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RESEARCH ARTICLE

An amalgamation property for metric groups

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Abstract. In this paper we show that under some mild assumptions two copies of a metric group can be freely amalgamated over any central subgroup so that the distance between them is sufficiently small.

1. Introduction

The main result of paper [4] is an amalgamation property which roughly states that if A and B are finite metric spaces which are sufficiently similar then there is a metric on $A \cup B$ extending the metrics of A and B so that A and B are sufficiently close in $A \cup B$. The proof is based on the following proposition (Proposition 1 in [4]).

Proposition 1. *Let a finite metric space $A = \{a_1, ..., a_n\}$ and numbers $0 \leqslant q < n$ and $\varepsilon > 0$ satisfy all inequalities of the form*

$$4\varepsilon < d(a_i, a_j) \text{ for pairs } i < j \leqslant n \text{ with } q < j \text{ and}$$

$$4\varepsilon < d(a_i, a_j)+d(a_i, a_k)-d(a_j, a_k) \text{ for triples } a_i, a_j, a_k \text{ with } |\{i, j, k\}| = 3$$
$$\text{and } k \leqslant q < min(i, j) < n.$$

Let B be an n-element metric space consisting of elements b_i so that for each pair $i < j \leqslant n$, $|d(b_i, b_j) - d(a_i, a_j)| \leqslant \varepsilon$. We assume that $a_1 = b_1$,..., $a_q = b_q$, $A \cap B = \{a_1, ..., a_q\}$, and the metric defined on $\{b_1, ..., b_q\}$ in the space B coincides with the metric defined on $\{a_1, ..., a_q\}$ in A.

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Then there is a metric on $A \cup B$ extending metrics in A and B so that for each $q < i \leqslant n$, $d(a_i, b_i) = \varepsilon$.

It is clear that the second collection of inequalities given in the formulation is just a version of the triangle inequality.

The main result of our note is an algebraic version of this proposition. Removing the assumption of finiteness of A, B and $A \cap B$ we add an operation making A and B metric groups (i.e. the metric is invariant) and $A \cap B$ a subgroup of both A and B. Since in this case $A \cup B$ is not a group we replace it by an amalgamation over $A \cap B$. One of the simplest possibilities is the amalgamated sum $A \oplus_{A \cap B} B$. In this case we obviously need the assumption that $A \cap B$ is contained in the intersection of the centres of A and B. Furthermore we replace the condition of similarity $|d(b_i, b_j) - d(a_i, a_j)| \leqslant \varepsilon$, $i < j \leqslant n$, just by the assumption that B is an isometric copy of A. Finaly this proposition looks as follows.

Theorem 1. *Let (G, d_G) be a a metric group, H be a subgroup of $Z(G)$ and ε be a positive real number which satisfies all inequalities of the following form*

$$4\varepsilon < d_G(a, b) \text{ for pairs } a \neq b \text{ of } G \text{ and}$$

$$4\varepsilon < d_G(a, c) + d_G(c, b) - d_G(a, b) \text{ for triples } a, b, c \in G$$

$$\text{with } |\{a, b, c\}| = 3.$$

Then there is a metric $\hat{d}$ on $G \oplus_H G$ extending d_G on every summand so that for each $a \in G \setminus H$, $\hat{d}((a, 1), (1, a)) = \varepsilon$.

Moreover the metric $\hat{d}$ is invariant if and only if G is abelian.

In this formulation the left copy of G consists of the set $\{(a, 1) : a \in G\}$ and the right one is $\{(1, a) : a \in G\}$. For $a \in H$ we add a relation identifying each $(a, 1)$ with $(1, a)$. It is worth noting here that inequalities of the formulation are equivalent to ones where a is replaced by the unit 1. Thus they exactly correspond to inequalities of Proposition 1.

2. Proof of Theorem 1. Amalgamation

We define $\hat{d}$ on $G \oplus_H G$ as follows:

- when $((a')^{-1}a, b'b^{-1})$ belongs to $(G \times H) \cup (H \times G)$ then let $\hat{d}((a, b), (a', b')) = d_G((a')^{-1}a, b'b^{-1})$;
- when $((a')^{-1}a, b'b^{-1}) \notin (G \times H) \cup (H \times G)$ then let $\hat{d}((a, b), (a', b')) = d_G((a')^{-1}a, b'b^{-1}) + \varepsilon$.

It is clear that $d((a,1),(1,a)) = \varepsilon$ for all $a \in G \setminus H$. It is worth noting that by invariantness of d_G we have $d_G((a')^{-1}a, b'b^{-1}) = d_G(ab, a'b')$. Thus when $\hat{d}((a,b),(a',b')) = 0$, we have $ab = a'b'$ and $(a')^{-1}a = b'b^{-1}$. Moreover the latter value must belong to H; we see $((a')^{-1}a, 1) = (1, b'b^{-1})$ and

$$(a,b) = (a',1)\cdot((a')^{-1}a,1)\cdot(1,b) = (a',1)\cdot(1,b'b^{-1})\cdot(1,b) = (a',b').$$

To verify that $\hat{d}$ is a metric we need to show the triangle inequality. Let $(a_i, b_i) \in G \times G$, where $i \in \{1,2,3\}$. We start with the observation that when $\hat{d}((a_1,b_1),(a_2,b_2)) = \varepsilon$, then

$$a_2^{-1}a_1 = b_2b_1^{-1} \notin H.$$

Using this and metric invariance we see that

$$d_G(a_2^{-1}a_3, b_2b_3^{-1}) = d_G(b_2b_1^{-1}a_1^{-1}a_3, b_2b_3^{-1}) = d_G(a_1^{-1}a_3, b_1b_3^{-1}).$$

When this number is not 0, it is greater than ε. Thus the definition of $\hat{d}$ says that

- the triangle inequality for $\hat{d}$ holds in any triangle where one of the sides is ε.

Assume that all sides are greater than ε. Then for any $i, j \in \{1,2,3\}$ we have

$$|\hat{d}((a_i,b_i),(a_j,b_j)) - d_G(a_ib_i, a_jb_j)| \leqslant \varepsilon.$$

In particular the value

$$\hat{d}((a_1,b_1),(a_2,b_2)) + \hat{d}((a_3,b_3),(a_2,b_2)) - \hat{d}((a_1,b_1),(a_3,b_3))$$

does not differ from

$$d_G(a_1b_1, a_2b_2) + d_G(a_3b_3, a_2b_2) - d_G(a_1b_1, a_3b_3)$$

more than 3ε. Since the second one is greater than 4ε we see that the triangle inequality holds.

To see the last statement note that when G is abelian, it is straightforward that

$$\hat{d}((a_1,b_1)(a_3,b_3),(a_2,b_2)(a_3,b_3)) = \hat{d}((a_1,b_1),(a_2,b_2)),$$

i.e. $\hat{d}$ is invariant.

For the contrary direction assume that $a \in G \setminus Z(G)$ and $ab \neq ba$ for some $b \in G$. Then $\{a,b\} \cap H = \varnothing$ and $\varepsilon = \hat{d}((1,a),(a,1))$. On the other hand $d_G(a, a^b) > 0$, i.e. $\varepsilon \neq \hat{d}((1,a^b),(a,1))$. Thus

$$\hat{d}((1,a),(a,1)) \neq \hat{d}((1,b^{-1})(1,a))(1,b),((1,b^{-1})(a,1)(1,b))). \quad \square$$

Remark 1. The final part of the proof above shows that G is abelian if the relation $\hat{d}(\bar{g}_1, \bar{g}_2) = \varepsilon$ is invariant in $G \oplus_H G$.

Remark 2. An easy example for the theorem. When the cyclic group $C(n)$ is realized by complex powers of $e^{\frac{2\pi i}{n}}$ one can consider the invariant metric $d_G(z_1, z_2) = \| z_1 - z_2 \|$. Then the real number $\frac{1}{5} \sin \frac{\pi}{n}$ works as ε.

Remark 3. It is natural to compare Theorem 1 with the approach of the paper [4]. When (G, d_G), H and ε satisfy the assumptions of Theorem 1 one can try to amalgamate two copies of (G, d_G) over H just like metric spaces by the method of the proof of Proposition 1 given in [4]. It is curious that then we obtain a metric, say d_{IMI}, on the set $\{(a, 1) : a \in G\} \cup \{(1, a) : a \in G\} \subseteq G \oplus_H G$, which coincides with $\hat{d}$ provided by Theorem 1. The corresponding verification is rather straightforward by the definition of d_{IMI} given in the proof of Proposition 1 in [4].

3. Comments

(I) The material of our paper obviously follows the direction of the survey paper [1]. It is also related to some questions discussed in [2]. Possible model theoretic connections are indicated in [4].

(II) P. Niemiec has proved in [5] that there is an elementary abelian 2-group $\mathbb{G}$ with an invariant metric d which coincides with the Urysohn space $\mathbb{U}$ as a metric space. We remind the reader that $\mathbb{U}$ is a ultrahomogeneous Polish space such that any separable metric space embed into $\mathbb{U}$ isometrically. Thus *any separable metric space* (X, d) *can be considered as a subspace of a metric abelian group*, i.e. an object which can be treated by our methods. For further adjustment to the situation of Theorem 1 we need a way to view a metric space with a distinguished subspace as a metric group with a distinguished subgroup. In particular given finite subspace $X_0 \subset X$ and isometric emedding $\rho_0 : X_0 \to \mathbb{G}$ is there an isometric embedding $\rho : X \to \mathbb{G}$ extending ρ_0 such that the subgroup $\langle \rho(X_0) \rangle$ does not intersect $\rho(X \setminus X_0)$? We now explain why the answer is positive. Indeed, applying free amalgamation (i.e. Theorem 2.1 of [1]) we see that there is a metric space X' containg X_0 which is a copy of X by an isometry fixing X_0 pointwise such that $\langle \rho_0(X_0) \rangle \cup X'$ has a metric extending the ones of the summands and $\langle \rho_0(X_0) \rangle \cap X' = X_0$. One of definitions of ultrahomogeneity (applied to $\mathbb{U}$) states that X' can be chosen in $\mathbb{U}$. This gives the required isometry ρ.

(III). Remark (II) can be considered in the context of papers [3] and [6]. Roughly they provide a viewpoint of metric structures where the notion of isometric isomorphism is relpaced by the notion where isomorphisms just preserve equalities and inequalities between lengths of metric intervals in the structure. It is proved in [6] that any finite metric space can be embedded by such an isomorphism into a Hamming space. Since a Hamming space is a metric abelian group we arrive at the situation of (II) where the original X is finite and $\mathbb{G}$ is replaced by the Hamming space, say H_m. It is worth noting that in this case the situation is slightly more complicated. On the one hand we obviously lose control of the metric. On the other one we cannot repeat the argument of (II) concerning the finite subspace $X_0 \subset X$ and isometric emedding ρ_0 of X_0 into H_m (instead of $\mathbb{G}$). It seems to the authors that it is an interesting question if given finite metric spaces $X_0 \subset X$ there is an isomorphic embedding ρ of X into some H_m such that $\langle \rho(X_0) \rangle$ does not intersect $\rho(X \setminus X_0)$.

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