\ker \varphi) \geqslant 2$, then there is a nontrivial direct-sum decomposition $\ker \varphi = A \oplus B$, so that $V_k = A \oplus B \oplus \mathrm{im}(\varphi)$. Then φ is the composite mapping $\varphi = \psi\omega$ of the endomorphism ω of V_k that is zero on A and the identity on $B \oplus \mathrm{im}(\varphi)$ and the endomorphism ψ that is zero on B and the identity on $A \oplus \mathrm{im}(\varphi)$.

Now suppose $\dim(\ker \varphi) = 1$ and that φ decomposes as $\varphi = \psi\omega$. Then $\ker \omega \subseteq \ker \varphi$ and $\mathrm{im}\, \varphi \subseteq \mathrm{im}\, \psi$, so that $\dim(\ker \omega) \leqslant 1$ and $\dim(\mathrm{im}\, \psi) \geqslant n-1$. If $\dim(\ker \omega) = 0$, then ω is an automorphism and the factorization $\varphi = \psi\omega$ is trivial. If $\dim(\mathrm{im}\, \psi) = n$, then ψ is an automorphism and the factorization $\varphi = \psi\omega$ is also trivial. Hence it remains to consider the factorizations $\varphi = \psi\omega$ with $\dim(\ker \omega) = 1$ and $\dim(\mathrm{im}\, \psi) = n - 1$. Now $\dim(\ker \omega) = \dim(\ker \varphi) = 1$ implies that $\ker \omega = \ker(\psi\omega)$, so that $\ker \psi \cap \mathrm{im}\, \omega = 0$. As $\dim(\mathrm{im}\, \omega) = n - 1$ and $\dim(\ker \psi) = 1$, we get that $V_k = \ker \psi \oplus \mathrm{im}\, \omega$. In matrix notation, it follows that $\omega \colon V_k = \ker \varphi \oplus \mathrm{im}\, \varphi \to V_k = \ker \psi \oplus \mathrm{im}\, \omega$ is of the form $\omega = \left(\begin{smallmatrix} 0 & 0 \\ 0 & f \end{smallmatrix}\right)$, where $f \colon \mathrm{im}\, \varphi \to \mathrm{im}\, \omega$ is an isomorphism, and $\psi \colon V_k = \ker \psi \oplus \mathrm{im}\, \omega \to V_k = \ker \varphi \oplus \mathrm{im}\, \varphi$ is of the form $\psi = \left(\begin{smallmatrix} 0 & 0 \\ 0 & g \end{smallmatrix}\right)$, where g is the inverse of f. Let $u \colon V_k = \ker \varphi \oplus \mathrm{im}\, \varphi \to V_k = \ker \psi \oplus \mathrm{im}\, \omega$ be the isomorphism $u = \left(\begin{smallmatrix} h & 0 \\ 0 & f \end{smallmatrix}\right)$, where h is any isomorphism between the one-dimensional vector spaces $\ker \varphi$ and $\ker \psi$. Then $\omega = u\varphi$ and $\psi = \varphi u^{-1}$, so that the factorization $\varphi = \psi\omega$ is also trivial in this case. □

Laffey proved that every singular $n \times n$ matrix with entries in a division ring k can be expressed as a product of idempotents over k [17].

Clearly, every idempotent ε in $R := \mathrm{End}(V_k)$ is a product of t irreducible idempotents, where $t = \dim(\ker \varepsilon)$. It follows that every noninvertible element of R is a product of finitely many irreducible idempotents. As far as uniqueness of such a decomposition is concerned, assume that $\varphi \in R$ can be written as $\varphi = \varepsilon_1 \dots \varepsilon_m = e_1 \dots e_t$, where the ε_i and the e_j are all irreducible idempotents. To determine the factors $\varepsilon_1 \dots \varepsilon_{i-1}R/\varepsilon_1 \dots \varepsilon_i R$ and $e_1 \dots e_{j-1}R/e_1 \dots e_j R$ of the series of principal right ideals corresponding to the two factorizations of φ, we need the following lemma.

Lemma 13. *Let k be a division ring, V_k a right vector space over k of finite dimension n, $R := \mathrm{End}(V_k)$ its endomorphism ring, $g \in R$ an endomorphism of V_k and $e \in R$ an irreducible idempotent. Then exactly one of the following two cases occurs:*

(a) $\dim(\ker(ge)) = \dim(\ker g)$ *and* $gR/geR = 0$; *or*

(b) $\dim(\ker(ge)) = \dim(\ker g) + 1$ *and gR/geR is a simple right R-module.*

Notice that R is a simple artinian ring, so that all its simple right R-modules are isomorphic.

Proof. We have that $\dim(\mathrm{im}(ge)) = \dim(g(eV)) = \dim(g(eV + \ker g))$. Now $eV + \ker g$ contains eV, which has dimension $n-1$ and is contained in V, that has dimension n. Hence either $eV + \ker g$ has dimension $n-1$, and in this case $eV + \ker g = eV$ and $\ker g \subseteq eV$, or $eV + \ker g$ has dimension n, in which case $eV + \ker g = V$ and $\ker g \not\subseteq eV$. Let us distinguish these two cases:

(a) Case $eV + \ker g = V$ and $\ker g \not\subseteq eV$. In this case, $\dim(\mathrm{im}(ge)) = \dim(g(eV + \ker g)) = \dim(g(V)) = \dim(\mathrm{im}(g))$, so that $\dim(\ker(ge)) = \dim(\ker g)$.

(b) Case $\ker g \subseteq eV$. Since g induces a lattice isomorphism between the lattice of all subspaces of V_k that contain $\ker g$ and the lattice of all subspaces of $\mathrm{im}(g)$ and for every subspace W of V_k with $W \supseteq \ker g$ we have $\dim g(W) = \dim(W) - \dim(\ker g)$, it follows that $\dim(\mathrm{im}(ge)) = \dim(g(eV)) = \dim(eV) - \dim(\ker g) = n - 1 - \dim(\ker g)$, so that $\dim(\ker(ge)) = \dim(\ker g) + 1$.

Now in both cases, left multiplication by g is a right R-module epimorphism $R_R \to gR$, which induces a right R-module epimorphism $R/eR \to gR/geR$. But R/eR is a simple right R-module, so that either $gR/geR = 0$ or gR/geR is simple. Hence, in order to conclude the proof of the lemma, it suffices to show that $gR/geR = 0$ if and only if $\dim(\ker(ge)) = \dim(\ker g)$.

Now if $gR/geR = 0$, then $gR = geR$, so that $g \in geR$. Hence there exists $h \in R$ with $g = geh$. Then $\mathrm{im}(g) = \mathrm{im}(geh) \subseteq \mathrm{im}(ge) \subseteq \mathrm{im}(g)$. Thus $\mathrm{im}(ge) = \mathrm{im}(g)$, from which $\dim(\mathrm{im}(ge)) = \dim(\mathrm{im}(g))$, and $\dim(\ker(ge)) = \dim(\ker g)$.

Conversely, suppose $\dim(\ker(ge)) = \dim(\ker g)$. Then $\dim(\mathrm{im}(ge)) = \dim(\mathrm{im}(g))$ and $\mathrm{im}(ge) \subseteq \mathrm{im}(g)$, so $\mathrm{im}(ge) = \mathrm{im}(g)$. It follows that $eV + \ker g = V$. Let x be an element in $\ker g$ not in eV, so that $eV \oplus xk = V$. With respect to this direct-sum decomposition the matrices of e and g are, respectively,

$$g = \begin{pmatrix} g_{11} & 0 \\ g_{21} & 0 \end{pmatrix} \qquad \text{and} \qquad g = \begin{pmatrix} 1 & e_{12} \\ 0 & e_{22} \end{pmatrix}.$$

Let $h \in R$ be the endomorphism of V with matrix

$$h = \begin{pmatrix} 1 & -e_{12} \\ 0 & 1 \end{pmatrix}.$$

It is easily seen that $g = geh$, so that $gR = geR$, and $gR/geR = 0$. □

Proposition 5. *Let k be a division ring, V_k a right vector space over k of finite dimension n, $R := \mathrm{End}(V_k)$ its endomorphism ring and $f = e_1 \dots e_m$ a product decomposition of an endomorphism $f \in R$ into irreducible idempotents $e_1, \dots, e_m$ of R. Let t be the dimension of the kernel of f. Then t of the factors $e_1 \dots e_{i-1}R/e_1 \dots e_iR$ are simple R-modules, and the other $m - t$ are zero.*

Proof. Induction of m. If $m = 1$, then the kernel of $f = e_1$ is 1, so that $t = m = 1$, and the unique factor R/e_1R is a simple right R-module.

Suppose $m > 1$ and that the proposition is true for endomorphisms that are products of $m - 1$ irreducible idempotents. Suppose that the dimension of the kernel of $f = e_1 \dots e_m$ is t. Set $g := e_1 \dots e_{m-1}$, so that the m factors $e_1 \dots e_{i-1}R/e_1 \dots e_iR$ relative to the factorization $f = e_1 \dots e_m$ are the $m-1$ factors relative to the factorization $g := e_1 \dots e_{m-1}$ plus the module gR/ge_mR. By the previous lemma, we have one of the following two cases:

(a) $\dim(\ker(ge)) = \dim(\ker g)$ and $gR/geR = 0$. In this case, $t = \dim(\ker g)$, so that, by the inductive hypothesis, t factors relative to the factorization $g := e_1 \dots e_{m-1}$ are simple, and all the other $m - t$ factors relative to factorization $f = e_1 \dots e_m$ are zero.

(b) $\dim(\ker(ge)) = \dim(\ker g) + 1$ and gR/geR is a simple right R-module. In this case, $\dim(\ker g) = t - 1$, so that by the inductive hypothesis $t-1$ factors relative to the factorization $g := e_1 \dots e_{m-1}$ are simple and the other $m-t$ are zero. The other factor relative to factorization $f = e_1 \dots e_m$ is the simple module gR/geR. □

Thus if $\varphi = \varepsilon_1 \dots \varepsilon_m = e_1 \dots e_t$ are two factorizations with the idempotents ε_i and e_j irreducible idempotents, then the series of principal right ideals associated to the two factorizations have the same number of simple factors and all the other factors are zero. This remark justifies the following definitions.

Let R be any ring and $a = a_1 \dots a_n$ a factorization in R, where $a, a_1, \dots, a_n \in R$. We call

$$aR = a_1 \dots a_n R \subseteq a_1 \dots a_{n-1} R \subseteq \dots \subseteq a_1 R \subseteq R$$

the series of principal right ideals of R associated to the factorization and

$$R/a_1R,\ a_1R/a_1a_2R,\ \dots,\ a_1a_2 \dots a_{n-1}R/a_1a_2 \dots a_nR$$

the *factors* of the series. We call *right length* of the factorization the number of nonzero factors. Similarly, we can consider the series

$$Ra = Ra_1a_2 \dots a_n \subseteq Ra_2a_3 \dots a_n \subseteq \dots \subseteq Ra_n \subseteq R$$

of principal left ideals of R associated to the factorization, and define its factors $Ra_i \dots a_n / Ra_{i-1} \dots a_n$ and the *left length* of the factorization. In our last example of this first paper, we show that the right length and the left length of a factorization can be different.

Example 1. Let R be the ring $\left(\begin{smallmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & \mathbb{R} \end{smallmatrix}\right)$. For every element $r = \left(\begin{smallmatrix} q & 0 \\ \alpha & \beta \end{smallmatrix}\right)$, the principal right ideal generated by r is

$$\begin{aligned} rR &= \begin{pmatrix} q & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R + \begin{pmatrix} q & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R = \\ &= \begin{pmatrix} q & 0 \\ \alpha & 0 \end{pmatrix} R + \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} R = \begin{pmatrix} q & 0 \\ \alpha & 0 \end{pmatrix} \mathbb{Q} + \beta \begin{pmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix} \end{aligned}$$

so that rR is the improper right ideal R if and only if $q\beta \neq 0$, and rR is maximal among the proper principal right ideals if and only if either $\beta \neq 0$ and $q = 0$, or $\beta = 0$ and $q \neq 0$. Thus the elements r with $q\beta \neq 0$ are right invertible, and those with $q = 0$ or $\beta = 0$ but not both q and β equal to zero are the left irreducible elements of R. For the elements r with $q = \beta = 0$ and $\alpha \neq 0$, the series $rR \subset \left(\begin{smallmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{smallmatrix}\right) \subset R$ is a series of principal right ideals that cannot be properly refined, so that the proper left divisors of r are exactly the generators of the principal right ideal $\left(\begin{smallmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{smallmatrix}\right)$, that is, the elements $s = \left(\begin{smallmatrix} 0 & 0 \\ \alpha' & \beta' \end{smallmatrix}\right)$ with $\beta' \neq 0$. The factorizations of r in this case are

$$r = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha' & \beta' \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \tag{3}$$

where $a \in \mathbb{Q}$ and $b \in \mathbb{R}$ are any two numbers with $\alpha' a + \beta' b = \alpha$. The factorization (3) has right length 2 and its right factors are the simple module $R/\left(\begin{smallmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{smallmatrix}\right)$ and the nonsimple module $\left(\begin{smallmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{smallmatrix}\right) / \left(\begin{smallmatrix} 0 & 0 \\ \alpha\mathbb{Q} & 0 \end{smallmatrix}\right)$.

Let us compute the left length of the factorization (3). It is easily seen that, for $r = \left(\begin{smallmatrix} q & 0 \\ \alpha & \beta \end{smallmatrix}\right)$, the principal left ideal generated by r is

$$Rr = q \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix}.$$

The series of left principal ideals associated to the factorization (3) of $r = \left(\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right)$ is

$$Rr \subseteq R \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \subseteq R.$$

Now $Rr = R\left(\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{smallmatrix}\right)$ and $R\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) = a\left(\begin{smallmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & 0 \end{smallmatrix}\right) + \mathbb{R}\left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right)$, so that we have two cases according to when $a = 0$ or $a \neq 0$:

(a) If $a = 0$, then $b \neq 0$, so that $R\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{smallmatrix}\right) = Rr$, and the factorization (3) of r has left length 1.

(b) If $a \neq 0$, then $R\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & 0 \end{smallmatrix}\right) \supset Rr$, so that the factorization (3) of r has left length 2.

In our next paper, we will treat the factorizations of arbitrary elements of a ring, including the case of zerodivisors.

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