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On Herstein's identity in prime rings

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In the memory of my loving mother

Abstract. A celebrated result of Herstein [10, Theorem 6] states that a ring R must be commutative if $[x, y]^{n(x,y)} = [x, y]$ for all $x, y \in R$, where $n(x, y) > 1$ is an integer. In this paper, we investigate the structure of a prime ring satisfies the identity $F([x, y])^n = F([x, y])$ and $\sigma([x, y])^n = \sigma([x, y])$, where F and σ are generalized derivation and automorphism of a prime ring R, respectively and $n > 1$ a fixed integer.

introduction

Throughout this article, R will always denote an associative prime ring with center $Z(R)$, Utumi quotient ring U and the extended centroid C. Note that in case R is prime, C is a field. For more details of these notions, one can see [3]. Recall, a ring R is said to be prime if $aRb = (0)$ for any $a, b \in R$, implies either $a = 0$ or $b = 0$. In other words, a ring in which (0) is prime ideal is called prime ring. A mapping $d : R \to R$ is called derivation of R if $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An immediate example of a derivation is the mapping $x \mapsto qx - xq$, where $q \in R$ is a fixed element; such a mapping is called the inner derivation associated with the element q. Moreover, for fixed elements $p, q \in R$, a mapping $x \mapsto px + xq$ is called the generalized inner derivation. In general, an additive mapping $F : R \to R$ is called

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generalized derivation if there exists a unique derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. For any $x, y \in R$, the Lie product in R is defined as $xy - yx$ and is denoted by $[x, y]$. By a Lie ideal of R, we mean an additive subgroup L of R, which satisfies the condition $[x, r] \in L$ for all $x \in L$ and $r \in R$. It is straight forward to see that every ideal is a Lie ideal but converse is not true in general.

During the decade 1940-1950, after the development of general structure theory for rings, much attention has been devoted to explore the conditions that finally imply the commutativity of rings. In this view, a classical theorem of Jacobson states that a ring R is commutative if there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$ for all $x \in R$. Later, Herstein [9] gave a complete generalization of this result as: *If R is a ring with center $Z(R)$, and if $x^n - x \in Z(R)$ for all $x \in R$, $n > 1$ a fixed integer, then R is commutative.* In 1957, Herstein [10] proved another result of same flavour in a more general way. Precisely, Herstein obtained the commutativity of rings that satisfy the condition $[x, y]^{n(x,y)} = [x, y]$, where $n(x, y) > 1$ is a fixed positive integer. These results has led to the development of several techniques to find the conditions that force a ring to be commutative; for instance, generalizing Herstein's conditions, using certain polynomial constraints, using restrictions on automorphisms, introducing identities involving derivations and generalized derivations etc. Continuing in this line of investigation, recently, Scudo and Ansari [19] studied generalized derivations of prime rings that satisfy an idempotent valued condition. More precisely, they proved the following theorem: *Let R be a noncommutative prime ring with $Char(R) \neq 2$, U the Utumi quotient ring of R, C the extended centroid of R and L a noncentral Lie ideal of R. If G is a generalized derivation of R associated with a derivation d such that $[G(u), u]^n = [G(u), u]$ for all $u \in L$, where $n > 1$ a fixed positive integer, then one of the following holds true:*

(i) *R satisfies the s_4 (the standard identity in four noncommuting variables), and there exists $a \in U$ and $\lambda \in C$ such that $G(x) = ax + xa + \lambda x$ for all $x \in R$.*
(ii) *there exists $\gamma \in C$ such that $G(x) = \gamma x$ for all $x \in R$.*

Further, Ashraf et al. [1] obtained a result with automorphisms in this direction. They proved the following: *Let R be a prime ring with $Char(R) \neq 2, 3$ and L a noncentral Lie ideal of R. If σ is an automorphism of R such that $[\sigma(x), x]^m = [\sigma(x), x]$ for all $x \in L$, where $m > 1$ a fixed integer, then R is commutative.*

Motivated by the above cited papers, we present the study of generalized derivations and automorphisms of prime rings that satisfy the identity $\mathcal{G}(u)^n = \mathcal{G}(u)$ on Lie ideals, where $n > 1$ is a fixed integer.

1. A result on generalized derivations

Fact 1. [5, THEOREM 2] If I is a nonzero ideal of a prime ring R, then I, R and U satisfy the same generalized polynomial identities with coefficients in U.

Fact 2. [16, THEOREM 2] If I is a nonzero ideal of a prime ring R, then I, R and U satisfy the same differential identities.

Fact 3. [17, THEOREM 4] Let R be a semiprime ring and D_r a dense right ideal of R. Then every generalized derivation $F : D_r \to U$ can be uniquely extended to U and assumes the form $F(x) = ax + \delta(x)$ for some $a \in U$ and a derivation δ of U.

Fact 4. [4, LEMMA 1] If R is a prime ring of $\mathrm{Char}(R) \neq 2$ and L a noncentral Lie ideal of R, then there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$.

Theorem 1. *Let R be a prime ring with $Char(R) \neq 2$, L a noncentral Lie ideal of R, U the Utumi quotient ring and C the extended centroid of R. Suppose that R admits a generalized derivation F associated with a derivation δ such that $F(u)^n = F(u)$ for all $u \in L$, where $n > 1$ is a fixed integer, then R satisfies s_4.*

Proof. Suppose that R does not satisfy s_4. By hypothesis, we have

$$F(u)^n = F(u), \ \forall \ u \in L.$$

In view of Fact 3, it follows that

$$(au + \delta(u))^n = au + \delta(u), \ \forall \ u \in L.$$

By Fact 4, there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Therefore, we find that

$$(a[x, y] + \delta([x, y]))^n = a[x, y] + \delta([x, y]), \ \forall \ x, y \in I.$$

In light of Fact 2, we have

$$(a[x, y] + \delta([x, y]))^n = a[x, y] + \delta([x, y]), \ \forall \ x, y \in U.$$

That can be rewritten as

$$(a[x,y]+[\delta(x),y]+[x,\delta(y)])^n = a[x,y]+[\delta(x),y]+[x,\delta(y)],\ \forall\ x,y \in U. \tag{1}$$

We now apply Kharchenko's theory of differential identities [13, Theorem 2] and divide the proof into the following two parts.

Case 1. We first assume that δ is the inner derivation of U, i.e., there exists some $q \in U$ such that $\delta(x) = [q,x]$ for all $x \in U$. In this case we shall show that $q \in C$ and hence $\delta = 0$. With this, equation (1) yields that U satisfies the generalized polynomial identity

$$\Omega(x,y) = (a[x,y]+[[q,x],y]+[x,[q,y]])^n - a[x,y]-[[q,x],y]-[x,[q,y]].$$

In case C is infinite, we have that $\Omega(x,y)$ is a generalized polynomial identity for $U \otimes_C \overline{C}$, where $\overline{C}$ denotes the algebraic closure of C. Notice that, $\Omega(x,y)$ is a generalized polynomial identity for U if and only if it is so for R (see Fact 1). Therefore in order to prove our claim, we may replace R by U or $U \otimes_C \overline{C}$ according as C is finite or infinite. In fact both U and $U \otimes_C \overline{C}$ are prime and centrally closed (see Theorem 2.5 and Theorem 3.5 of [8]). Thus we may assume that R is centrally closed over C (i.e., $R_C = R$) which is either finite or algebraic closed and $\Omega(x,y) = 0$ for all $x,y \in R$. By a result due to Martindale [[18], Theorem 3], R_C (and so is R) is a primitive ring having nonzero socle H associated with the division ring D. Hence R is isomorphic to a dense ring of linear transformations of some vector space V over D. If V is finite dimensional over D, then density of R on V implies that $R \cong M_j(D)$, where $j =$dim(V_D). Clearly if $j = 1$, R is commutative, a contradiction.

We now suppose that $\dim(V_D) \geqslant 3$. Our first goal is to show that for any $v \in V$, the set $\{v, qv\}$ is linearly D-dependent. For this purpose, we assume that v and qv are linearly D-independent vectors in V. Since $\dim(V_D) \geqslant 3$, there exists some $w \in V$ such that the set $\{v, qv, w\}$ is a linearly D-independent set. By density of R, there exist $x, y \in R$ such that

$$xv = 0, \quad xqv = w, \quad xw = v, \quad yv = 0, \quad yqv = w, \quad yw = v.$$

With all this, we see that $a[x,y]v = 0$, $[[q,x],y]v = v$ and $[x,[q,y]] = v$. Thus we have

$$\begin{aligned} 0 &= ((a[x,y]+[[q,x],y]+[x,[q,y]])^n - a[x,y]-[[q,x],y]-[x,[q,y]])v \\ &= (2^n-2)v, \end{aligned}$$

a contradiction. It implies that v and qv are linearly D-dependent and so there exists $\xi_v \in D$ such that $qv = v\xi_v$. We now show that ξ_v does not depend on the choice of v, i.e., there exists $\xi \in D$ such that $qv = v\xi$ for all $v \in V$. Let us choose linearly D-independent vectors v and w in V. Since $\dim(V_D) \geqslant 3$, we can find $u \in V$ such that the set $\{u, v, w\}$ is linearly D-independent. Then there exist $\xi_u, \xi_v, \xi_w \in D$ such that

$$qu = u\xi_u, \quad qv = v\xi_v, \quad qw = w\xi_w.$$

It implies that

$$\begin{aligned} q(u+v+w) &= (u+v+w)\xi_{u+v+w} \\ qu + qv + qw &= u\xi_{u+v+w} + v\xi_{u+v+w} + w\xi_{u+v+w} \\ u\xi_u + v\xi_v + w\xi_w &= u\xi_{u+v+w} + v\xi_{u+v+w} + w\xi_{u+v+w} \\ 0 &= ua_1 + va_2 + wa_3, \end{aligned}$$

where $a_1 = \xi_u - \xi_{u+v+w}, a_2 = \xi_v - \xi_{u+v+w}$ and $a_3 = \xi_w - \xi_{u+v+w}$. It implies that $a_i = 0$ for all $i = 1, 2, 3$. Thus $\xi_u = \xi_{u+v+w}, \xi_v = \xi_{u+v+w}$ and $\xi_w = \xi_{u+v+w}$. It proves our claim. Now we see that for any $r \in R$ and $v \in V$,

$$rqv = r(qv) = r(v\xi) = (rv)\xi = q(rv) = qrv.$$

It implies that $[r, q]v = 0$ for all $r \in R$ and $v \in V$. Since V is a left faithful irreducible module, we find that $[r, q] = 0$ for all $r \in R$. Hence $q \in R$ and so $\delta = 0$.

With this, we get the situation $(a[x, y])^n = a[x, y]$ for all $x, y \in R$. Suppose that there exists $v \in V$ such that the vectors v and av are linearly D-independent. Again as $\dim(V_D) \geqslant 3$, there exists $u \in V$ such that the set $\{v, av, u\}$ is linearly D-independent. By density of R, we have $x, y \in R$ such that

$$vx = 0, \quad vax = w, \quad wx = v, \quad vy = w, \quad vay = 0, \quad wy = 2v.$$

It implies that $0 = v((a[x, y])^n - a[x, y]) = (2^n - 2)v$, which is a contradiction. Therefore, one can easily find that $a \in C$, by repeating the similar arguments as above. Thus, we obtain $a^{n-1}[x, y]^n - [x, y] = 0$ for all $x, y \in R$. Again choosing linearly D-independent variables $\rho_1, \rho_2, \rho_3 \in V$ such that

$$x\rho_1 = 0, \quad x\rho_2 = \rho_3, \quad x\rho_3 = 0, \quad y\rho_1 = \rho_2, \quad y\rho_2 = \rho_3, \quad y\rho_3 = 0.$$

It yields that $\rho_3 = (a^{n-1}[x, y]^n - [x, y])\rho_1 = 0$, a contradiction.

We now assume that $\dim(V_D) \leqslant 2$. In this case R is a simple GPI-ring with unity and so is a central simple finite dimensional algebra over its center. In light of Lemma 2 of [14], it follows that there exists a suitable field F (say) such that $R \subseteq M_i(F)$, ring of $i \times i$ matrices with entries from F, moreover R and $M_i(F)$ satisfy same generalized polynomial identities. Therefore $M_i(F)$ satisfies $\Omega(x, y)$. If $i \geqslant 3$, a contradiction follows as above. If $i = 1$, then R is commutative and if $i = 2$, then R satisfies s_4, both of these cases also take us to a contradiction.

Finally, we consider the case when $\dim_D(V) = \infty$. By Wong [20, Lemma 2], R satisfies $(ax + [q, x])^n - (ax + [q, x]) = 0$. Let v and qv be the linearly D-independent vectors. By density of R, we have $x \in R$ such that $xv = 0$ and $xqv = 2v$. It follows that

$$0 = (ax + [q, x])^n - (ax + [q, x])v = (2^n - 2)v,$$

since $\mathrm{Char}(R) \neq 2$, we get $(2^{n-1} - 1)v = 0$, again a contradiction.

Case 2. Suppose that δ is not the inner derivation of U, i.e., δ is outer. By Kharchenko's result [13], U must satisfies the following generalized polynomial identity

$$(a[x, y] + [r, y] + [x, s])^n - (a[x, y] + [r, y] + [x, s]). \tag{2}$$

As we mentioned earlier, R may be replaced by U and $U \otimes_C \overline{C}$ according as C is finite or infinite, and assume that R is centrally closed over C. Therefore R satisfies the GPI (2). In particular R satisfies the blended component $[r, y]^n - [r, y]$. That means, R is a PI-ring. With the aid of Lanski's result [14, Lemma 2], we find a suitable filed F such that $R \subseteq M_i(F)$ and $M_i(F)$ satisfies the identity $[r, y]^n - [r, y]$. Obviously $k \neq 1, 2$. For $k > 2$, we choose $r = e_{ij}$ and $y = e_{jj}$, we get $0 = [e_{ij}, e_{jj}]^n - [e_{ij}, e_{jj}] = -e_{ij} \neq 0$, which is again a contradiction. It completes the proof of the theorem. □

Proceeding in same way with necessary variations, the following theorem can be easily proved.

Theorem 2. *Let R be a prime ring with Char$(R) \neq 2$. If R admits a generalized derivation F associated with a derivation δ such that $F(x)^n = F(x)$ for all $x \in R$, where $n > 1$ is a fixed integer, then R is commutative.*

Corollary 1. *Let R be a prime ring with Char$(R) \neq 2$. If for some fixed integer $n > 1$, $[x, y]^n = [x, y]$ for all $x, y \in R$, then R is commutative.*

Proof. Let us fix x. Then we have $I_x(y)^n = I_x(y)$ for all $y \in R$, where I_x denotes the inner derivation associated with x. In view of Theorem 2, R is commutative. □

2. A result on automorphisms

It is well-known that every automorphism of R can be uniquely extended to U. An automorphism of R is called U-inner if there exists an invertible element $q \in U$ such that $\sigma(x) = qxq^{-1}$ for all $x \in R$. Otherwise σ is called U-outer. Let us denote the group of all automorphisms of R by G and the group consisting of all the U-inner automorphisms of R by G_{Inn}. Recall that a subset A of G is said to be independent (modulo G_{Inn}) if for any $u_1, u_2 \in A$, $u_1 u_2^{-1} \in G_{Inn}$ implies $u_1 = u_2$. We begin our discussion with some important facts of this subject that will be used in the development of our main proof.

Fact 5. [7, THEOREM 3] Let $\phi = \Phi(x_i^{u_j})$ be a generalized polynomial identity with automorphisms of R reduced with respect to A. If for all $x_i \in X$ and $u_j \in A$, the $x_i^{u_j}$-word degree of $\phi = \Phi(x_i^{u_j})$ is strictly less than $\mathrm{Char}(R)$, when $\mathrm{Char}(R) \neq 0$, then $\Phi(y_{ij}) = 0$ is a generalized polynomial identity of R.

Fact 6. [7, THEOREM 1] Let R be a prime ring and I a two sided ideal of R. Then I, R and U satisfy the same generalized polynomial identities with automorphisms.

Fact 7. [2, LEMMA 7.1] Let V_D be a vector space over a division ring D with $\dim_D(V) \geqslant 2$ and $\psi \in End(V)$. If v and ψv are D-dependent for every $v \in V$, then there exists $\ell \in D$ such that $\psi v = \ell v$ for every $v \in V$.

Fact 8. [12] Let R be a domain and σ be an automorphism of R which is outer. If R satisfies a GPI $\Phi(x_i, \sigma(x_i))$, then R also satisfies the nontrivial GPI $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.

Let V_D be a right vector space over a division ring D. We denote by $End(V_D)$ the ring of D-linear transformations on V_D. A map $q : V_D \to V_D$ is said to be a semi-linear transformation if q is additive and there exists an automorphism σ of D such that $q(v\lambda) = (qv)\sigma(\lambda)$ for every $v \in V$ and $\lambda \in D$. By a theorem of Jacobson [11, Isomorphism Theorem, p.79], $\sigma(x) = qxq^{-1}$ for every $x \in End(V_D)$, where σ is an automorphism of $End(V_D)$ and q is the invertible semi-linear transformation.

We first prove the following lemmas which are crucial in our discussion.

Lemma 1. *Let $\sigma : End(V_D) \to End(V_D)$ be an automorphism such that for some fixed integer $n \geqslant 2$, $\sigma([x, y])^n = \sigma([x, y])$ for all $x, y \in End(V_D)$. If dim$(V_D) \geqslant 2$, then σ is the identity map.*

Proof. As discussed above, we have $\sigma(x) = qxq^{-1}$ for all $x \in End(V_D)$ and $q(v\lambda) = (qv)\sigma(\lambda)$ for all $v \in V$ and $\lambda \in D$. By the given hypothesis, we find that

$$(q[x,y]q^{-1})^n - q[x,y]q^{-1} = 0 \ \forall \ x,y \in End(V_D).$$

Since $\dim(V_D) \geqslant 2$, suppose that for any $v \in V$, the vectors v, qv are linearly D-independent. Further, we assume that $q^{-1}v \notin \mathrm{Span}_D\{v, qv\}$, so the set $\{v, qv, q^{-1}v\}$ is linearly D-independent. By density of R, there exist $x, y \in End(V_D)$ such that

$$xv = v, \quad xqv = v, \quad xq^{-1}v = 0, \quad yv = 0, \quad yqv = v, \quad yq^{-1}v = v.$$

In this view, it follows that

$$0 = ((q[x,y]q^{-1})^n - q[x,y]q^{-1})v = -qv \neq 0,$$

a contradiction. Therefore the set $\{v, qv, q^{-1}v\}$ is linearly D-dependent and so there exist $\alpha, \beta \in D$ such that $q^{-1}v = v\alpha + qv\beta$. We notice that $\beta \neq 0$, because otherwise we have $q^{-1}v = v\alpha$ implies $v = qv\alpha$, which is an absurd. Now we choose $x, y \in End(V_D)$ such that

$$xv = v, \quad xqv = v, \quad yv = 0, \quad yqv = v.$$

It implies that

$$0 = ((q[x,y]q^{-1})^n - q[x,y]q^{-1})v = -\beta qv,$$

a contradiction. Therefore v and $q^{-1}v$ are linearly D-dependent. By Fact 7, $q^{-1}v = v\ell$, where $\ell \in D$ and for all $v \in V$. Thus for each $x \in End(V_D)$, we have $q^{-1}(xv) = xv\ell$. It implies that

$$xv = q^{-1}(qxv) = (qxv)\ell = qx(v\ell) = qx(q^{-1}v) = (qxq^{-1})v = \sigma(x)v$$

for all $x \in End(V_D)$ and $v \in V$. It implies that $(\sigma(x) - x)V = (0)$, and hence we get $\sigma(x) = x$ for all $x \in End(V_D)$. □

Lemma 2. *Let R be a prime ring with $Char(R) \neq 2$. If R admits an outer automorphism σ such that for a fixed integer $n > 1$, $\sigma([x,y])^n = \sigma([x,y])$ for all $x, y \in R$, then R is commutative.*

Proof. In case σ is the identity map, we have $[x,y]^n = [x,y]$ for all $x, y \in R$. In view of Corollary 1, R is commutative. Suppose that σ is a non-identity map. Thus $\sigma([x,y])^n - \sigma([x,y])$ is a nontrivial differential identity for R, by Chaung [6, Main Theorem], R must satisfy a nontrivial generalized polynomial identity. In fact, U satisfies the same generalized polynomial identity (see Fact 1). Moreover, U is a primitive ring and isomorphic to a dense ring of linear transformations of some vector space V over a division ring D (see [18, Theorem 3]).

In case U is a domain, by Kharchenko [12], U satisfies the polynomial identity $[x,y]^m - [x,y]$. With the aid of Corollary 1, we are done.

Assume that U is not a domain, then we have $\sigma(x) = qxq^{-1}$ for all $x \in U$. In this view, it follows that U satisfies the nontrivial GPI

$$(q[x,y]q^{-1})^n - q[x,y]q^{-1}.$$

Notice that if for any $v \in V$, the vectors v and $q^{-1}v$ are linearly D-dependent, then σ becomes the identity map by Lemma 1, and we have $[x,y]^n = [x,y]$ for all $x, y \in R$, hence the conclusion follows as above. Thus we assume that there exists some $v \in V$ such that v and $q^{-1}v$ are linearly D-independent vectors. Let us first assume that $\dim_D(V) \geqslant 3$. Then there exists some $w \in V$ such that the set $\{v, q^{-1}v, w\}$ is linearly D-independent. In view of density of U, we find $x, y \in U$ such that

$$xv = 0, \quad xq^{-1}v = -v, \quad xw = 0, \quad yv = v, \quad yq^{-1}v = 0, \quad yw = q^{-1}v.$$

From the hypothesis, we find that $0 = ((q[x,y]q^{-1})^n - q[x,y]q^{-1})v = qv$, and hence $v = 0$, a contradiction. Now assume that $\dim(V_D) = 2$, i.e., $U \cong M_2(D)$. Therefore $\sigma([x,y])^n - \sigma([x,y]) = 0$ for all $x, y \in U$, since s^σ-word degree is 2 and characteristic of R is > 2, invoking Fact 5, U satisfies the polynomial identity $[x,y]^n - [x,y]$. As above, R is commutative, it completes the proof. □

Finally, we are ready to prove our main result.

Theorem 3. *Let R be a prime ring with Char$(R) \neq 2$, L a nonzero Lie ideal of R, U the Utumi quotient ring and C the extended centroid of R. Suppose that R admits an automorphism σ such that $\sigma(u)^n = \sigma(u)$ for all $u \in L$, where $n > 1$ is a fixed positive integer, then either $L \subseteq Z(R)$ or L is commutative and R satisfies s_4.*

Proof. If $L \subseteq Z(R)$, then we are done. Let us suppose that $L \nsubseteq Z(R)$. By hypothesis, we have $\sigma(u)^n = \sigma(u)$ for all $u \in L$. If L is not commutative, then by Fact 4 and Fact 6, it follows that $\sigma([x,y])^n = \sigma([x,y])$ for all $x \in R$. We now have the following two cases (see [12]).

Case 1. Let us assume that σ is not U-inner, i.e., σ is U-outer. Then a contradiction follows from Lemma 2.

Case 2. If σ is U-inner, i.e., there exists an invertible $q \in U$ such that $\sigma(x) = qxq^{-1}$ for all $x \in R$, then for all $x, y \in R$, we have

$$(q[x,y]q^{-1})^n = q[x,y]q^{-1}.$$

Further if $q \in C$, then we have nothing to prove. Thus $q \notin C$. It implies that $\Lambda(x,y) = (q[x,y]q^{-1})^n - q[x,y]q^{-1}$ is a nontrivial generalized polynomial identity for R as well as for U.

As in the case of generalized derivations, we mention that $\overline{C}$ is the algebraic closure of C if C is infinite and $C = \overline{C}$ if C is finite. One may observe that $U \cong U \otimes_C C \subseteq U \otimes_C \overline{C}$, and $U \otimes_C \overline{C}$ is a prime ring with extended centroid $\overline{C}$ (see [8, Theorem 3.5]). Thus $\Lambda(x,y)$ is a nontrivial generalized polynomial identity for $U \otimes_C \overline{C}$. In view of Theorem 6.4.4 of [3], U' (the Utumi quotient ring of $U \otimes_C \overline{C}$) also satisfies the nontrivial generalized polynomial identity $\Lambda(x,y)$. In addition, by Martindale [18], we find that $U' \cong End(V_D)$. Therefore by invoking Lemma 1, we have $[x,y]^n = [x,y]$ for all $x, y \in R$, which implies that R is commutative, a contradiction. Hence L is commutative and by [15, Theorem 4], R satisfies s_4. □

Corollary 2. *Let R be a prime ring with $Char(R) \neq 2$. If R admits an automorphism σ such that $\sigma(x)^n = \sigma(x)$ for all $x \in R$, where $n > 1$ is a fixed positive integer, then R is commutative.*

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