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RESEARCH ARTICLE

Modules which have a rad-supplement that is a direct summand in every extension

Burcu Nişancı Türkmen and Ergül Türkmen

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Abstract. In this paper, we introduce the concept of modules with the properties (RE) and (SRE), and we provide various properties of these modules. In particular, we prove that a semisimple module M is Rad-supplementing if and only if M has the property (SRE). Moreover, we show that a ring R is a left V-ring if and only if every left R-module with the property (RE) is injective. Finally, we characterize the rings whose modules have the properties (RE) and (SRE).

1. Introduction

In this paper all rings are associative with identity and all modules are unital left modules. Let R be such a ring and let M be an R-module. The notation $(K \subset M)$ $K \subseteq M$ means that K is a (proper) submodule of M. In [10, 17.1], a submodule $K \subseteq M$ is called *essential* in M, written as $K \trianglelefteq M$, if $K \cap L \neq 0$ for every non-zero submodule of M. Dually, a submodule $S \subset M$ is called *small* in M, denoted by $S \ll M$, if $M \neq S + K$ for every proper submodule K of M ([10, 19.1]). Let U and V be submodules of M. U is said to *have a supplement* V in M, or V is said to be a *supplement* of U in M if V is minimal with respect to $M = U + V$. A submodule $V \subseteq M$ is a supplement of some submodule U in M if and only if $M = U + V$ and $U \cap V \ll V$ ([10, 41]).

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Recall from [8, Theorem 2.15] that a module M is *injective* if and only if it is a direct summand of every extension N. Here N is *an extension* of M provided $M \subseteq N$. Since every direct summand is a supplement, Zöschinger generalized in [12] injective modules to modules with the property (E). A module M is said to *have the property (E)* if M has a supplement in every extension N. He studied on modules with the property (E) in the same paper. Considering different kinds of supplements, modules having the following conditions are studied in [7];

(SE) In any extension N of M, M has a supplement that is a direct summand of N.

(SSE) M has a strongly supplement in every extension N, that is, $N = L \oplus K$ and $M \cap K \ll K$ for some submodules L and K of N with $L \subseteq M$.

By the radical of a module M, denoted by $\operatorname{Rad} M$, we will indicate the sum of all small submodules of a module M, or, equivalently the intersection of all maximal submodules of M. Weakening the notion of "supplement", one calls a submodule K of M a Rad-*supplement* of N in M if $M = N + K$ and $N \cap K \subseteq \operatorname{Rad} K$ ([3, pp.100]). Adapting the notion of modules with the property (E), M is called Rad-*supplementing* if M has a Rad-supplement in every extension N ([6]).

By these definitions, we have these implications on modules:

injective $\Longrightarrow$ module with (SSE) $\Longrightarrow$ module with (SE) $\Longrightarrow$ module with (E) $\Longrightarrow$ Rad-supplementing

In this study, we generalize the concept of modules with the property (SE) (respectively, the property (SSE)) to modules with the property (RE) (respectively, the property (SRE)) using the notion of Rad-supplements. We prove that over a left max ring every left module with the property (RE) (respectively, the property (SRE)) has the property (SE) (respectively, the property (SSE)). We show that a semisimple module M is Rad-supplementing if and only if it has the property (RE). We give a characterization of left V-rings via modules with the property (RE). Over any ring R, every left R-module is $\oplus$-supplemented if and only if every left R-module is Rad-$\oplus$-supplemented if and only if every left R-module has the property (RE). Using this fact we also prove that a ring R is artinian serial if and only if every left and right R-module has the property (RE), and R is an artinian serial ring with $J^2 = 0$, where J is the Jacobson radical of the ring R, if and only if every left R-module has the property (SRE).

2. Modules with the properties (re) and (sre)

Definition 2.1. We define the following conditions for a module M:

(RE) In every extension N of M, M has a Rad-supplement that is a direct summand of N.

(SRE) In every extension N of M, N has the decomposition $N = L \oplus K$ such that $L \subseteq M$ and $M \cap K \subseteq \operatorname{Rad} K$.

It is clear that every module with the property (SE) (respectively, the property (SSE)) has the property (RE) (respectively, the property (SRE)). Now we give an example which shows the converse is not always true. Firstly we need the following fact.

Let M be a module. M is said to be *radical* in case $\operatorname{Rad} M = M$. By $\mathcal{P}(M)$ we will denote the sum of all radical submodules of M. Note that $\mathcal{P}(M)$ is the largest radical submodule of M and $\mathcal{P}(M) \subseteq \operatorname{Rad} M$.

Lemma 2.2. *Every radical module has the property* (SRE). *In particular,* $\mathcal{P}(M)$ *has the property (SRE) for every module* M.

Proof. Let $Y = \operatorname{Rad} Y$, that is, Y is radical. For any extension N of Y, we have $N = 0 \oplus N$ and $Y \cap N = Y = \operatorname{Rad} Y \subseteq \operatorname{Rad} N$. It means that Y has the property (SRE).

Let M be any module. Since $\mathcal{P}(M)$ is a radical module, it has the property (SRE). □

A non-zero module M is said to be *hollow* if every proper submodule of M is small in M, and it is said to be *local* if it is hollow and finitely generated. M is local if and only if it is finitely generated and $\operatorname{Rad} M$ is the unique maximal submodule of M (see [3, 2.12 and 2.15]). A ring R is *local* if $J = \operatorname{Rad} R$ is the unique maximal left (right) ideal of R, that is, $\frac{R}{J}$ is a simple R-module.

The following example shows that, in general, a module with the property (SRE) need not have the property (E).

Example 2.3. Let K be a field. In the polynomial ring $K[x_1, x_2, \ldots]$ with countably many indeterminates x_n, $n \in \mathbb{N}$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_3, \cdots)$ generated by x_1^2 and $x_{n+1}^2 - x_n$ for each $n \in \mathbb{N}$. Then as shown in [1, Example 6.2], the quotient ring $R = \frac{K[x_1, x_2, \ldots]}{I}$ is a local ring with the unique maximal ideal $J = \frac{(x_1, x_2, \ldots)}{I}$, $J^2 = J$, $\operatorname{Rad} J = J$ and J is not left T-nilpotent which then implies that the ring R is not left perfect. Let $N = R^{(\mathbb{N})}$ and $M = \operatorname{Rad} N = \operatorname{Rad}(R^{(\mathbb{N})}) = (\operatorname{Rad} R)^{(\mathbb{N})} = J^{(\mathbb{N})}$. Then

$\operatorname{Rad} M = \operatorname{Rad}(J^{(\mathbb{N})}) = (\operatorname{Rad} J)^{(\mathbb{N})} = J^{(\mathbb{N})} = M$ since $\operatorname{Rad} J = J$. By Lemma 2.2, we obtain that the radical module M has the property (SRE). But by [1, Theorem 1], $M = \operatorname{Rad}(R^{(\mathbb{N})})$ does not have a supplement in $N = R^{(\mathbb{N})}$ since R is a(semi)local ring but not left perfect.

Recall from [11] that a module M over an arbitrary ring is said to be *coatomic* if every proper submodule of M is contained in a maximal submodule of M. Note that finitely generated (in particular, local) modules are coatomic. It is easy to see that if, for any module M, a submodule $X \subseteq \operatorname{Rad} M$ is coatomic, then X is a small submodule of M.

Proposition 2.4. *Let M be a coatomic module. If M has the property (SRE), it has the property (SSE).*

Proof. Let M be a coatomic module with the property (SRE) and N be any extension of M. Then, there exist submodules L and K of N such that $N = L \oplus K$ and $M \cap K \subseteq \operatorname{Rad} K$ and $L \subseteq M$. By the modular law, we can write $M = (M \cap K) \oplus L$. Since M is coatomic, $M \cap K$ is coatomic as it is isomorphic to the factor module $\frac{M}{L}$ of M. It follows that $M \cap K$ is a small submodule of K. Hence, M has the property (SSE). □

Proposition 2.5. *For a module M, assume that every submodule of M is coatomic. If M has the property (RE), it has the property (SE).*

Proof. Let N be any extension of the module M having the property (RE). Therefore, we can write $N = M + K$ and $M \cap K \subseteq \operatorname{Rad} K$ for a direct summand K of N. It follows from the assumption that $M \cap K$ is coatomic. So, we obtain that $M \cap K$ is a small submodule of K. It means that M has the property (SE). □

Let R be a ring. R is said to be a *left max ring* if every nonzero left R-module has maximal submodules. It is well known that R is a left max ring if and only if every non-zero left R-module M is coatomic. Using this fact, Proposition 2.4 and Proposition 2.5 we have the following fact:

Proposition 2.6. *Let R be a left max ring and M be an R-module with the property (RE) (respectively, the property (SRE)). Then, M has the property (SE) (respectively, the property (SSE)).*

A ring R is called *left perfect* if every left R-module has a projective cover [10, 43.9]. Left perfect rings are left max. Using this fact along with the above proposition we obtain the following corollary:

Corollary 2.7. *Let R be a left perfect ring and M be an R-module. Then,*
(1) *M has the property (RE) if and only if it has the property (SE).*
(2) *M has the property (SRE) if and only if it has the property (SSE).*

Proposition 2.8. *If a module M is noetherian and has the property (RE), it has the property (SE).*

Proof. Let M be a noetherian module with the property (RE). Then, for any extension N of M, there exists a direct summand K of M such that K is a Rad-supplement of M in N, that is, $N = M + K$ and $M \cap K \subseteq \operatorname{Rad} K$. Since M is noetherian, the submodule $M \cap K$ of M is finitely generated. Since every finite sum of small submodules is small, we get $M \cap K \ll K$. Consequently, M has the property (SE). □

Let M be an R-module and let U and V be any submodules of M with $M = U + V$. If $U \cap V \subseteq \operatorname{Rad} M$, then V is called a *weak* Rad-*supplement* of U in M.

Proposition 2.9. *Let M be a semisimple R-module. Then, the following statements are equivalent:*
(1) *M has the property (SRE).*
(2) *M has the property (RE).*
(3) *M is a* Rad-*supplementing module.*
(4) *M has a weak* Rad-*supplement in every extension N.*

Proof. (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) are clear.

(4) $\Longrightarrow$ (1) It follows by [7, Proposition 2.1] that M has (SSE) and so (SRE). □

Using Proposition 2.9 and [7, Proposition 2.1] we get the following fact.

Corollary 2.10. *For a semisimple module M, the following statements are equivalent:*
(1) *M has the property (SRE).*
(2) *M has the property (RE).*
(3) *M is a* Rad-*supplementing module.*
(4) *M has a weak* Rad-*supplement in every extension N.*
(5) *M has the property (SSE).*
(6) *M has the property (SE).*
(7) *M has the property (E).*

Corollary 2.11. *Every finitely generated semisimple module has the property (SRE).*

Proof. Clear by Corollary 2.10 and [6, Corollary 2.13]. □

The proof of the next result is similar to [7, Theorem 2.1].

Theorem 2.12. *For a module M, the following are equivalent:*

(1) *For any extension N of M and for every submodule K of N such that $M + K = N$, K contains a* Rad*-supplement of M in N that is a direct summand of K.*
(2) *Every submodule of M has the property (RE).*

A ring R is called a *left V-ring* if every simple left R-module is injective. It is well known that R is a left V-ring if and only if $\operatorname{Rad} M = 0$ for every left R-module M ([10]).

Lemma 2.13. *Let M be a module with the property (RE). Suppose that N is an extension of M such that* $\operatorname{Rad} N = 0$*. Then, M is a direct summand of N.*

Proof. Let N be any extension of M. Since M has the property (RE), there exist submodules K and $K^{'}$ of N such that $N = M + K$, $M \cap K \subseteq \operatorname{Rad} K$ and $N = K \oplus K^{'}$. By the hypothesis, $M \cap K \subseteq \operatorname{Rad} N = 0$. It follows that $N = M \oplus K$. □

Theorem 2.14. *Let R be a ring. Then, R is a left V-ring if and only if every left R-module with the property (RE) is injective.*

Proof. ($\Longrightarrow$) Let M be a module with the property (RE) and N be an extension of M. Therefore, for a direct summand K of N, we can write $N = M + K$ and $M \cap K \subseteq \operatorname{Rad} N$. Since R is a left V-ring, we have $\operatorname{Rad} N = 0$. By Lemma 2.13, M is a direct summand of N. Thus, M is injective.

($\Longleftarrow$) Since simple modules are finitely generated semisimple, it follows from Corollary 2.10 and Corollary 2.11. □

Recall that a module M is (Rad-) $\oplus$-*supplemented* if every submodule has a (Rad-) supplement that is a direct summand of M ([5] and [9]).

Lemma 2.15. *Let R be a ring. Suppose that every left R-module has the property (RE). Then, R is left perfect.*

Proof. Let M be any projective left R-module and $U \subseteq M$. By the assumption, there exist submodules V and V' of M such that $M = U + V$, $U \cap V \subseteq \operatorname{Rad} V$ and $M = V \oplus V'$. So M is Rad-$\oplus$-supplemented. It follows from [9, Corollary 2.3] that R is left perfect. □

Lemma 2.16. *The following statements are equivalent for a ring R:*
(1) *Every left R-module has the property (RE).*
(2) *Every left R-module has the property (SE).*
(3) *Every left R-module is $\oplus$-supplemented.*
(4) *Every left R-module is* Rad-$\oplus$*-supplemented.*

Proof. (1) $\Longrightarrow$ (2) It follows from Lemma 2.15 and Corollary 2.7 (1).

(2) $\Longrightarrow$ (3) Let M be any left R-module. By (2), every submodule U of M has the property (SE). In particular, U has a supplement that is a direct summand of M. It means that M is $\oplus$-supplemented.

(3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (1) are clear. □

In [3, p. 17] a module M is called *uniserial* if its lattice of submodules is a chain. M is said to be *serial* if M is a direct sum of uniserial modules. A ring R is *left serial* (respectively, *right serial*) if the left R-module ${}_RR$ (respectively, the right R-module R_R) is serial. A ring R is said to be *artinian serial* if it is left and right artinian, and left and right serial.

Theorem 2.17. *The following statements are equivalent for a ring R:*
(1) *R is artinian serial.*
(2) *Every left and right R-module has the property (RE).*

Proof. By Lemma 2.16 and [5, Theorem 3.11]. □

Using Corollary 2.7 (2), Lemma 2.15 and [3, 29.10], we give the structure of rings whose modules have the property (SRE).

Corollary 2.18. *The following statements are equivalent for a ring R:*
(1) *R is an artinian serial ring and $J^2 = 0$, where J is the Jacobson radical of the ring R.*
(2) *Every left R-module has the property (SRE).*

Finally, consider the commutative local artinian serial ring $R = \mathbb{Z}_8$. Then, it follows from Theorem 2.17 that every left R-module has the property (RE). Note that the radical J of R is the maximal ideal $\{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$, and so $J^2 \neq 0$. This means that, by Corollary 2.18, there exists a left R-module which hasn't the property (SRE).

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Contact information

B. N. Türkmen,
E. Türkmen

Amasya University, Faculty of Art and Science, Department of Mathematics, Ipekköy, Amasya, Turkey
E-Mail(s): burcunisancie@hotmail.com,
ergulturkmen@hotmail.com

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