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Enumeration of strong dichotomy patterns

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Abstract. We apply the version of Pólya-Redfield theory obtained by White to count patterns with a given automorphism group to the enumeration of strong dichotomy patterns, that is, we count bicolor patterns of $\mathbb{Z}_{2k}$ with respect to the action of $\mathrm{Aff}(\mathbb{Z}_{2k})$ and with trivial isotropy group. As a byproduct, a conjectural instance of phenomenon similar to cyclic sieving for special cases of these combinatorial objects is proposed.

1. Introduction

In a short and beautiful paper [10], White proved an analogue of Cauchy-Frobenius-Burnside lemma tailored for the purpose of counting patterns with a fixed group of automorphisms. Before stating it, we warn the reader that we will use the Iverson bracket[1] as defined by Graham, Knuth and Patashnik [4, p. 24]: if P is a property, then

$$[P] = \begin{cases} 1, & P \text{ is true}, \\ 0, & \text{otherwise}. \end{cases}$$

Theorem 1 (D. E. White, 1975). *Let S be a finite set, G a finite group acting on S and Δ a system of orbit representatives for G acting on S.*

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[1] White uses the similar notation $\chi(P)$ in his articles, which was introduced first by Adriano Garsia in a paper from 1979 according to Knuth [6], although White's paper predates it by four years.

Suppose $\{G_1, \ldots, G_N\}$ *is a traversal of the orbits of the subgroups of* G *under the conjugation action, such that*

$$|G_1| \geqslant \cdots \geqslant |G_N|.$$

Given a weight function $w : S \to T$ *such that* $w(\sigma s) = w(s)$ *for all* $\sigma \in G$ *and all* $s \in S$*, we have*

$$\sum_{s\in\Delta} w(s)[G_s \sim G_i] = \sum_{j=1}^{N} b_{i,j} \sum_{s\in S} w(s)[G_j s = s],$$

where $B = (b_{i,j})$ *is the inverse of the* table of marks *matrix*

$$M_{i,j} = \frac{1}{|G_j|} \sum_{\sigma\in G} [\sigma G_i \sigma^{-1} \subseteq G_j].$$

Note that the table of marks matrix is invertible because it is triangular and no element of the diagonal is 0.

Let $S = R^D$, where both R and D are finite sets. If G acts on R (the set of *colors*), it is well known that the action can be extended to S defining $\sigma \cdot f = f \circ \sigma^{-1}$, where σ is reinterpreted as a member of the permutation group of D. We know that the action of G_i on D defines a set of disjoint orbits

$$O_{G_i:D} := \{G_i x_1, \ldots, G_i x_\ell\}$$

which is a partition of D, so we can define

$$q_{G_i}(d) = \sum_{i=1}^{\ell} [|Gx_i| = d].$$

This allows us to define the *orbit index monomial* as

$$P_i(z_1, z_2, \ldots, z_{|D|}) := \prod_{d\in D} z_d^{q_{G_i}(d)},$$

which can be used in a straightforward manner to obtain a pattern inventory polynomial.

Theorem 2 (D. E. White, 1975)**.** *The pattern inventory polynomial for patterns fixed by the subgroup* G_i *is*

$$Q_i = \sum_{j=1}^{N} b_{i,j} P_j(z_1, z_2, \ldots, z_{|D|}),$$

where the substitution $y_i = \sum_{r\in R} x_r^i$ *is made.*

We will use White's results (and one further generalization obtained by him that we will discuss later) to count bicolor patterns under a group G which are of particular interest for mathematical musicology. The usual choices for G are cyclic, dihedral and general affine groups, since they model common musically meaningful transformations, such as transpositions, inversions, retrogradations and others related to twelve-tone techniques, to name a few (see [7, Chapter 8] for more examples).

One reason to study this kind of combinatorial objects is that they represent rhythmic patterns if they are interpreted as onsets in a measure (see [2] and [5] and the references therein for more information). Another reason is that they can be seen as abstractions of the concepts of consonance and dissonance in Renaissance counterpoint. In particular, *self-complementary* (that is, those whose complement belongs to its orbit) and *rigid* (which means that they are invariant only under the identity) patterns, hereafter called *strong*, are known to be used in both Western and Eastern music [7, Part VII], and that their combinatorial structure lead to significant musicological results [7, Chapter 31]. Note in passing that self-complementarity forces the patterns to be subsets of cardinality k of sets of even cardinality $2k$. In general, *dichotomy* patterns are those of cardinality k within a set of cardinality $2k$.

In the following section we provide simple examples of White's theory in action to explain the algorithms we use in the main computations. The results of these calculations appear in Section 3, and in the final section we provide some further comments regarding them.

2. Two easy examples

Suppose we color black or white the vertices of a rectangle that is not a square. The group of symmetries acting on the colorings of the vertices is the Klein four-group

$$V = \langle a, b | a^2 = b^2 = (ab)^2 = e\rangle.$$

We will find the patterns that are invariant under $G_1 = V$, $G_2 = \langle a\rangle$, $G_3 = \langle b\rangle$, $G_4 = \langle ab\rangle$ and $G_5 = \langle e\rangle$ using White's formulas. Since all the proper subgroups of V are normal, we easily calculate the table of marks matrix

$$M_V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 2 & 2 & 2 & 4 \end{pmatrix},$$

whose inverse is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

It is illustrative to make explicit the formula of Theorem 1 for this simple example. Let us code the colorings of the vertices with the strings $u_1u_2u_3u_4$ over the alphabet $\{n, b\}$, using the clockwise order and beginning from the upper left corner. Then

$$\Delta = \{nnnn, nnnb, nnbb, nbbn, nbnb, nbbb, nnnn\}.$$

For G_1 the formula trivially asserts that the only patterns invariant under the full group are the monochromatic ones. For G_2 we have

$$\begin{aligned} &-\tfrac{1}{2}(w(nnnn) + w(bbbb)) \\ &+\tfrac{1}{2}(w(nnnn) + w(nnbb) + w(bbnn) + w(bbbb)) \\ =\ &\tfrac{1}{2}(w(bbnn) + w(nnbb)) = w(nnbb) \end{aligned}$$

because $w(bbnn) = w(nnbb)$, by hypothesis. The colorings $bbnn$ and $nnbb$ are precisely those who represent the only pattern which is invariant under the reflection with vertical axis. The cases of G_3 and G_4 are analogous. Finally, the case of the trivial subgroup is more interesting:

$$\begin{aligned} &\tfrac{1}{2}(w(nnnn) + w(bbbb)) \\ &-\tfrac{1}{4}(w(nnnn) + w(nnbb) + w(bbnn) + w(bbbb)) \\ &-\tfrac{1}{4}(w(nnnn) + w(nbbn) + w(bnnb) + w(bbbb)) \\ &-\tfrac{1}{4}(w(nnnn) + w(nbnb) + w(bnbn) + w(bbbb)) \\ &+\tfrac{1}{4}\sum_{\text{all strings}} w(s) \\ =\ &\tfrac{1}{4}(w(nnnb) + w(nnbn) + w(nbnn) + w(nbbb) \\ &\quad + w(bnnn) + w(bnbb) + w(bbnb) + w(bbbn)) \\ =\ &w(nnnb) + w(nbbb), \end{aligned}$$

and it informs us of the two patterns that are invariant under the action of the trivial subgroup only; they are precisely those with only one black or only one white vertex.

Let us confirm the former using the orbit index polynomials for each subgroup. For G_1, we have only one orbit of four elements, thus

$$P_1 = z_4.$$

The orbits defined by G_2, G_3 and G_4 are all of cardinality two, thus

$$P_2 = P_3 = P_4 = z_2^2.$$

Finally, there are four orbits of cardinality one for the trivial subgroup, hence

$$P_5 = z_1^4.$$

Using these polynomials, we can calculate all the pattern inventories at once:

$$\begin{aligned}
\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} z_4 \\ z_2^2 \\ z_2^2 \\ z_2^2 \\ z_1^4 \end{pmatrix} \\
&= \begin{pmatrix} z_4 \\ \frac{1}{2}z_2^2 - \frac{1}{2}z_4 \\ \frac{1}{2}z_2^2 - \frac{1}{2}z_4 \\ \frac{1}{2}z_2^2 - \frac{1}{2}z_4 \\ \frac{1}{2}z_4 - \frac{3}{4}z_2^2 + \frac{1}{4}z_1^4 \end{pmatrix}.
\end{aligned}$$

Upon the substitution $z_i = 1 + x^i$, that allows us to count the number of bicolor patterns according to the number of black elements (say), we find

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix} = \begin{pmatrix} 1 + x^4 \\ x^2 \\ x^2 \\ x^2 \\ x + x^3 \end{pmatrix}.$$

We proceed now with a more complicated example that introduces the group we will use in our main computation. Define

$$\mathrm{Aff}(\mathbb{Z}_{2k}) = \mathbb{Z}/2k\mathbb{Z} \ltimes \mathbb{Z}/2k\mathbb{Z}^\times.$$

Denote an element $(u, v) \in \mathrm{Aff}(\mathbb{Z}_{2k})$ by $e^u.v$. The action of $\mathrm{Aff}(\mathbb{Z}_{2k})$ on $\mathbb{Z}/2k\mathbb{Z}$ is given by

$$e^u.v(x) = vx + u.$$

Let us compute the number of patterns of the action of $\mathrm{Aff}(\mathbb{Z}_6)$. We have the following sequence of normal subgroups,

$$G_1 = \langle e^1.1, e^0.5\rangle, \quad G_2 = \langle e^4.1, e^5.5\rangle, \quad G_3 = \langle e^1.1, e^0.5\rangle, \quad G_4 = \langle e^1.1\rangle,$$
$$G_5 = \langle e^3.1, e^0.5\rangle, \quad G_6 = \langle e^2.1\rangle, \quad G_7 = \langle e^5.5\rangle, \quad G_8 = \langle e^0.5\rangle,$$
$$G_9 = \langle e^3.1\rangle, G_{10} = \{e^0.1\},$$

The computation of the table of marks matrix is not as direct as before, in part because the subgroups G_5, G_7 and G_9 are not normal. But using GAP [3] we readily find

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 4 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 6 & 0 \\ 1 & 2 & 2 & 2 & 3 & 4 & 6 & 6 & 6 & 12 \end{pmatrix}$$

whose inverse is

$$B = \begin{pmatrix} \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{12} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{12} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & -\frac{1}{6} & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & -\frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

The orbit index polynomials are

$$P_1 = z_6, \quad P_2 = z_6, \quad P_3 = z_3^2, \quad P_4 = z_6, \quad P_5 = z_2 z_4, \quad P_6 = z_3^2,$$
$$P_7 = z_2^3, \quad P_8 = z_1^2 z_2^2, \quad P_9 = z_2^3, \quad P_{10} = z_1^6$$

whence

$$Q(z_1,\dots,z_6) = BP = \begin{pmatrix} z_6 \\ 0 \\ \frac{1}{2}z_3^2 - \frac{1}{2}z_6 \\ 0 \\ z_2z_4 - z_6 \\ 0 \\ -\frac{1}{2}z_2^3 - \frac{1}{2}z_2z_4 \\ \frac{1}{2}z_6 - \frac{1}{2}z_2z_4 - \frac{1}{2}z_3^2 + \frac{1}{2}z_1^2z_2^2 \\ \frac{1}{3}z_6 - \frac{1}{2}z_2z_4 + \frac{1}{6}z_2^3 \\ -\frac{1}{6}z_6 + \frac{1}{2}z_2z_4 + \frac{1}{6}z_3^2 - \frac{1}{3}z_2^3 - \frac{1}{4}z_1^2z_2^2 + \frac{1}{12}z_1^6 \end{pmatrix}$$

thus

$$Q(1+x,\dots,1+x^6) = \begin{pmatrix} x^6+1 \\ 0 \\ x^3 \\ 0 \\ x^4+x^2 \\ 0 \\ x^4+x^2 \\ x^5+x^4+x^3+x^2+x \\ 0 \\ x^3 \end{pmatrix}.$$

It is interesting to learn that there are no patterns that are exclusively invariant under the subgroups generated, respectively, by the translations $e^1.1$, $e^2.1$, $e^3.1$. In other words: arithmetic progressions with common difference 1, 2 and 3 are invariant under symmetries that are not translations.

3. Main calculations

Denoting by $\mathcal{D}$ the set of dichotomies, and by $\mathcal{S}$ and $\mathcal{R}$ the subsets of the self-complementary and rigid dichotomies (respectively), we know by the principle of inclusion and exclusion (PIE) that

$$|\mathcal{D}| \geqslant |\mathcal{S}\cup\mathcal{R}| = |\mathcal{S}| + |\mathcal{R}| - |\mathcal{S}\cap\mathcal{R}|$$

where $|\mathcal{S}\cap\mathcal{R}|$ is precisely the number of strong dichotomies. Hence

$$|\mathcal{S}\cap\mathcal{R}| \geqslant |\mathcal{S}| + |\mathcal{R}| - |\mathcal{D}|.$$

We can calculate $|\mathcal{D}|$ and $|\mathcal{S}|$ with the classical Pólya-Redfield theory, and $|\mathcal{R}|$ with White's formulas, so we may expect this inequality to provide reasonably good bounds on the number of strong dichotomies. But, unfortunately, in general it does not, as we can readily see in Table 1, since many of them are negative.

However, not everything is lost. After examining the cases when the PIE yields a nontrivial bound, we discover that this happens when k is a power of a prime and, more importantly, the value of $|Q_1(-1)|$ coincides with the number of strong dichotomy patterns calculated by direct construction for these cases (see [1]). On the other hand, it is known that the classical pattern inventory polynomials of the Pólya-Redfield theory exhibit a form of the *cyclic sieving phenomenon* [8, Corollary 6.2], which means that if $p(x)$ is the generating function of the number of patterns according to its number of black elements, then $p(-1)$ yields the number of self-complementary patterns.

Since the polynomials for White's formulas do not count cycles but orbits, in general they fail to cyclically sieve patterns, but we may expect it to work when $\mathbb{Z}_{2k}^{\times}$ is cyclic. Indeed, if the group of units is generated by a single element, it is plausible to think that all the orbits are cycles of $e^1.1$ and a generator of $\mathbb{Z}_{2k}^{\times}$. Furthermore, it is a well-known fact that the group of units of $\mathbb{Z}_n$ is cyclic precisely when $n = 1, 2, p^k$, where p is a prime number. This discussion, however, is not a full proof, so we formalize it as a conjecture. Hopefully, it will be proved soon.

Conjecture 1. Let $G_N = \mathrm{Aff}(\mathbb{Z}_{2k})$ and $\{G_i\}$ be a set of representatives of the orbits of the conjugation action such that $|G_N| \geqslant \cdots \geqslant |G_1|$ and let $B = (b_{i,j})$ be the inverse of its table of marks. If k is equal to 1, 2 or a power of an odd prime number, then the pattern inventory polynomial for bicolor patterns fixed by the subgroup G_i

$$Q_i = \sum_{j=1}^{N} b_{i,j} P_j(1 + x, 1 + x^2, \ldots, 1 + x^{|D|}),$$

is such that $|Q_i(-1)|$ counts the number of self-complementary dichotomies with automorphism group G_i. In particular, $|Q_1(-1)|$ counts the number of strong dichotomies.

For the general case, we can use another formula of White [9]. Now we need to consider the swapping action on the colors of the patterns simultaneously with that of the affine group, so we see $G = \mathrm{Aff}(\mathbb{Z}_{2k}) \times \mathbb{Z}_2$

$2k$	$\lvert\mathcal{D}\rvert$	$\lvert\mathcal{S}\rvert$	$\lvert\mathcal{R}\rvert$	PIE bound	$\lvert Q_1(-1)\rvert$
2	1	1	1	1	1
4	2	2	0	0	0
6	3	3	0	0	1
8	6	4	1	−1	1
10	9	7	5	3	3
12	34	18	10	−6	4
14	47	15	37	5	9
16	129	21	83	−25	1
18	471	55	436	20	40
20	1280	134	1052	−94	66
22	3235	115	3181	61	105
24	15008	440	13331	−1237	33
26	33429	385	33253	209	355
28	121466	1194	117422	−2850	886
30	648819	3365	643901	−1153	3007
32	1182781	2189	1165498	−15094	1432
34	4290533	4375	4288913	2755	4305
36	21082620	18404	20933318	−130898	15518
38	51677171	15347	51671611	9787	15267
40	215804540	49684	214972319	−782537	25659
42	1068159497	133285	1067785287	−240925	130839
44	2392981542	171662	2389064994	−3744886	155346
46	8135833183	198943	8135769049	134809	198753
48	42007923187	786707	41970277573	−36858907	643019
50	126410742103	872893	126410471144	601934	871992

TABLE 1. Summary of the information that can be obtained via the classical Pólya-Redfield theory and White's extension, for $1 \leqslant k \leqslant 25$.

as acting both in R and D, according to

$$(\sigma,\tau)\cdot r = \tau \cdot r \quad \text{and} \quad (\sigma,\tau)\cdot d = \sigma \cdot d;$$

hence G acts doubly on R^D in the following manner

$$g \cdot f = g \circ f \circ g^{-1}.$$

We provide a quick sketch of White's reasoning to obtain the counting formula, in part because our problem's conditions lead to a simpler statement and in part because his original paper has some minor (but misleading) typographical errors.

In order to apply Theorem 1, we must characterize first the patterns f such that $H' \subseteq G_f$. If a $g \in H'$ leaves a pattern invariant, it means that it sends an element of $O_{H':D}$ to an element of $O_{H':R}$.

Thus, let $B \in O_{H':D}$ and $f(B) = C \in O_{H':R}$. Taking arbitrary elements $b \in B$ and $c \in C$, we deduce that f must be defined by $f(\gamma_1 b) = \gamma_1 c$. This relation, however, might not be functional, for it may happen that $f(\gamma_1 b) \neq f(\gamma_2 b)$ when $\gamma_1 b = \gamma_2 b$, unless $\gamma_1 c = \gamma_2 c$, or $\gamma_1^{-1}\gamma_2 \in H_c$. In other words, if the function is well defined then

$$\gamma_1^{-1}\gamma_2 \in H_b \quad \text{implies that} \quad \gamma_1^{-1}\gamma_2 \in H_c \tag{1}$$

or, equivalently,

$$H_b \subseteq H_c$$

(note that these isotropy groups are relative to H). To check that (1) is also sufficient is direct, like the fact that the election of b is irrelevant.

We have

$$\sum_{f\in S} w(s)[G_j f = f] = \sum_{\hat{f}\in O_{H:R}^{O_{H:D}}} \prod_{B\in O_{H:D}} \sum_{j=0}^{|\hat{f}(B)|-1} [H_b \subseteq H_{\tau_j c}] w(f)$$

and, reorganizing the terms (in what White calls *sum-product interchange*), we get

$$\sum_{f\in S} w(f)[G_j f = f] = \prod_{B\in O_{H:D}} \sum_{C\in O_{H:R}} \sum_{j=0}^{|C|-1} [H_b \subseteq H_{\tau_j c}] w(f) \tag{2}$$

where

$$w(f) = \prod_{i=0}^{|B|-1} x_{\tau_j c}. \tag{3}$$

For bicolor patterns we have $C = \{0, 1\}$, therefore the invariance under the action of the whole group reduces the weights $w(f)$ to the following choices:

$$w(f) = \begin{cases} x_0^{|B|/2} x_1^{|B|/2}, & H \text{ swaps colors,} \\ x_0^{|B|} = x_1^{|B|}, & \text{otherwise.} \end{cases}$$

Thus we can reuse the previous algorithm that involves the inverse of the table of marks matrix, but with the larger group $\mathrm{Aff}(\mathbb{Z}_{2k}) \times \mathbb{Z}_2$ and calculating the corresponding vector of polynomials with (2) and (3). The only remaining detail is that no longer we may read the total number

$2k$	$\lvert\mathcal{S}\cap\mathcal{R}\rvert$
6	1
8	1
12	$2+4=6$
16	$1+14=15$
20	$3+6+54+27=90$
24	$14+54+63+228=359$
28	$38+76+326+652=1092$
32	$120+2032=2152$
36	$560+1120+5382+10764=17826$
40	$1572+6357+8100+32520=48549$
42	$3936+12135+28320+86448=130839$
44	$4662+9324+52278+104556=170820$
48	$21435+65040+172410+521760=780645$

Table 2. Summary of the information that can be obtained via White's extension of Pólya-Redfield theory for strong dichotomy patterns and selected values of k. The totals of strong dichotomies are displayed as sums, where each summand represents the number of patterns with a specific polarity. Thus, the number of summands on each row is the number of polarities.

of strong dichotomy patterns in a single entry of the output vector, for such patterns have automorphism groups of cardinality two; namely, the identity $(e^0.1, 0)$ and the color swap $(e^0.1, 1)$ composed with a unique symmetry of $\mathrm{Aff}(\mathbb{Z}_{2k})$, which is called the *polarity* of the pattern. Hence, we gain a feature and not an inconvenience, for now we can know the number of strong dichotomy patterns for each polarity.

The first case that is not covered by Conjecture 1 is $n = 8$, but the table of marks matrix is of size 148×148, so we will not display it here. Let us simply state that there is only one strong dichotomy, whose polarity is $e^5.-1$. In Table 2 we summarize the information that can be calculated with this algorithm up to $2k = 48$.

4. Concluding remarks

The enumerations of strong dichotomies done here coincide with the explicit ones performed in [1] and subsequent verifications done by the author, with a variation of the original algorithm presented in [1]. It is interesting to note that Conjecture 1 is of practical interest, since it significantly simplifies the computation of the table of marks: we should consider that the volume of calculations is exacerbated when we have to calculate with the product $\mathrm{Aff}(\mathbb{Z}_{2k}) \times \mathbb{Z}_2$; its table of marks can be much bigger that the one of its largest factor.

Harald Fripertinger noted in a personal communication with the author that the number of self-complementary patterns $|\mathcal{S}|$ seems to approach asymptotically to the number of the strong ones (or, equivalently, that the vast majority of dichotomies is rigid). In particular, $|\mathcal{S}|$ provides a direct and fast way (it does not require to compute the table of marks) to determine a very good upper bound for the number of strong patterns, a useful fact in order to partially validate the exact (but lengthy) calculations.

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