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An elementary description of $K_1(R)$ without elementary matrices

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Abstract. Let R be a ring with unit. Passing to the colimit with respect to the standard inclusions $\mathrm{GL}(n,R) \longrightarrow \mathrm{GL}(n+1,R)$ (which add a unit vector as new last row and column) yields, by definition, the stable linear group $\mathrm{GL}(R)$; the same result is obtained, up to isomorphism, when using the "opposite" inclusions (which add a unit vector as new first row and column). In this note it is shown that passing to the colimit along both these families of inclusions simultaneously recovers the algebraic K-group $K_1(R) = \mathrm{GL}(R)/E(R)$ of R, giving an elementary description that does not involve elementary matrices explicitly.

Let R be an associative ring with unit element 1, and let $\mathrm{GL}(n,R)$ denote the group of invertible $n \times n$-matrices with entries in R. The usual stabilisation maps

$$i^n_{n+1} \colon \mathrm{GL}(n,R) \longrightarrow \mathrm{GL}(n+1,R) \ , \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

are used to define the stable general linear group $\mathrm{GL}(R) = \bigcup_{n \geqslant 3} \mathrm{GL}(n,R)$, or, phrased in categorical language,

$$\mathrm{GL}(R) = \operatorname{colim} \Big(\mathrm{GL}(3,R) \xrightarrow{i^3_4} \mathrm{GL}(4,R) \xrightarrow{i^4_5} \mathrm{GL}(5,R) \xrightarrow{i^5_6} \ \dots \ \Big) \, . \tag{1}$$

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The canonical group homomorphisms $\iota_n \colon \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(R)$ are injective and satisfy the relation

$$\iota_{n+1} \circ i^n_{n+1} = \iota_n \ . \tag{2}$$

There are other "block-diagonal" embeddings

$$i^n_j \colon \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n+1, R),$$

for $1 \leqslant j \leqslant n+1$, characterised by saying that the jth row and jth column of $i^n_j(A)$ are jth unit vectors, and that deleting these from $i^n_j(A)$ recovers the matrix A. We will determine the result of stabilising over first and last embeddings simultaneously, that is, we identify the categorical colimit M of the following group-valued infinite diagram:

$$\mathrm{GL}(3, R) \overset{i^3_1}{\underset{i^3_4}{\rightrightarrows}} \mathrm{GL}(4, R) \overset{i^4_1}{\underset{i^4_5}{\rightrightarrows}} \cdots \overset{i^{n-1}_1}{\underset{i^{n-1}_n}{\rightrightarrows}} \mathrm{GL}(n, R) \overset{i^n_1}{\underset{i^n_{n+1}}{\rightrightarrows}} \cdots \ . \tag{3}$$

By the general theory of colimits, the group M comes equipped with canonical group homomorphisms $\alpha_n \colon \mathrm{GL}(n, R) \longrightarrow M$ satisfying the relations

$$\alpha_{n+1} \circ i^n_j = \alpha_n \quad (j = 1,\, n+1) \ . \tag{4}$$

Theorem. *The group M is canonically isomorphic to $K_1(R)$.*

Proof. First we observe that in M we have the commutation relation

$$\alpha_n(X)\alpha_n(Y) = \alpha_n(Y)\alpha_n(X) \quad \text{for all } X, Y \in \mathrm{GL}(n, R) \ . \tag{5}$$

Indeed, by (4) we can re-write

$$\alpha_n(X) = \alpha_{2n}\big(i^{2n-1}_{2n} i^{2n-2}_{2n-1} \dots i^n_{n+1}(X)\big)$$

and

$$\alpha_n(Y) = \alpha_{2n}\big(i^{2n-1}_1 i^{2n-2}_1 \dots i^n_1(Y)\big) \ ,$$

and the arguments of α_{2n} are block-diagonal matrices of the form

$$\begin{pmatrix} X & 0 \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_n & 0 \\ 0 & Y \end{pmatrix}$$

which commute in $\mathrm{GL}(2n, R)$; hence their images under α_{2n} must commute as well.

Let $E(n,R)$ denote the subgroup of $\mathrm{GL}(n,R)$ generated by the elementary matrices [1, §V.1]. Since $E(n,R) = \big[E(n,R),\ E(n,R)\big]$ for all $n \geqslant 3$, cf. [1, Corollary V.1.5], the commutation relation (5) implies

$$E(n,R) \subseteq \ker(\alpha_n) \ . \tag{6}$$

Since the diagram (1) defining $\mathrm{GL}(R)$ is contained in the diagram (3) defining M there is a canonical group homomorphism $\alpha\colon \mathrm{GL}(R) \longrightarrow M$ described completely by $\alpha \circ \iota_n = \alpha_n$, that is, $\alpha|_{\mathrm{GL}(n,R)} = \alpha_n$.

Let $E(R) = \bigcup_{n\geqslant 3} E(n,R) \subseteq \mathrm{GL}(R)$ be the stabilisation *via* the embeddings i^n_{n+1}. In view of (6) above we have the inclusion

$$E(R) = \bigcup_{n\geqslant 3} E(n,R) \subseteq \ker\alpha \ . \tag{7}$$

The group $E(R)$ is normal in $\mathrm{GL}(R)$, cf. [1, Theorem V.2.1], and $K_1(R) = \mathrm{GL}(R)/E(R)$ is an ABELian group [1, p. 229]. We write $\pi\colon \mathrm{GL}(R) \longrightarrow K_1(R)$ for the canonical projection. Let $\pi_n = \pi \circ \iota_n$ denote the restriction of π to $\mathrm{GL}(n,R)$, and write $[X] = \pi_n(X)$ for the class of $X \in \mathrm{GL}(n,R)$ in $K_1(R)$. By (7) we obtain a factorisation $\lambda\colon K_1(R) \longrightarrow M$ of α with $\lambda \circ \pi = \alpha$. Explicitly, λ is described by the formula

$$\lambda\colon [X] = \pi_n(X) \ \mapsto\ \alpha \circ \iota_n(X) = \alpha_n(X)\ , \quad \text{for } X \in \mathrm{GL}(n,R)\ . \tag{8}$$

We observe the relation $\pi_{n+1} \circ i^n_j = \pi_n$. Indeed, for $X \in \mathrm{GL}(n,R)$ the matrices $i^n_j(X)$ and $i^n_{n+1}(X)$ are related by the expression

$$i^n_j(X) = P^{-1} i^n_{n+1}(X) P$$

for a permutation matrix $P \in \mathrm{GL}(n+1,R)$. It follows that said two matrices have the same image under π_{n+1} in the ABELian group $K_1(R)$ whence, using (2),

$$\pi_{n+1} \circ i^n_j(X) = \pi_{n+1} \circ i^n_{n+1}(X) = \pi \circ \iota_{n+1} \circ i^n_{n+1}(X) = \pi \circ \iota_n(X) = \pi_n(X)\ .$$

The various maps π_n thus form a "cone" on the diagram (3) and induce a map $\varrho\colon M \longrightarrow K_1(R)$ such that

$$\pi_n = \varrho \circ \alpha_n \ . \tag{9}$$

We verify the equality $\lambda \circ \varrho = \mathrm{id}_M$. By the universal property of colimits, it is enough to show that $\lambda \circ \varrho \circ \alpha_n = \alpha_n$ for all n. But for $X \in \mathrm{GL}(n, R)$ we calculate

$$\lambda \circ \varrho \circ \alpha_n(X) \underset{(9)}{=} \lambda \circ \pi_n(X) \underset{(8)}{=} \alpha_n(X) \ ,$$

using relation (9) and the explicit description (8) of λ above.

Using the same relations again, in opposite order, we finally verify that $\varrho \circ \lambda = \mathrm{id}_{K_1(R)}$. Let $X \in \mathrm{GL}(n, R)$ represent the element $[X] \in K_1(R)$ as before; then $\varrho \circ \lambda\big([X]\big) \underset{(8)}{=} \varrho \circ \alpha_n(X) \underset{(9)}{=} \pi_n(X) = [X]$. □

With minor changes the argument also shows that the the colimit of the group-valued diagram

$$\mathrm{GL}(3,R) \overset{i^3_1}{\underset{i^3_4}{\rightrightarrows\!\!\rightrightarrows}} \mathrm{GL}(4,R) \overset{i^4_1}{\underset{i^4_5}{\rightrightarrows\!\!\rightrightarrows\!\!\to}} \cdots \begin{matrix} \xrightarrow{i^{n-1}_1} \\ \vdots \\ \xrightarrow[i^{n-1}_n]{} \end{matrix} \mathrm{GL}(n,R) \begin{matrix} \xrightarrow{i^{n}_1} \\ \vdots \\ \xrightarrow[i^{n}_{n+1}]{} \end{matrix} \cdots$$

is canonically isomorphic to $K_1(R)$.

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