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The Thue–Morse substitutions and self-similar groups and algebras

L. Bartholdi, J. M. R. Caballero, and T. Ahmed

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Abstract. We introduce self-similar algebras and groups closely related to the Thue–Morse sequence, and begin their investigation by describing a character on them, the "spread" character.

1. Introduction

Fix an alphabet $X = \{x_0, \dots, x_{q-1}\}$. The *Thue–Morse* substitution is the free monoid morphism $\theta\colon X^* \to X^*$ given by

$$\theta(x_i) = x_i x_{i+1} \dots x_{q-1} x_0 \dots x_{i-1},$$

and the Thue–Morse word $W_q \in X^\omega$ is the limit of all words $\theta^n(x_0)$. For example, if $q = 2$ then $\theta(x_0) = x_0x_1$ and $\theta(x_1) = x_1x_0$ and $W_2 = x_0x_1x_1x_0x_1x_0x_0x_1\dots$ is the classical, ubiquitous Thue–Morse sequence, see [1, 6].

We construct some self-similar algebraic objects — groups and associative algebras — and report on a curious connection between them and the Thue–Morse substitution.

Fix an alphabet $A = \{a_0, \dots, a_{q-1}\}$. Recall that a *self-similar group* is a group G endowed with a group homomorphism $\phi\colon G \to G \wr_A \mathfrak{S}_A$, the *decomposition*: every element of G may be written, via ϕ, as an

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A-tuple of elements of G decorating a permutation of A. Likewise, a *self-similar algebra* is an associative algebra $\mathscr{A}$ endowed with an algebra homomorphism $\phi\colon \mathscr{A} \to M_q(\mathscr{A})$ also called the *decomposition*: every element of $\mathscr{A}$ may be written as an $A \times A$ matrix with entries in $\mathscr{A}$. For more details see [3, 9].

We insist that self-similarity is an attribute of a group or algebra, and not a property: it is legal to consider for G or $\mathscr{A}$ a free group (respectively algebra), and then the decomposition ϕ may be defined at will on G or $\mathscr{A}$'s generators. There will then exist a maximal quotient (called the *injective quotient*) of G or $\mathscr{A}$ on which ϕ induces an injective decomposition. This is the approach we follow in defining our self-similar group.

Consider the free group $F = \langle x_0, \dots, x_{q-1}\rangle$, the alphabet $A = \mathbb{Z}/q$, and define $\phi\colon F \to F \wr_A \mathfrak{S}_A$ by

$$\phi(x_0) = \langle\!\langle x_0, \dots, x_{q-1}\rangle\!\rangle(j \mapsto j+1)$$

and

$$\phi(x_i) = \langle\!\langle 1, \dots, 1\rangle\!\rangle(j \mapsto j+1) \quad \text{for all } i \geqslant 1.$$

Here and below we denote by $\langle\!\langle g_0, \dots, g_{q-1}\rangle\!\rangle\pi$ the element of $F \wr \mathfrak{S}_A$ with decorations g_i on the permutation π. We denote by G_q the injective quotient of F, with self-similarity structure still written ϕ. Note that it is a proper quotient; for example, the image of x_1 has order q in G_q.

There is a standard construction of a self-similar algebra from a self-similar group, by mapping decorated permutations to monomial matrices. Fix a commutative ring $\Bbbk$, consider the free associative (tensor) algebra $T = \Bbbk\langle x_0, \dots, x_{q-1}\rangle$, and define $\phi\colon T \to M_q(T)$ by

$$\phi(x_0) = \begin{pmatrix} 0 & \cdots & 0 & x_{q-1} \\ x_0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & x_{q-2} & 0 \end{pmatrix}, \qquad \phi(x_i) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

We denote by $\mathscr{A}_q$ the injective quotient of T, with self-similarity structure still written ϕ. Our main result is a description of a natural character, the "spread", on $\mathscr{A}_q$, see §3.1; roughly speaking, it measures the number of non-zeros in matrix rows or column.

Theorem A. *The "spread" character on $\mathscr{A}_q$ has image $\mathbb{Z}[1/q] \cap \mathbb{R}_+$.*

The proof crucially uses the fact that the decomposition of G_q admits a partial splitting defined using the Thue–Morse endomorphism θ; the same holds for $\mathscr{A}_q$. This is embodied in the following Lemma, proved in the next section:

Lemma 1.1. *For all $w \in F$ we have $\phi(\theta(w)) = \langle\!\langle w, \gamma(w), \dots, \gamma^{q-1}(w) \rangle\!\rangle$, where $\gamma\colon F \to F$ is the automorphism permuting cyclically the generators $x_i \mapsto x_{i+1 \bmod q}$.*

We conclude with some variants of the construction, and in particular relations to iterated monodromy groups of rational functions in one complex variable.

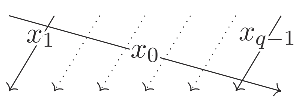
2. The groups

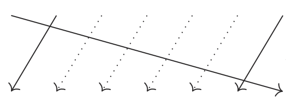
As sketched in the introduction, a *self-similar group* is a group G endowed with a homomorphism $\phi\colon G \to G \wr_A \mathfrak{S}_A$, the *decomposition*. The range of ϕ is the permutational wreath product of G with A; its elements may be represented as permutations of A with a decoration in G on each strand. We write $\phi(g) = \langle\!\langle g_0, \dots, g_{q-1} \rangle\!\rangle \pi$.

Starting from the free group $F = \langle x_0, \dots, x_{q-1} \rangle$ and the alphabet $A = \{a_0, \dots, a_{q-1}\}$, we define $\phi\colon F \to F \wr_A \mathfrak{S}_A$ by

$$\phi(x_0) = \langle\!\langle x_0, \dots, x_{q-1} \rangle\!\rangle (j \mapsto j+1), \qquad \phi(x_i) = \langle\!\langle 1, \dots, 1 \rangle\!\rangle (j \mapsto j+1),$$

turning F into a self-similar group. Write $K_0 = 1$ and $K_{n+1} = \phi^{-1}(K_n^A)$; these form then an ascending sequence of normal subgroups of F, and $G \coloneqq F / \bigcup_n K_n$ is again a self-similar group, but now on which the map induced by ϕ is injective. We christen the group G just constructed the *qth Thue–Morse group*. The decompositions may be written, using permutations, as

$\phi(x_0) =$ 

, $\phi(x_i) =$ 
.

Note that in the injective quotient G_q the generators $x_1, \dots, x_{q-1}$ coincide and have order q. We thus have a presentation

$$G_q = \langle x_0, x_1 \mid x_1^q, [(x_0 x_1^{-1})^q, (x_1^{-1} x_0)^q], \dots \rangle,$$

where producing an explicit presentation of the group is beyond our current goals, but could be done following the lines of [2].

It is straightforward to prove Lemma 1.1: for generator x_i, we have $\phi(\theta(x_i)) = \langle\!\langle x_i, x_{i+1}, \dots, x_{i-1} \rangle\!\rangle = \langle\!\langle x_i, \gamma(x_i), \dots, \gamma^{q-1}(x_i) \rangle\!\rangle$, so

$$\phi(\theta(w)) = \langle\!\langle w, \gamma(w), \dots, \gamma^{q-1}(w) \rangle\!\rangle \text{ for all } w \in F.$$

A self-similar group G is called *contracting* if there exists a finite subset $N \subseteq G$ with the following property: for every $g \in G$ there exists $n \in \mathbb{N}$, such that if one iterates the decomposition at least n times on g then all entries belong to N. The minimal admissible such N is called the *nucleus*.

Lemma 2.1. *The Thue–Morse group G_q is contracting with $N = \{x_0^{\pm 1}, x_1^{\pm 1}\}$.*

Proof. It suffices to check contraction on words in N^2, and this is direct. □

Let G be a self-similar group, and consider an element $g \in G$. Iterating n times the map ϕ on g yields a permutation of A^n decorated by $\#A^n$ elements. The element g is called *bounded* if only a bounded number of these decorations are non-trivial, independently of n. The group G itself is called *bounded* if all its elements are bounded; by an easy argument, it suffices to check this property on generators of G. It is classical [5] that if G is bounded and finitely generated then it is contracting.

2.1. Characters

Recall that a character $\chi\colon G \to \mathbb{C}$ on a group is a function that is normalized ($\chi(1) = 1$), central ($\chi(gh) = \chi(hg)$ for all $g, h \in G$) and positive semidefinite ($\sum_{i,j=1}^n \chi(g_i g_j^{-1})\lambda_i \overline{\lambda_j} \geqslant 0$ for all $g_i \in G, \lambda_i \in \mathbb{C}$). A model example of character are the "fixed points": if G acts on a measure space (X, μ), set $\chi(g) = \mu(\{x \in X : g(x) = x\})$. By the Gelfand-Naimark-Segal construction, every character may be written as $\chi(g) = \langle \xi, \pi(g)\xi \rangle$ for some unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ and some unit vector $\xi \in \mathcal{H}$.

Let now G be self-similar, with decomposition $\phi\colon G \to G \wr_A \mathfrak{S}_A$. A character χ will be called *self-similar* if there exists a positive semidefinite kernel $k(\cdot, \cdot) \in \mathbb{C}^{A \times A}$ such that

$$(\#A)\chi(g) = \sum_{a \in A} k(a, \pi(a))\chi(g_a) \text{ whenever } \phi(g) = \langle\!\langle g_a \rangle\!\rangle \pi.$$

We also note the following easy property of characters.

Lemma 2.2. *If G is a contracting, self-similar group, then every self-similar character on G is determined by its values on the nucleus. If moreover G is bounded and finitely generated, then every self-similar character on G is determined by the kernel k.*

Proof. For each element $g \in G$, write the linear relation imposed on $\chi(g)$ by self-similarity of the character χ. Substituting sufficiently many times, $\chi(g)$ may be expressed in terms of $\chi \restriction N$.

If G is bounded, then furthermore the nucleus may be decomposed as $N = N_0 \sqcup N_1$ with the property that for every $g \in N_0$, all decorations of g are eventually trivial, while if $g \in N_1$, then a single decoration g' of g is in N_1 and all the others are in N_0. Clearly $\chi \restriction N_0$ is determined by k, while for $g \in N_1$ we obtain a linear relation $\chi(g) = \chi(g')/\#A + C_g$ with C_g depending only on k; this linear system is non-degenerate, yielding a unique solution for $\chi \restriction N_1$. □

Let us check that G_q is bounded. For the generators $x_1, \dots, x_{q-1}$ this is obvious, since all their decorations are trivial starting from level $n = 1$. Then x_0 has a single decoration which is x_0 itself on top of the $x_1, \dots, x_{q-1}$, so in fact for all $n \in \mathbb{N}$ there are at most q non-trivial decorations in the n-fold decomposition of x_0.

Note that every self-similar group acts on a $\#A$-regular rooted tree, as follows. The group fixes the empty sequence ε. To determine the action of $g \in G$ on a word $v = v_1 v_2 \dots v_n$, compute $\phi(g) = \langle\!\langle g_a \rangle\!\rangle \pi$; then define recursively $g(v) = \pi(v_1)\, g_{v_1}(v_2 \dots v_n)$.

This action extends naturally to the boundary of the rooted tree, which is identified with the space of infinite sequences A^∞. This space comes naturally equipped with the Bernoulli measure μ, assigning mass $1/\#A$ to each of the elementary cylinders $C_{i,a} = \{v \in A^\infty : v_i = a\}$, and G acts by measure-preserving transformations. It is easy to see that the constant kernel ($k(a, b) = 1/\#A$ for all a, b) induces the trivial self-similar character $\chi(g) \equiv 1$, and that the identity kernel ($k(a, b) = \delta_{a=b}$) induces the fixed-point self-similar character $\chi(g) = \mu\{v \in A^\infty : g(v) = v\}$.

Recall that every self-similar group G admits an *injective quotient*, on which the decomposition ϕ induces an injection $G \hookrightarrow G \wr_A \mathfrak{S}_A$. The group G also admits a *faithful quotient*, defined as the quotient of G by the kernel of the natural map to $\mathfrak{S}_{A^\infty}$ given by the action defined above; it is the largest self-similar quotient of G that acts faithfully on A^∞. Clearly the faithful quotient is a quotient of the injective quotient, but they need not coincide.

It is easy to see that, for G_q, the injective and faithful quotients coincide, using the contraction property and the fact that the action on A^∞ is faithful on the nucleus.

3. The algebras

We fix once and for all a commutative ring $\Bbbk$. We are particularly interested in the example $\Bbbk = \mathbb{F}_q$.

As in the case of groups, we start by considering the free associative (tensor) algebra $T = \Bbbk\langle x_0, \dots, x_{q-1}\rangle$, and define $\phi\colon T \to M_q(T)$ by

$$\phi(x_0) = \begin{pmatrix} 0 & \cdots & 0 & x_{q-1} \\ x_0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & x_{q-2} & 0 \end{pmatrix}, \qquad \phi(x_i) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Write $J_0 = 0$ and $J_{n+1} = \phi^{-1}(M_q(J_n))$; these form then an ascending sequence of ideals in T, and $\mathscr{A}_q \coloneqq T/\bigcup_n J_n$ is a self-similar algebra, on which the map induced by ϕ is injective.

The construction of $\mathscr{A}_q$ from G_q should be transparent: both algebraic objects have the same generating set, and if $\phi(g) = \langle\!\langle g_a \rangle\!\rangle \pi$ in G_q, then the decomposition $\phi(g)$ in $\mathscr{A}_q$ is a monomial matrix with permutation π and non-zero entries g_a.

It may be convenient to extend $\mathscr{A}_q$ into a *-algebra, namely an algebra $\mathscr{B}_q$ equipped with an anti-involution $x \mapsto x^*$. This may easily be done by extending T to $\Bbbk F$, the group ring of F, and extending the decomposition by

$$\phi(x_0^{-1}) = \begin{pmatrix} 0 & x_0^{-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & x_{q-2}^{-1} \\ x_{q-1}^{-1} & 0 & \cdots & 0 \end{pmatrix}, \qquad \phi(x_i^{-1}) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

We then have a natural group homomorphism $G_q \to \mathscr{B}_q^\times$ given by $x_i \mapsto x_i$ on the generating set. In particular, $\mathscr{B}_q$ is a quotient of the group ring $\Bbbk G_q$. A presentation of $\mathscr{B}_q$ begins as

$$\mathscr{B}_q = \langle x_0^{\pm 1}, x_1 \mid x_1^q - 1, (x_0 x_1^{-1})^q - 1)(x_1^{-1} x_0)^q - 1), \dots \rangle;$$

we see in particular that $\mathscr{B}_q$ is a proper quotient of $\Bbbk G_q$, since in $\Bbbk G_q$ the elements $(x_0 x_1^{-1})^q - 1$ and $(x_1^{-1} x_0)^q - 1$ commute while in $\mathscr{B}_q$ their product vanishes, being a product of two matrices each with a single non-zero entry. As in the case of groups, a presentation of $\mathscr{A}_q$ and of $\mathscr{B}_q$ could be computed following the techniques in [3], but this is beyond our purposes.

We naturally extend the Thue–Morse endomorphism θ to T; and note then, similarly to Lemma 1.1, the easy

Lemma 3.1. *We have*

$$\phi(\theta(w)) = \begin{pmatrix} w & 0 & \cdots & 0 \\ 0 & \gamma(w) & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma^{q-1}(w), \end{pmatrix}$$

where γ is the endomorphism of T permuting cyclically the generators $x_i \mapsto x_{i+1 \bmod q}$. □

A self-similar algebra $\mathscr{A}$ is called *contracting* if there exists a finite-rank submodule $N \leqslant \mathscr{A}$ with the following property: for every $s \in \mathscr{A}$ there exists $n \in \mathbb{N}$, such that iterating the decomposition at least n times on s gives a matrix with all entries in N. The minimal admissible such N is called the *nucleus*.

Lemma 3.2. *The Thue–Morse algebras $\mathscr{A}_q$ and $\mathscr{B}_q$ are contracting, with respective nuclei $\Bbbk\{x_0, x_1\}$ and $\Bbbk\{x_0^{\pm 1}, x_1^{\pm 1}\}$.*

Proof. It suffices to check contraction on monomials in N^2, and this is direct. □

Let $\mathscr{A}$ be a self-similar algebra, and consider an element $x \in \mathscr{A}$. Iterating n times the map ϕ on x yields an $A^n \times A^n$-matrix with entries in $\mathscr{A}$. The element x is called *row-bounded* if only a bounded number of entries are non-trivial on each row of that matrix, independently of n and the row; and is called *column-bounded* if the same property holds for columns. The algebra $\mathscr{A}$ itself is called *bounded* if all its elements are bounded. Evidently, the product of row-bounded elements in row-bounded, and the same holds for column-bounded elements; so it suffices, to prove that $\mathscr{A}$ is bounded, to check that property on its generators. The same argument as in the case of groups shows that row-bounded or column-bounded self-similar algebras are contracting.

It is again easy to see that the algebras $\mathscr{A}_q$ and $\mathscr{B}_q$ are bounded. This will play a major role in the computations below.

3.1. Characters

We begin by introducing some concepts. A *character* on $\Bbbk$ is a semigroup homomorphism $\chi\colon (\Bbbk, \cdot) \to \mathbb{C}$ satisfying $\chi(1) = 1$ and $\chi(0) = 0$. Recall that the group of units in $\mathbb{F}_q$ is cyclic; so may be embedded in $\mathbb{C}^\times$ by mapping a generator to a primitive $(q-1)$th root of unity. The trivial character, mapping all non-zero elements to 1, is also a valid choice.

By *characters* we think of extensions to a group ring $\Bbbk G$ of Brauer characters, rather than algebra homomorphisms. For our purposes, the following definition suffices:

Definition 3.3. A *character* on a $\Bbbk$-self-similar algebra $\mathscr{A}$ is a map $\chi\colon \mathscr{A} \to \mathbb{C}$ satisfying, for some character χ_0 on $\Bbbk$,

1) $\chi(1) = 1$;
2) $\chi(\lambda s) = \chi_0(\lambda)\chi(s)$ for all $\lambda \in \Bbbk, s \in \mathscr{A}$;
3) $\chi(x^* x) \geqslant 0$ for all $x \in \mathscr{A}$, if $\mathscr{A}$ is a *-algebra.

Note in particular that we do not require $\chi(xy) = \chi(x)\chi(y)$ (this holds only for "linear characters") nor $\chi(x+y) = \chi(x) + \chi(y)$ (this would be meaningless if $\Bbbk$ has positive characteristic), and we also do not require $\chi(xy) = \chi(yx)$ (this holds only for "diagonalizable elements").

A character χ on $\mathscr{A}$ is called *self-similar* if there is a character χ_0 on $\Bbbk$ and a positive semidefinite kernel $k(\cdot,\cdot) \in \mathbb{C}^{q\times q}$ such that

$$q \cdot \chi(s) = \sum_{i,j=0}^{q} k(i,j)\chi(\phi(s)_{i,j}).$$

We also note the following easy property of characters:

Lemma 3.4. *If $\mathscr{A}$ is a contracting, self-similar algebra, then every self-similar character on $\mathscr{A}$ is determined by its values on the nucleus. If moreover $\mathscr{A}$ is row- or column-bounded, then every self-similar character on $\mathscr{A}$ is determined by the kernel k.* □

We concentrate on two specific characters, which are both self-similar, with trivial character $\chi_0(\lambda) = 1 - \delta_{\lambda=0}$, and determined (via Lemma 3.4) respectively by the kernels $k(i,j) = \delta_{i=j}$ and $k(i,j) \equiv 1$. We denote the first character by χ_f since it measures in some sense the fixed points of an element, and the second one by χ_s since it measures in some sense the "spread" of an element. For ease of reference, the "spread" character is characterized by

$$q \cdot \chi_s(\lambda s) = \sum_{i,j=0}^{q} \chi_s(\phi(s)_{i,j}) \quad \text{for all } \lambda \in \Bbbk^\times.$$

3.2. The "spread" character

We embark in the proof of Theorem A, which will occupy this whole subsection.

The "spread" character is in fact tightly connected to the boundedness property of $\mathscr{A}$. In the case of $\mathscr{A}_q$, or more generally self-similar algebras whose generators decompose as monomial matrices, the recursion formula of χ_s implies $\chi_s(x_0) = \chi_s(x_1) = 1$, and in fact in $\mathscr{B}_q$ we have $\chi_s(x) = 1$ for any monomial $x \in G_q$.

It follows that χ_s may be related to the growth of languages in $(A \times A)^*$: for each $x \in \mathscr{A}$, set

$$L_x = \{(u, v) \in A^k \times A^k \mid \phi^k(x)_{u,v} \in \Bbbk^\times \cup \Bbbk^\times x_0 \cup \Bbbk^\times x_1\}.$$

Lemma 3.5. *For all $x \in \mathscr{A}$, the language L_x is related to the "spread" character $\chi_s(x)$ as follows: there is a constant C such that*

$$\#((A \times A)^k \cap L_x) = q^k \chi_s(x) - C \text{ for all } k \text{ large enough.}$$

Proof. This follows from a slight refinement of the contraction property: in fact, for every $x \in \mathscr{A}$, if one iterates sufficiently many times ϕ on x then the resulting matrix (of size $q^k \times q^k$) has entries in $\Bbbk \cup \Bbbk x_0 \cup \Bbbk x_1$, and the language L_x counts those entries that are not trivial. On the other hand, the "spread" character also counts (up to normalizing by a factor q^k) the number of non-trivial entries. From then on, increasing k multiplies the number of words in L_x by q so the relationship between the growth of L_x and $\chi_s(x)$ remains the same. □

Note that we could have considered a large number of different other languages: counting the number of entries $(u, v) \in A^k \times A^k$ such that the (u, v)-coefficient of $\phi^k(x)$ is, at choice,

- a scalar in $\mathscr{A}$;
- a non-zero element in $\mathscr{A}$;
- an element not in the augmentation ideal $\langle x_i - 1\rangle$ of $\mathscr{A}$;
- a monomial in $\mathscr{A}$;
- an invertible element of $\mathscr{A}$;
- a unitary element of $\mathscr{A}$.

All these choices would yield essentially equivalent languages, with comparable growth.

Lemma 3.6. *For all integers $k \geqslant 1$, the "spread" character satisfies*

$$\chi_s(1 - x_0^{q^k}) = 2/q^{k-1}, \qquad \chi_s(1 - \gamma^i (x_0 \cdots x_{q-1})^{q^k}) = 2/q^k.$$

Proof. We compute recursively some values of χ_s. First, $\chi_s(x_1) = 1$ since $\phi(x_1)$ is a permutation matrix. Then $\chi_s(x_0) = 1$ since self-similarity of χ_s yields $q\chi_s(x_0) = \chi_s(x_0)+q-1$. We next note $\chi_s(1-x_0) = \chi_s(1-x_1) = 2$; indeed self-similarity yields $q\chi_s(x_0) = 2q = q\chi_s(x_1)$.

Next, $\phi(x_0^q) = \langle\!\langle x_0 \cdots x_{q-1}, x_1 \cdots x_{q-1}x_0, \dots, x_{q-1}x_0 \cdots x_{q-2}\rangle\!\rangle$, and $\phi(x_0 \cdots x_{q-1}) = \langle\!\langle x_0, \dots, x_{q-1}\rangle\!\rangle$ and similarly for its cyclic permutations; so self-similarity yields

$$q\chi_s(1-\gamma^i(x_0 \cdots x_{q-1})) = 2q, \quad q\chi_s(1-x_0^q) = 2q$$

so $\chi_s(1-\gamma^i(x_0 \cdots x_{q-1})) = \chi_s(1-x_0^q) = 2$.

This is the beginning of induction: for $k \geqslant 1$, the matrix $\phi(x_0^{q^{k+1}})$ is diagonal, with diagonal entries $\gamma^i(x_0 \cdots x_{q-1})^{q^k}$, and $\phi(\gamma^i(x_0 \cdots x_{q-1})^{q^k})$ is also diagonal, with diagonal entries $x_0^{q^k}, \dots, x_{q-1}^{q^k}$; so self-similarity yields

$$q\chi_s(1-x_0^{q^{k+1}}) = \sum_{i=0}^{q-1} \chi_s(1-\gamma^i(x_0 \cdots x_{q-1})^{q^k}),$$

$$q\chi_s(1-(x_0 \cdots x_{q-1})^{q^k}) = \chi_s(1-x_0^{q^k}) + q(q-1)\chi_s(1-x_1^{q^k}).$$

Now $x_1^q = 1$ so the last term vanishes because $k \geqslant 1$, and we get

$$\chi_s(1-x_0^{q^{k+1}}) = \chi_s(1-\gamma^i(x_0 \cdots x_{q-1})^{q^k}) = \chi_s(1-x_0^{q^k})/q. \qquad \square$$

Consider next the map $\sigma\colon T \times \cdots \times T \to T$ given by

$$\sigma(s_0, \dots, s_{q-1}) = \theta(s_0) + x_1\theta(s_1) + \cdots + x_1^{q-1}\theta(s_{q-1}).$$

Recalling that γ is the automorphism of T permuting cyclically all generators, we get

$$\phi(\sigma(s_0, \dots, s_{q-1})) = \begin{pmatrix} s_0 & \gamma(s_{q-1}) & \cdots & \gamma^{q-1}(s_1) \\ s_1 & \gamma(s_0) & \cdots & \gamma^{q-1}(s_2) \\ \vdots & \vdots & \ddots & \vdots \\ s_{q-1} & \gamma(s_{q-2}) & \cdots & \gamma^{q-1}(s_0) \end{pmatrix}.$$

We are ready to prove Theorem A. Define subsets Ω_n of T by

$$\Omega_0 = \{0, 1-\gamma^i(x_0 \cdots x_{q-1})^{q^k} \text{ for all } i, k\},$$

$$\Omega_{n+1} = \bigcup_{i=0}^{q-1} \gamma^i\sigma(\Omega_n^q)$$

and finally $\Omega = \bigcup_{n \geqslant 0} \Omega_n$.

Lemma 3.7. *For all $x \in \Omega$ and all i the matrix $\phi(x)$ is diagonal and $\chi_s(s) = \chi_s(\gamma^i(x))$.*

Lemma 3.8. *For all $s_0, \dots, s_{q-1} \in \Omega$ we have*

$$\chi_s(\sigma(s_0, \dots, s_{q-1})) = \chi_s(s_0) + \cdots + \chi_s(s_{q-1}).$$

Proof. This follows directly from the form of $\phi(\sigma(s_0, \dots, s_{q-1}))$ given above, and the fact that χ_s is γ-invariant on Ω. □

Proof of Theorem A. Since $\mathscr{A}_q$ is contracting, every element $s \in \mathscr{A}$ decomposes in finitely many steps into elements of the nucleus; and χ_s takes values in $\mathbb{Z}[1/q] \cap \mathbb{R}_+$ on the nucleus; so $\chi_s(\mathscr{A})$ is contained in $\mathbb{Z}[1/q] \cap \mathbb{R}_+$.

On the other hand, by Lemma 3.6 the values of χ_s include all $2/q^k$, and Lemma 3.8 its values form a semigroup under addition. It follows (considering separately q even and q odd) that all fractions of the form i/q^k with $i, k \geqslant 0$ are in the range of χ_s. □

4. Variants

Essentially the same methods apply to numerous other examples; we have concentrated, here, on the one with the closest connection to the Thue–Morse sequence.

Here is another example we considered: write the alphabet $A = \{a_0, \dots, a_{q-1}\}$, and define $\phi\colon F \to F \wr_A \mathfrak{S}_A$ by

$$\phi(x_0) = \langle\!\langle x_0, \dots, x_{q-1} \rangle\!\rangle (a_i \mapsto a_{i-1 \bmod q}), \quad \phi(x_i) = \langle\!\langle 1, \dots, 1 \rangle\!\rangle (a_0 \leftrightarrow a_i),$$

or in terms of matrices

$$\phi(x_0) = \begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 \\ 0 & 0 & x_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & x_{q-1} \\ x_0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad \phi(x_i) = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & 1 & \vdots & \cdots & \vdots \\ 1 & \cdots & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}.$$

If furthermore one applies the automorphism that inverts every generator (noting that the x_i are involutions for $i \geqslant 1$), we may define an injective self-similar group H_q, isomorphic to the above, by

$$\phi(x_0) = \langle\!\langle x_0^{-1}, \dots, x_{q-1}^{-1} \rangle\!\rangle (a_i \mapsto a_{i-1 \bmod q}),$$
$$\phi(x_i) = \langle\!\langle 1, \dots, 1 \rangle\!\rangle (a_0 \leftrightarrow a_i).$$

We now note that H_q is a contracting "iterated monodromy group". As such, it possesses a limit space — a topological space equipped with an expanding self-covering, whose iterated monodromy group is isomorphic to H_q. Note that H_2 and G_2 are isomorphic. It is tempting to try to "read" the Thue–Morse sequence, and in particular the Thue–Morse word, within the dynamics of the self-covering map.

Iterated monodromy groups

Let f be a rational function, seen as a self-map of $\mathbb{P}^1(\mathbb{C})$, and write $P = \{f^n(z) : n \geqslant 1, f'(z) = 0\}$ the *post-critical set* of f. For simplicity, assume that P is finite. Choose a basepoint $* \in \mathbb{P}^1(\mathbb{C}) \setminus P$, and write $F = \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus P, *)$, a free group of rank $\#P - 1$.

The choice of a family of paths $\lambda_x \colon [0,1] \to \mathbb{P}^1(\mathbb{C}) \setminus P$ from $*$ to $x \in f^{-1}(*)$ for all choices of x naturally leads to a self-similar structure on F, following [7]: the decomposition of $\gamma \in F$ has as permutation the monodromy action of F on $f^{-1}(*)$, and the $\deg(f)$ elements of F are all $\lambda_x \# f^{-1}(\gamma) \# \lambda_{\gamma \cdot x}^{-1}$, with $\#$ denoting concatenation of paths. The faithful quotient of F is called the *iterated monodromy group* of G.

Proposition 4.1. *The Thue–Morse group H_q is the iterated monodromy group of a degree-q branched covering of the sphere.*

Proof. This follows from the general theory of [4]. The branched covering, and its iterated monodromy group, may be explicitly described as follows.

Consider as post-critical set $\{0, \infty, \zeta^0, \dots, \zeta^{q-2}\}$ for the primitive $(q-1)$th root of unity $\zeta = \exp(2\pi i/(q-1))$. Put the basepoint $*$ inside the unit disk, in such a way that it sees $\zeta^0, \zeta^1, \dots, \zeta^{q-2}, 0, \infty$ in cyclic CCW order. Put the preimages of $*$ at $*$ and points $*_i$ inside the unit disk but very close to ζ^i. As connections between $*$ and its preimages choose paths ℓ_i as straight lines. Consider as generators g_x a straight path from $*$ to x, following by a small CCW loop around x, and back, in the order mentioned above.

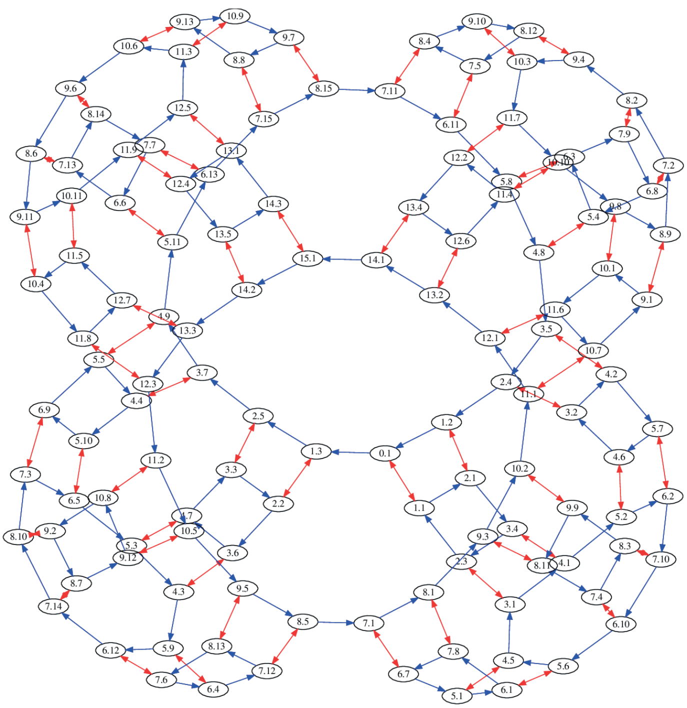
The lift of each g_{ζ^i} will be two homotopic paths exchanging $*$ and $*_i$ (all other lifts are trivial) and the lifts of g_∞ will be g_0 and a straight path from $*_i$ to ζ_i encircling it once CCW before coming back. It is clear that we have defined a branched covering of the sphere with the appropriate recursion. □

Conjecture 4.2. *The branched covering described above is isotopic to a rational map of degree q.*

We could verify this conjecture for small q; the maps corresponding to $q \leqslant 5$ are

$$
\begin{aligned}
f_2 &\approx \frac{1}{z - 0.5z^2},\\
f_3 &\approx \frac{0.128775 + 0.0942072i}{z + (-1.74702 + 0.285702i)z^2 + (0.831347 - 0.190468i)z^3},\\
f_4 &\approx \frac{0.0232438 + 0.0757918i}{z + (-2.67804 + 1.10938i)z^2 + (2.37852 - 1.93187i)z^3},\\
&\quad + (-0.694865 + 0.89421i)z^4\\
f_5 &\approx \frac{-0.00877156 + 0.0526634i}{\begin{array}{c} z + (-3.22614 + 2.0417i)z^2 + (3.13076 - 5.12089i)z^3 \\ + (-0.677772 + 4.35662i)z^4 + (-0.245783 - 1.22944i)z^5 \end{array}}.
\end{aligned}
$$

For $q = 2$, when the groups H_2 and G_2 agree, it would be particularly interesting to relate the Thue–Morse word W_2 with the geometry of the Julia set of f_2. Here is a graph approximating this Julia set; the path W_2 may be traced in it, and may be seen to explore neighbourhoods of the large Fatou regions:



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Contact information

Laurent Barthold Universität des Saarlandes, Germany
E-Mail(s): laurent.bartholdi@gmail.com

José Manuel Rodríguez Caballero University of Tartu, Estonia
E-Mail(s): jose.manuel.rodriguez.caballero@ut.ee

Tanbir Ahmed Université du Québec à Montréal, Canada
E-Mail(s): tanbir@gmail.com

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