Algebra and Discrete Mathematics
Volume **22** (2016). Number 2, pp. 153–170 RESEARCH ARTICLE


# Quadratic residues of the norm group in sectorial domains

**Lyubov Balyas and Pavel Varbanets**

Communicated by D. Simson

Abstract. In the article the distribution of quadratic residues in the ring $G_{p^n}$, in the norm subgroup $E_n$ of multiplicative group $G^*_{p^n}$, is investigated. The asymptotic formula for the number $R(x,\phi)$ of quadratic residues in the sectorial domain of a special form has been constructed.

## 1. Introduction

In 1918 I.M. Vinogradov and G. Polya built the asymptotic formula for the number of quadratic residues modulo prime number on the segment $1 \leqslant n \leqslant x < p$, which was nontrivial for every $x > \sqrt{p}\log p$. It was the first result about incomplete residue system in analytic number theory. Henceforth Vinogradov-Polya theorem was firstly sharpened by D. Burgess [1]. After this on the assumption under extended Weil hypothesis H. Montogomery and R. Vaughan [3] got the unimprovable result for the theorem.

The research of analogous issue over the ring of Gaussian integers is, evidently, a difficult problem by the virtue of the fact, that geometry of points of a plane is richer than geometry of points on a line. In this article the distribution of quadratic residues in the norm subgroup $E_n$

**2010 MSC:** 11L05, 11L07, 11N25, 11S40.

**Key words and phrases:** quadratic residue, the norm group, Hecke Z-function,the gamma function, Dirichlet series,functional equation, Gaussian integers.

of multiplicative group $G^*_{p^n}$, is investigated. Here $p$ is a prime rational number of the type $p = 3 + 4k$ and $E_n$ can be written in the form:

$$E_n := \{\alpha \in G_{p^n} | N(\alpha) \equiv \pm 1 \pmod{p^n}\}.$$

This subgroup is cyclic, its order is equal to $2(p+1)p^{n-1}$. The numbers $p = 2$ and $p \equiv 1 \pmod 4$ are not prime in $G$. Thus, for $p \equiv 1 \pmod 4$ we have $p = \pi \cdot \bar{\pi}; \pi, \bar{\pi} \in \mathbb{Z}[i]$, and the residue class rings in $\mathbb{Z}[i]$ modulo $p^n$ (respectively, $\pi^n$) are isomorphic. So, this case was investigated in the works mentioned above. Similarly we have for $p = 2$. That is why we don't consider these $p$.

If $(u_0 + iv_0)$ is a generating element of the group $E_n$, then $N(u_0 + iv_0) \equiv -1 \pmod{p^n}$. It follows that only the elements of the type $(u_0 + iv_0)^{2a}$, where $a = 0, 1, \ldots, (p+1)p^{n-1}$, are quadratic residues modulo $p^n$ in $E_n$.

Our aim is to prove Theorems 1 and 2 stated in Section 3, and to obtain an asymptotic formula for the number $R(x, \phi)$ of quadratic residues in the sectorial domain

$$S(x, \phi) = \left\{\phi_1 \leqslant \arg w < \phi_2, 0 < N(w) \leqslant x, \phi_2 - \phi_1 = \phi < \frac{\pi}{2}\right\}. \quad (1)$$

The formula for $R(x, \varphi)$ is contained in Theorem 2 and has the following form

$$R(x; \phi) = \frac{\phi_2 - \phi_1}{2} \cdot \frac{p+1}{p} \cdot \frac{x}{p^n} + O\left(3^n \frac{x^{1-s}}{p^n} \log x\right).$$

The most interesting case is the case, when $\phi_2 - \phi_1 \to 0$ with $x \to \infty$, because the case $\phi_2 - \phi_1 \geqslant C$, $C > 0$ is a fixed constant, follows from the work [5] about the distribution of values of the function $r(n)$ (the number of representations of $n$ by the sum of two squares) in the arithmetic progression.

**Notations.** We will use the following notations:

- $G := \{a + bi | a, b \in \mathbb{Z}, i^2 = -1\}$ is the ring of Gaussian integers;
- $G_\gamma$ is the ring of residues of Gaussian integers modulo $\gamma$;
- $G^*_\gamma = \{\alpha \in G_\gamma, (\alpha, \gamma) = 1\}$;
- for $\alpha \in G$ we denote $N(\alpha) = |\alpha|^2, \mathrm{Sp}(\alpha) = 2\Re(\alpha)$;
- $E_n \subset G_{p^n}$ is the norm group;
- $\chi$ stands for a character of the group $E_n$;
- for $a \in \mathbb{Z}$ (or $\alpha \in G$) $\nu_p(a)$ (or $\nu_p(\alpha)$) stands that $p^{\nu_p(a)} | a, p^{\nu_p(a)+1}$ does not divide $a$;

- $s \in \mathbb{C}, s = \sigma + it, \sigma = \Re s, t = \Im s$;
- $\Gamma(z)$ is the Euler gamma-function;
- by $f \ll g (f = O(g))$ for $x \in X$, where $X$ is an arbitrary set, on which $f$ and $g$ are defined, we mean that there exists a constant $C > 0$ such that $|f(x)| \leqslant C \cdot g(x)$ for all $x \in X$;
- $\exp(x) = \mathrm{e}^x$ for $x \in \mathbb{C}$ (sometimes, instead of $\mathrm{e}^x$ we will use $\exp(x)$).

Let us denote

$$E_n^+ := \left\{ \alpha \in G_{p^n}^* | N(\alpha) \equiv 1 \pmod{p^n} \right\} \\ = \left\{ \alpha \in E_n | \alpha = (u_0 + iv_0)^{2a}, a = 0, 1, \ldots, (p+1)p^{n-1} \right\}.$$

Then

$$R(x, \phi) = \sum_{\alpha \in E_n^+} \sum_{\substack{w \in G \\ w \equiv \alpha \pmod{p^n} \\ w \in S(x,\phi)}} 1. \tag{2}$$

We consider Dirichlet series

$$F_m(s) = \sum_{\alpha \in E_n^+} \sum_{w \equiv \alpha \pmod{p^n}} \frac{\mathrm{e}^{4mi \arg w}}{N(w)^s}, \quad \Re s > 1.$$

We have

$$F_m(s) = \sum_{\alpha \in E_n^+} \frac{1}{N(p^n)^s} \zeta_m \left( s; \frac{\alpha}{p^n}, 0 \right), \quad \Re s > 1, \tag{3}$$

where $\zeta_m \left( s; \frac{\alpha}{p^n}, 0 \right)$ is a special case of Hecke zeta-function $\zeta_m(s; \delta_0, \delta)$ with a shift. In the domain $\Re s > 1$ the last is defined by absolutely convergent Dirichlet series

$$\zeta_m(s; \delta_0, \delta) = \sum_{w \in G} \frac{\mathrm{e}^{4mi \arg(w + \delta_0)}}{N(w + \delta_0)^s} e^{\pi i Sp(\delta w)},$$

where $\delta_0, \delta$ are Gaussian numbers from the field $\mathbb{Q}(i)$; $\mathrm{Sp}(\beta)$ is a trace of an element $\beta$ from $\mathbb{Q}(i)$ to $\mathbb{Q}$.

## 2. Auxiliary results

In the following lemmas we bring necessary information about Hecke zeta-function for the next steps.

**Lemma 1.** *The Hecke zeta-function* $\zeta_m(s;\delta_0,\delta)$ *satisfies the functional equation*

$$\pi^{-s}\Gamma(2|m|+s)\zeta_m(s;\delta_0,\delta) = \pi^{-(1-s)}\Gamma(2|m|+1-s)\cdot\zeta_{-m}(1-s;\delta_0,-\delta)\,e^{-\pi i\,\mathrm{Sp}(\delta\bar{\delta}_0)}.$$

*Moreover,* $\zeta_m(s;\delta_0,\delta)$ *is an entire function if* $m\neq 0$ *or* $m=0$ *and* $\delta$ *is not a Gaussian integer. For* $m=0$ *and* $\delta\in G$ *it is holomorphic except for the point* $s=1$*, where it has a simple pole with the residue* $\pi$*.*

*Proof.* For $\delta_0=\delta=0$ and $m=4m_1$, we get the well-known Hecke zeta-function $Z_m(s)$ with the Hecke character of the first kind with the exponent $m$ (see, [2]). In [8] this lemma has been stated without a proof. But for the completeness of treatment we restore a proof of this statement.

In the general case, for the proof of statement of the lemma we start from the relation

$$\Gamma(s)\,|w\delta_0|^{-2s} = \int_0^\infty \exp(-x|w+\delta_0|^2)x^{s-1}dx.$$

It is evident that for $\Re s>1$ and $m\in\mathbb{Z}$ we can write

$$\Gamma(2|m|+s)Z_m(s;\delta_0;\delta) = \int_0^\delta \sum_{\substack{w\in G\\ w\neq-\delta_0}} \mathrm{e}^{-x|w+\delta_0|^2}x^{s-1}dx.$$

Let us denote $\delta_0=\delta_{01}+\delta_{02}$. Then a groundtruthing shows that the functions

$$f(u_1,u_2) = \exp\left(-x\left(u_1^2+u_2^2\right)+2\pi i(\delta_{01}u_1+\delta_{02}u_2)\right),$$
$$\hat{f}(v_1,v_2) = \frac{\pi}{x}\exp\left(-\frac{\pi^2}{x}\left[(\delta_{01}+v_1)^2+(\delta_{02}+v_2)^2\right]\right)$$

satisfy the conditions of Poisson summation formula (see, e.g. [6], Ch. VII, Corollary 2.6).

Hence, denoting

$$\Theta_m(x,\delta_0,\delta) = \sum_{w\in G}\exp\left(-x|w+\delta_0|^2\right)(w+\delta_0)^{4m}\exp\left(\pi i\,\mathrm{Sp}(\bar{\delta}w)\right)$$

and using Poisson summation formula, we find

$$\Theta_0(x,\delta_0,\delta) = \frac{\pi}{x}\Theta_0\left(\frac{\pi^2}{x},\delta,-\delta_0\right)\exp\left(-\pi i\,\mathrm{Sp}(\delta_0\bar{\delta})\right).$$

Consider the operator

$$\frac{d}{d\delta_0} := \frac{\partial}{\partial \delta_{01}} + i\frac{\partial}{\partial \delta_{02}}, \quad \delta_0 = \delta_{01} + \delta_{02}.$$

Then the following equalities for $m \geqslant 0$

$$(-2x)^{4m}\Theta_m(x, \delta_0, \delta) = \frac{d^m}{d\delta^m}\Theta_0(x, \delta_0, \delta)$$

and

$$\begin{aligned}&\frac{\pi}{x}(-2\pi i)^{4m}\Theta_m\left(\frac{\pi^2}{x}, \delta_0, -\delta\right)\exp\left(-\pi i\,\mathrm{Sp}\left(\delta_0\overline{\delta}\right)\right)\\ &\qquad = \frac{d^m}{d\delta_0^m}\left(\frac{\pi}{x}\Theta_0\left(\frac{\pi^2}{x}, \delta, -\delta_0\right)\exp\left(-\pi i\,\mathrm{Sp}\left(\delta_0\overline{\delta}\right)\right)\right)\end{aligned}$$

hold.

So, for any $m \in \mathbb{Z}$ the following functional equation

$$\Theta_m(x, \delta_0, \delta) = \left(\frac{\pi}{x}\right)^{4m+1}\Theta_m\left(\frac{\pi^2}{x}, \delta, \delta_0\right)\exp\left(-\pi i\,\mathrm{Sp}\left(\delta_0\overline{\delta}\right)\right) \qquad (4)$$

is true.

Now, applying reasoning used for the proof of the functional equation for Riemann zeta-function by the functional equation for a theta-function $\Theta_m$ we easily infer

$$\Gamma(2|m| + s)\zeta_m(s, \delta_0, \delta) = \pi^{-(1-2s)}\exp\left(-\pi i\,\mathrm{Sp}\left(\overline{\delta}_0\delta\right)\right)\mathfrak{I}_m(\delta_0, \delta),$$

where

$$\begin{aligned}&\mathfrak{I}_m(\delta_0, \delta)\\ &\quad = \int_0^\infty \sum_{\substack{w\\ w\neq-\delta_0}} \exp\left(-x|w+\delta_0|^2\right)(w+\delta_0)^{4m}\exp\left(\pi i\,\mathrm{Sp}(\overline{\delta}w)\right)x^{s+2m-1}dx\\ &\quad = \int\limits_0^\pi + \int\limits_\pi^\infty := \mathfrak{I}_{m,1} + \mathfrak{I}_{m,2}.\end{aligned}$$

In the integral $\mathfrak{I}_{m,1}$ we apply the functional equation (4) for $\Theta_m(x,\delta_0,\delta)$ and make the substitution $x = \pi^2 y^{-1}$. This gives the equality

$$\begin{aligned}\Gamma(2|m|+s)\zeta_m(s,\delta_0,\delta) &= \pi^{2s-1}\exp\left(-\pi i\operatorname{Sp}(\overline{\delta}_0\delta)\right)\times\\ &\times\int_\pi^\infty \sum_{\substack{w\in G\\ w\neq-\delta}} \exp\left(-x|w+\delta|^2\right)(w+\delta)^{4m}\exp\left(-\pi i\operatorname{Sp}(\overline{\delta}_0 w)\right)x^{-s+2m}dx\\ &+\int_\pi^\infty \sum_{\substack{w\in G\\ w\neq-\delta_0}} \exp\left(-x|w+\delta_0|^2\right)(w+\delta_0)^{4m}\exp\left(-\pi i\operatorname{Sp}(\overline{\delta} w)\right)x^{s+2m-1}dx\\ &+\varepsilon(m,\delta)\frac{\pi^s}{s-1}-\varepsilon(m,\delta_0)\exp\left(-\pi i\operatorname{Sp}(\delta_0,\overline{\delta})\right)\frac{\pi^s}{s},\end{aligned}\tag{5}$$

where

$$\varepsilon(m,a)=\begin{cases}1 & \text{if } m=0 \text{ and } a\in G\\ 0 & \text{otherwise.}\end{cases}$$

The equality (5) was obtained for $\Re s > 1$. However, the right part of this equality is an analytic function in all complex $s$-planes except maybe the points $s=0$ and $s=1$, which can be the poles.

Now, multiplying the equality (5) by $\exp(\pi i\operatorname{Sp}(\overline{\delta}_0\delta))\pi^{-2s+1}$ and making the substitution $s\to 1-s$, $\delta_0\to\delta$, $\delta\to\delta_0$, we obtain that the right part doesn't vary, and hence, we have proved the following functional equation

$$\begin{aligned}&\pi^{-s}\Gamma(2|m|+s)\zeta_m(s;\delta_0,\delta)\\ &\quad=\pi^{-(1-s)}\Gamma(2|m|+1-s)\zeta_m(1-s;-\delta,\delta_0)\exp(-\pi i\operatorname{Sp}(\delta_0\overline{\delta})).\end{aligned}$$

If $m=-m'$, $m'>0$, we put $\delta_0=-\delta_0'$, $\delta=-\delta'$, and then we have

$$\zeta_m(s,\delta,\delta_0)=\zeta_{m'}(s,-\delta,-\delta_0)\Rightarrow\zeta_{m'}(1-s,\delta_0,-\delta)=\zeta_m(1-s,-\delta_0,\delta).$$

So, for any $m\in\mathbb{Z}$,

$$\begin{aligned}\pi^{-s}\Gamma(2|m|+s)\zeta_m(s;\delta,\delta_0)&=\pi^{-(1-s)}\Gamma(2|m|+1-s)\zeta_{-m}(1-s,-\delta_0,\delta)\\ &=\pi^{-(1-s)}\Gamma(2|m|+1-s)\zeta_{-m}(1-s;\delta_0,-\delta).\end{aligned}$$

This completes the proof of Lemma 1. □

**Corollary 1.** *If $\delta$ is not a Gaussian integer, then $\zeta_0(0;\delta_0,\delta)=0$.*

**Lemma 2.** *In the strip $\varepsilon \leqslant \Re s \leqslant 1+\varepsilon$, $\varepsilon > 0$, the following estimate*

$$(s-1)\cdot\zeta_m(s;\delta_0,\delta) \ll (|t|+1)(t^2+m^2)^{\frac{(1-2\sigma)(1+\varepsilon-\sigma)}{1+2\varepsilon}}\,|N(\delta)|^{-\frac{\sigma+\varepsilon}{1+2\varepsilon}}$$

*holds.*

This lemma follows from Phragmen-Lindelof principle and the estimates for $\zeta_m(s;\delta_0,\delta)$ on the boundaries of the strip $\varepsilon \leqslant \Re s \leqslant 1+\varepsilon$, which can be received with the usage of the functional equation for $\zeta_m(s;\delta_0,\delta)$ and Stirling formula for $\Gamma(z)$.

**Lemma 3.** *Let $y \geqslant k \in \{0,1,2\}$. Let $a$ be a real number, $-1 < a \leqslant \frac{5}{4}$, $\eta(a) = \min_{j=0,1,\dots,k} |a-j| \neq 0$. Then for any real numbers $u, v$ the following estimate*

$$\int\limits_{a+iu}^{a+iv} \frac{y^s\psi(s,m)}{s(s+1)\dots(s+k)}ds$$
$$\ll N(\gamma)^{\frac{1}{2}}M\left(\left(\frac{y}{N(\gamma)}\cdot\frac{1}{M}\right)^a(\eta^{-1}(a)+\log M)+\left(\frac{y}{N(\gamma)M}\right)^{\frac{1}{2}-\frac{2k+1}{4}}\right),$$

*holds, where $\psi(s,m) = \left(\frac{1}{\pi}N(\gamma)^{\frac{1}{2}}\right)^{1-2s}\frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)}$, $M = |m|+10$.*

*Proof.* Apply [3, Lemma 8]. □

**Lemma 4** ([7], Theorem 1)**.** *Upon the condition $D^{\frac{1}{2}} \leqslant x < D^2$ the asymptotic formula*

$$\sum_{\substack{n\equiv 1 \,(\mathrm{mod}\ D)\\ n\leqslant x}} r(n) = \frac{\pi x}{D}\gamma_0\prod_{p|D}\left(1-\frac{\chi_4(p)}{p}\right)$$
$$+O\left(D^{\frac{1}{2}}\exp\left(c\frac{(\log D)^{\frac{1}{2}}}{\log\log D}\right)\right)+O\left(\frac{x^{\frac{1}{2}}}{D^{\frac{1}{2}}}\tau(D)\right),$$
$$\gamma_0 = \begin{cases} 1 & \text{if } D \not\equiv 0 \,(\mathrm{mod}\ 4),\\ 2 & \text{if } D \equiv 0 \,(\mathrm{mod}\ 4)\end{cases}$$

*is true.*

**Lemma 5.** *Let $p \equiv 3 \,(\mathrm{mod}\ 4)$. Then for $n = 1,2,3,\dots$ the estimate*

$$\sum_{\alpha\in E_n^+} e^{\pi i\,\mathrm{Sp}\,\frac{\alpha^2}{p^n}} \ll p^{\frac{n}{2}}$$

*holds.*

*Proof.* In the articles [5] and [9] the following description of elements $\alpha \in E_n^+, n \geqslant 2$ was given:

$$\alpha = (u_0 + iv_0)^{2(p+1)t+k} \equiv \sum_{j=0}^{n-1} (A_j(k) + iB_j(k))\, t^j \pmod{p^n}.$$

Here $(u_0 + iv_0)$ is a generator of the group $E_n^+$, $t = 0, 1, \ldots, p^{n-1}$, $k = 0, 1, \ldots, 2p+1$. Moreover,

$$A_0(k) = u(k), \quad B_0(k) = v(k);$$
$$A_1(k) \equiv -py_0 v(k) \pmod{p^3}, \quad B_1(k) \equiv py_0 u(k) \pmod{p^3},$$
$$A_2(k) \equiv -\frac{1}{2}p^2 y_0^2 u(k) \pmod{p^3}, \quad B_2(k) \equiv -\frac{1}{2}p^2 y_0^2 v(k) \pmod{p^3},$$
$$(u_0 + iv_0)^k \equiv u(k) + iv(k) \pmod{p^n}, (y_0, p) = 1.$$

Furthermore,

$$u(k) \equiv 0 \pmod{p}, \quad \text{when } k = \frac{p+1}{2},\ k = \frac{3(p+1)}{2};$$
$$v(k) \equiv 0 \pmod{p}, \quad \text{when } k = 0,\ k = p+1;$$
$$A_j(k) \equiv B_j(k) \equiv 0 \pmod{p^3}, \quad j = 3, 4, \ldots, m-1,\ k = 0, 1, \ldots, 2p+1.$$

Hence we easily conclude

$$\Re(\alpha^2) \equiv (A_0^2(k) - B_0^2(k)) + 2(A_0(k)A_1(k) - B_0(k)B_1(k))t$$
$$+ (A_1^2(k) - B_1^2(k)) + A_0(k)A_2(k) - B_0(k)B_2(k)t^2 \pmod{p^3}.$$

Then $\Re(\alpha^2) \equiv C_0 + C_1 t + C_2 t^2 \pmod{p^3}$ with the coefficients

$$C_1 \equiv -2py_0 u(k)v(k) \pmod{p^3}, \quad C_2 \equiv \frac{1}{2}p^2 y_0^2 (u^2(k) - v^2(k)) \pmod{p^3}$$
$$\text{or } C_2 \equiv \frac{1}{2}p^2 y_0^2 (1 - 2v^2(k)) \pmod{p^3}.$$

Let us note that $u(k)^2 + v(k)^2 \equiv (-1)^k \pmod{p}$. Therefore, it follows that $u(k)$ and $v(k)$ can not divide $p$ simultaneously. It is obvious that $\nu_p(C_2) \geqslant 2$ (the strict inequality holds for the cases $k=0, \frac{p+1}{2}, \frac{3(p+1)}{2}, p+1$). That is why, when $\nu_p(C_1) < \nu_p(C_2)$, $S = 0$. So, from the well-known relation, for $(b, p) = 1$, $f(x) \in \mathbb{Z}[x]$,

$$\left| \sum_{x \in \mathbb{Z}_{p^n}} e^{2\pi i \frac{ax + pbx^2 + p^2 f(x)}{p^n}} \right| = \begin{cases} 0 & if \quad (a,p) = 1, \\ 2p^{\frac{n-1}{2}} & if \quad a \equiv 0 \pmod{p} \end{cases}$$

we get

$$\left|\sum_{\alpha\in E_n^+} e^{\pi i \operatorname{Sp}\frac{\alpha^2}{p^n}}\right| \leqslant 4p^{\frac{n}{2}}.$$

In case $n=1$ we take into account that

$$E_1 = \left\{\pm 1, \pm i, \frac{a-i}{a+i}, i\frac{a-i}{a+i} \mid a = 1, 2, \ldots, p-1\right\}.$$

Thus, we conclude that $\operatorname{Sp}(\alpha^2)$ can be represented as the ratio of the polynomials of degree 2. Then, following Weil [11], we have

$$\left|\sum_{\alpha\in E_1} e^{\pi i \operatorname{Sp}\frac{\alpha^2}{p}}\right| \leqslant 2\sqrt{p}.$$

Hence, the assertion of lemma follows. □

**Lemma 6** ([7], Lemma 5)**.** *Let $p$ be a prime number, let $u_1, u_2$ be integers and $(u_1, u_2, p^n) = p^m$. Then*

$$\left|\sum_{l_1^2+l_2^2\equiv 1 \pmod{p^n}} e^{2\pi i\frac{u_1 l_1 + u_2 l_2}{p^n}}\right| \leqslant 2p^{\frac{n+m}{2}}.$$

**Corollary 2.** *For $m \neq 0$ the following estimate*

$$\sum_{\alpha\in E_n^+} \zeta_m\left(0; \frac{\alpha}{p^n}, 0\right) \ll p^{\frac{3}{2}n} M \log M, \qquad M = |m| + 10,$$

*holds.*

This statement follows immediately from the functional equation for $\zeta_m(s;\delta_0,\delta)$ for $m \neq 0$ and Lemma 6.

The following Lemma was proved in [10] (see Lemma 11, pp. 259–260).

**Lemma 7** (Vinogradov's 'glasses')**.** *Let $r \in \mathbb{N}$, $\Omega > 0$, $0 < \Delta < \frac{1}{2}\Omega$ and let $\phi_1, \phi_2$ be real numbers, $\Delta \leqslant \phi_2 - \phi_1 \leqslant \Omega - 2\Delta$. Then there exists a periodic function $f(\phi)$ with the period $\Omega$ such that:*

(i) *$f(\phi) = 1$, in the segment $\phi \in [\phi_1, \phi_2]$;*
*$0 \leqslant f(\phi) \leqslant 1$ in the segments $[\phi_1 - \Delta, \phi_1]$ and $[\phi_2, \phi_2 + \Delta]$;*
*$f(\phi) = 0$, in the segment $[\phi_2 + \Delta, \phi_1 + \Omega - \Delta]$;*

(ii) *$f(\phi)$ has the expansion in a Fourier series*

$$f(\phi) = \sum_{m=-\infty}^{+\infty} a_m e^{2\pi i \frac{m\phi}{\Omega}},$$

*where $a_0 = \frac{1}{\Omega}(\phi_2 - \phi_1 + \Delta)$, and for $m \neq 0$ and $r \in \mathbb{N}$ each of the following inequalities holds*

$$|a_m| \leqslant \begin{cases} \frac{1}{\Omega}(\phi_2 - \phi_1 + \Delta), \\ \frac{2}{\pi|m|}, \\ \frac{2}{\pi|m|}\left(\frac{r\Omega}{\pi|m|\Delta}\right)^r. \end{cases}$$

## 3. Main results

Let us consider the function of a natural argument

$$r_m(k) = \sum_{\substack{u,v\in\mathbb{Z} \\ u^2+v^2=k}} e^{4mi\arg(u+iv)}.$$

In view of (3) we can write

$$F_m(s) = \sum_{\substack{k\leqslant x \\ k\equiv 1 \pmod{p^n}}}^{\infty} \frac{r_m(k)}{k^s}.$$

**Theorem 1.** *Let $m \neq 0$, $p^n \leqslant x \leqslant p^{2n}$. Then*

$$\sum_{\substack{k\leqslant x \\ k\equiv 1 \pmod{p^n}}} r_m(k) \ll \frac{\sqrt{x}}{p^{\frac{n}{2}}} + p^{\frac{n}{2}}\log x + p^{\frac{n}{2}} M \log M.$$

*Proof.* Our assertion is trivial for $x \ll p^n M$. That is why we will assume that $x \geqslant C \cdot Mp^n$, $C > 0$. It follows from Lemma 1 that $\zeta_m(s; \delta_0, \delta)$ is an entire function. In view of the fact

$$\frac{1}{p^{2ns}}\zeta_m\left(s; \frac{\alpha}{p^n}, 0\right) = \sum_{\substack{w\in G \\ w\equiv\alpha \pmod{p^n}}} e^{\frac{4mi\arg w}{N(w)^s}}$$

for $\Re s > 1$ and every $\alpha \in G$ the usage of the theorem of the residues gives

$$\begin{aligned} &\frac{1}{2\pi i}\int_{2-i\cdot\infty}^{2+i\cdot\infty} \frac{x^{s+2}\zeta_m\left(s; \frac{\alpha}{p^n}, 0\right)}{p^{2ns}s(s+1)(s+2)} ds \\ &= \frac{x^2}{2}\delta_m\left(s; \frac{\alpha}{p^n}, 0\right) + \frac{1}{2\pi i}\int_{a-i\cdot\infty}^{a+i\cdot\infty} \frac{x^{s+2}\zeta_m\left(s; \frac{\alpha}{p^n}, 0\right)}{p^{2ns}s(s+1)(s+2)} \end{aligned} \tag{6}$$

for every $-1 < a < 0$.

Let us denote

$$S_2(x,\alpha) = \frac{1}{2} \sum_{\substack{0<N(w)\leqslant x \\ w\equiv\alpha \pmod{p^n}}} e^{4mi\arg w}(x-N(w))^2. \tag{7}$$

Using the relation

$$\frac{1}{2\pi i}\int_{2-i\cdot\infty}^{2+i\cdot\infty} \frac{y^{s+l}}{s(s+1)\dots(s+k)} = \begin{cases} \frac{1}{l!}(y-1)^l & if \quad y>1 \\ 0 & if \quad 0<y<1 \end{cases}$$

and taking into account the uniform convergence of the series for zeta-function $\zeta_m\left(s;\frac{\alpha}{p^n},0\right)$ in the semiplane $\Re s \geqslant 1+\varepsilon$, $\varepsilon>0$, we get

$$\begin{aligned} &\frac{1}{2\pi i}\int_{2-i\cdot\infty}^{2+i\cdot\infty} \frac{x^{s+2}\zeta_m\left(s;\frac{\alpha}{p^n},0\right)}{p^{2ns}s(s+1)(s+2)}ds \\ &= \sum_{\substack{w \\ w\equiv\alpha \pmod{p^n}}} \frac{e^{4mi\arg w}}{N(w)^{-2}}\cdot\frac{1}{2\pi i}\int_{2-i\cdot\infty}^{2+i\cdot\infty}\frac{\left(\frac{x}{N(w)}\right)^{s+2}}{s(s+1)(s+2)}ds \\ &= \frac{1}{2}\sum_{\substack{w\equiv\alpha \pmod{p^n} \\ N(w)\leqslant x}} e^{4mi\arg w}(x-N(w))^2 = S_2(x,\alpha). \end{aligned} \tag{8}$$

The application of the functional equation for $\zeta_m(s;\delta_0,\delta)$ (see Lemma 1) and the estimate $\zeta_m(s;\delta_0,\delta)$ in critical strip (see Lemma 2) give

$$\begin{aligned} &\frac{1}{2\pi i}\int_{a-i\cdot\infty}^{a+i\cdot\infty} \frac{x^{s+2}\zeta_m\left(s;\frac{\alpha}{p^n},0\right)}{p^{2ns}s(s+1)(s+2)} \\ &= \sum_{w\in G\setminus\{0\}} \mathrm{e}^{-4mi\arg w}\mathrm{e}^{\pi i\,\mathrm{Sp}\left(\frac{\alpha w}{p^n}\right)}N(w)^{-s}\frac{1}{2\pi i}\int_{a-i\cdot\infty}^{a+i\cdot\infty}\frac{(xN(w))^{s+2}\frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)}}{\pi^{1-2s}s(s+1)(s+2)p^{2ns}}ds. \end{aligned} \tag{9}$$

From (6)–(9) we deduce the formula:

$$\begin{aligned} S_2(x,\alpha) &= \frac{x^2}{2}\zeta_m\left(0;\frac{\alpha}{p^n},0\right) \\ &+ \sum_{w\in G\setminus\{0\}} \mathrm{e}^{-4mi\arg w}\mathrm{e}^{\pi i\,\mathrm{Sp}\left(\frac{\alpha w}{p^n}\right)}N(w)^{-s}W\left(\frac{xN(w)}{p^{2n}}\right)p^{2(n+1)}, \end{aligned}$$

where

$$W(y)=\frac{1}{2\pi i}\int\limits_{a-i\cdot\infty}^{a+i\cdot\infty}\frac{y^{s+2}\Gamma(2|m|+1-s)}{s(s+1)(s+2)\Gamma(2|m|+s)}ds.$$

We consider the following operator

$$\Delta_z F(x)=\sum_{j=0}^{2}(-1)^j F(x+jz)=\int\limits_{x}^{x+z}dy_1\int\limits_{y_1}^{y_1+z}F''(y_2)dy_2.$$

Then

$$\Delta_z\left(\frac{x^2}{2}\zeta_m\left(0;\frac{\alpha}{p^n},0\right)\right)=z^2\zeta_m\left(0;\frac{\alpha}{p^n},0\right).$$

It is obvious that for every $b$, $-1<b<0$, we have

$$W(y)=\frac{1}{2\pi i}\int\limits_{b-i\cdot\infty}^{b+i\cdot\infty}\frac{y^{s+2}\Gamma(2|m|+1-s)}{s(s+1)(s+2)\Gamma(2|m|+s)}ds.$$

We put $b=-1+\frac{1}{\log y}$, if $y>1$. Using Lemma 3, we conclude that

$$W(y)\ll K(y,m), \tag{10}$$

where

$$K(y,m)=p^{3n}M^3y\left(\log y+\log M\right).$$

It means that

$$\Delta_z W\left(\frac{xN(w)}{p^{2n}}\right)\ll K\left(\frac{xN(w)}{p^{2n}},m\right), \tag{11}$$

if only $z\ll\frac{xN(w)}{p^{2n}}$.

The value $\Delta_z W\left(\frac{xN(w)}{p^{2n}}\right)$ may be defined in a different way. We put

$$\Phi(y)=\frac{1}{2\pi i}\int\limits_{c-i\cdot\infty}^{c+i\cdot\infty}\frac{y^{s+2}\Gamma(2|m|+1-s)}{s(s+1)(s+2)\Gamma(2|m|+s)}ds,\quad c>1.$$

Then

$$\Phi(y)=\frac{y^2}{2}\frac{\Gamma(2|m|+1)}{\Gamma(2|m|)}+W(y).$$

For all $y > 0$ the integrals

$$\frac{1}{2\pi i}\int_{c-i\cdot\infty}^{c+i\cdot\infty}\frac{y^{s+2}\Gamma(2|m|+1-s)}{s\cdot\ldots\cdot(s+2-j)\Gamma(2|m|+s)}ds,\ \ j=0,1,2,$$

converge absolutely and uniformly. Hence, for the derivatives of $\Phi(y)$ we have

$$\Phi^{(j)}(y)=\frac{1}{2\pi i}\int_{c-i\cdot\infty}^{c+i\cdot\infty}\frac{y^{s+2-j}\Gamma(2|m|+1-s)}{s\cdot\ldots\cdot(s+2-j)\Gamma(2|m|+s)}ds,\quad j=0,1,2.$$

Thus,

$$W^{''}(y)=-\frac{\Gamma(2|m|+1)}{\Gamma(2|m|)}+\frac{1}{2\pi i}\int_{c-i\cdot\infty}^{c+i\cdot\infty}\frac{y^s}{s}\frac{\Gamma(2\,|m|+1-s)}{\Gamma(2|m|+s)}ds. \tag{12}$$

Now we will take into account that the subintegral function doesn't have singularities in the semiplane $\Re s > 0$. Then, transfering the contour of the integration in (12) to the line $\Re s = \frac{1}{\log y}$ and using Lemma 3, Stirling formula for the gamma-function, we get

$$W^{''}(y) \ll L(y,m),$$

where $L(y,m) = p^n(M\log M + y^{\frac{1}{4}})$. But then

$$\begin{aligned}\Delta_z\left(W\left(\frac{N(w)x}{p^{2n}}\right)\right) &= \int_{\frac{N(w)}{p^{2n}}x}^{\frac{N(w)}{p^{2n}}(x+z)} dy_1 \int_{y_1}^{y_1+\frac{N(w)}{p^{2n}}z} W^{''}(y_2)dy_2 \\ &\ll L\left(\frac{xN(\alpha)}{p^{2n}},m\right)\frac{z^2N(w)^2}{p^{4n}}.\end{aligned} \tag{13}$$

Let us denote as $S_2(x)$ the following sum

$$S_2(x)=\sum_{\alpha\in E_n^+}S_2(x,\alpha).$$

We have

$$
\begin{aligned}
S_2(x) &= \frac{x^2}{2}\sum_{\alpha\in E_n^+}\zeta_m\left(0;\frac{\alpha}{p^n},0\right)\\
&\quad+\sum_{\alpha\in E_n^+}\sum_{w\equiv\alpha\,(\mathrm{mod}\ p^n)}\mathrm{e}^{4mi\arg w}\mathrm{e}^{\pi i\,\mathrm{Sp}\left(\frac{\alpha w}{p^n}\right)}\frac{W\left(\frac{xN(w)}{p^{2n}}\right)}{N(w)^3}\\
&=\frac{x^2}{2}\sum_{\alpha\in E_n^+}\zeta_m\left(0;\frac{\alpha}{p^n},0\right)+\sum_{\chi_{E_n}}\frac{1}{|E_n|}\sum_{\alpha\in E_n}\bar\chi(\alpha)\\
&\quad\times\sum_{N(w)\equiv1\,(\mathrm{mod}\ p^n)}\frac{\chi(w)\mathrm{e}^{4mi\arg w}}{N(w)^3}\cdot\mathrm{e}^{\pi i\,\mathrm{Sp}\left(\frac{\alpha w}{p^n}\right)}W\left(\frac{xN(w)}{p^n}\right).
\end{aligned}
\tag{14}
$$

Applying the operator $\Delta_z$ to both parts of (14), we obtain

$$
\begin{aligned}
\Delta_z\left(S_2(x)\right) &= z^2\sum_{\alpha\in E_n^+}\zeta_m\left(0;\frac{\alpha}{p^n},0\right)\\
&\quad+\sum_{\substack{w\in G\\ N(w)\equiv1\,(\mathrm{mod}\ p^n)}}\mathrm{e}^{-4mi\arg w}N(w)^{-3}W\left(\frac{xN(w)}{p^{2n}}\right)\sum_{\alpha\in E_n^+}\mathrm{e}^{\pi i\,\mathrm{Sp}\,\frac{\alpha^2}{p^n}}.
\end{aligned}
$$

In virtue of (10), (11) and (13), Lemma 5 and Corollary 2 we infer

$$
\begin{aligned}
&\Delta_z\left(S_2(x)\right)\\
&\quad\ll z^2p^{\frac32 n}M\log M+p^{\frac n2}z^2\sum_{N(w)\leqslant x}N(w)^{-1}L\left(\frac{xN(w)}{p^{2n}},m\right)\\
&\qquad+p^{\frac n2}\sum_{N(w)>x}N(w)^{-3}K\left(\frac{xN(w)}{p^{2n}},m\right)\\
&\quad\ll z^2p^{\frac n2}p^nM\log M+z^2p^{\frac n2}\sum_{N(w)\leqslant x}p^n\left(M\log M+\frac{N(w)^{\frac14}x^{\frac14}}{p^{\frac n2}}\right)N(w)^{-1}\\
&\qquad+z^2p^{\frac n2}\sum_{N(w)>x}p^{3n}M^3N(w)^{-2}p^{-2n}\log N(w).
\end{aligned}
\tag{15}
$$

From this we get

$$
\begin{aligned}
&\Delta_z\left(S_2(x)\right)\ll\\
&\quad\ll p^{\frac32}\left\{z^2M\log M+z^2M\log M\log x+z^2p^{-\frac n2}\sqrt{x}+M^3\log x\right\}.
\end{aligned}
\tag{16}
$$

The application of the estimates (10), (14) requires that $z \ll \frac{xN(w)}{p^{2n}}$. Thus the condition $N(w) > x$ in the second sum of (15) allows to assume $z = p^n M \leqslant \frac{x^2}{p^{2n}}$ for $M \ll \frac{x^2}{p^{2n}}$. Then the following inequality

$$\Delta_z\left(S_2(x)\right) \ll z^2 p^{\frac{3}{2}n} M \log M$$

holds.

Let $H_2(x)$ stands for the sum

$$H_2(x) = \sum_{\alpha \in E_n^+} \sum_{\substack{w \in G \\ w \equiv \alpha \,(\mathrm{mod}\ p^n) \\ N(w) \leqslant x}} \mathrm{e}^{4mi \arg w}. \tag{17}$$

Then from the definition of $S_2(x)$ we easily find

$$H_2(x) = \frac{d^2}{dx^2}(S_2(x)).$$

It is clear that

$$\int\limits_{x}^{x+z} dy_1 \int\limits_{y_1}^{y_1+z} H_2(y_2) dy_2 = \Delta_z\left(S_2(x)\right).$$

By $x \leqslant y_1 \leqslant x + 2z$ and Lemma 4 we have

$$\begin{aligned} |H_2(y_2) - H_2(x)| &= |E_n^+| \cdot \left| \sum_{\substack{x < N(w) \leqslant y_2 \\ N(w) \equiv 1 \,(\mathrm{mod}\ p^n)}} \mathrm{e}^{4mi \arg w} \right| \\ &\leqslant (p+1)p^{n-1} \sum_{\substack{x < n \leqslant x+2z \\ n \equiv 1 \,(\mathrm{mod}\ p^n)}} r(n) \\ &\leqslant \frac{\pi z}{p^n} \cdot \frac{p+1}{p} + O\left(\frac{\sqrt{x}}{p^{\frac{n}{2}}}\right) + O\left(p^{\frac{n}{2}} \exp\left(c \frac{(\log p^n)^{\frac{1}{2}}}{\log\log p^n}\right)\right). \end{aligned}$$

Consequently,

$$|H_2(y_2) - H_2(x)| = O\left(\frac{z}{p^n}\right) + O\left(\sqrt{x} p^{-\frac{n}{2}}\right) + O\left(p^{\frac{n}{2}} \exp\left(c \frac{(\log p^n)^{\frac{1}{2}}}{\log\log p^n}\right)\right).$$

It follows that

$$H_2(y_2) = H_2(x) + O\left(\frac{z}{p^n}\right) + O\left(\sqrt{x} p^{-\frac{n}{2}}\right) + O\left(p^{\frac{n}{2}} \exp\left(c \frac{(\log p^n)^{\frac{1}{2}}}{\log\log p^n}\right)\right). \tag{18}$$

Now from (17) and (18) we get

$$z^2\left(H_2(x)+O\left(\frac{z}{p^n}\right)+O\left(\sqrt{x}p^{-\frac{n}{2}}\right)+O\left(p^{\frac{n}{2}}\exp\left(c\frac{(\log p^n)^{\frac{1}{2}}}{\log\log p^n}\right)\right)\right)$$
$$=O(z^2p^{\frac{n}{2}}M\log M).$$

Thus,

$$H_2(x)=x^{\frac{1}{2}}p^{-\frac{n}{2}}+p^{\frac{n}{2}}\exp\left(c\frac{(\log p^n)^{\frac{1}{2}}}{\log\log p^n}\right)+p^{\frac{n}{2}}M\log M.$$

So, the proof of Theorem 1 is completed. □

Now we can investigate the distribution of quadratic residues modulo $p^n$ in narrow sectors.

**Theorem 2.** *Let* $p^{\frac{3}{2}n}\leqslant x\leqslant p^{2n}$, $0\leqslant\phi_1<\phi_2\leqslant\frac{\pi}{2}$ *and let* $0<s\leqslant\frac{1}{8}$. *Then for* $\phi_2-\phi_1\geqslant x^{-s}$ *the asymptotic formula*

$$R(x;\phi)=\frac{\phi_2-\phi_1}{2}\cdot\frac{p+1}{p}\cdot\frac{x}{p^n}+O\left(3^n\frac{x^{1-s}}{p^n}\log x\right)$$

*holds.*

*Proof.* It is known that the distribution of the arguments of Gaussian integers (being considered up to the association) has the period $\frac{\pi}{2}$. In view of this fact the application of Lemma 7 with $\Omega=\frac{\pi}{2}$ gives for every $T\geqslant 1$

$$\sum_{\substack{\alpha\in E_n^+\\ \phi_1\leqslant\alpha<\phi_2\\ N(\alpha)\leqslant x}}1=\Phi(\phi_1,\phi_2)+\theta_1\Phi(\phi_1-\Delta,\phi_1)+\theta_2\Phi(\phi_2,\phi_2+\Delta),$$

$$|\theta_i|\leqslant 1,\quad i=1,2,\qquad \Phi(\phi_1,\phi_2)=\frac{1}{4}\sum_{\substack{w\in E_n^+\\ N(w)\leqslant x)}}f(\arg w)$$

and $f(\phi)$ is the function from Lemma 7, $0<\Delta=\frac{1}{2}\Omega$.

Furthermore

$$\Phi(\phi_1,\phi_2)=\sum_{\substack{w\in E_n^+\\ N(w)\leqslant x)}}\sum_{m=-\infty}^{+\infty}a_m e^{4mi\arg\alpha}=\sum_{m=-\infty}^{+\infty}a_m\sum_{\substack{k\equiv 1\ (\mathrm{mod}\ p^n)\\ k\leqslant x}}r_m(x),$$

where $a_m$ is the Fourier coefficient from Lemma 7.

We put $\delta = x^s$, $0 < s < 1$ (we will find the more precise estimate for $s$ later). Let us use the estimates for the coefficients $a_m$ (see Lemma 7 with $r = 2$):

$$|a_m| \ll \begin{cases} \frac{1}{|m|}, & |m| \leqslant \delta = \Delta^{-1}; \\ \frac{1}{|m|^3\Delta^2}, & |m| > \delta. \end{cases}$$

After simple calculations we get

$$\Phi(\phi_1, \phi_2) = \frac{\phi_2 - \phi_1}{2} \cdot \frac{p+1}{p} \cdot \frac{x}{p^n} + O\left(\frac{x^{1-s}}{p^n}\right) + O\left(s\frac{\sqrt{x}}{p^{\frac{n}{2}}} \log^2 x\right)$$
$$+ O\left(sp^{\frac{n}{2}} \log^2 x\right) + O\left(\frac{x^{\frac{1}{2}+s}}{p^{\frac{n}{2}}}\right) + O\left(3^n p^{\frac{n}{2}} x^s \log x\right).$$

In view of the assumption of the theorem the following inequalities

$$\frac{x^{1-s}}{p^n} \gg p^{\frac{n}{2}} x^s, \qquad \frac{x^{1-s}}{p^n} \gg \frac{x^{\frac{1}{2}+s}}{p^{\frac{n}{2}}}$$

hold. Therefore,

$$\Phi(\phi_1, \phi_2) = \frac{\phi_2 - \phi_1}{2} \cdot \frac{p+1}{p} \cdot \frac{x}{p^n} + O\left(3^n \frac{x^{1-s}}{p^n} \log x\right). \tag{19}$$

It follows from (19) that

$$\Phi(\phi_1 - \Delta, \phi_1), \Phi(\phi_2, \phi_2 + \Delta) \ll 3^n \frac{x^{1-s}}{p^n} \log x. \tag{20}$$

The relations (19) and (20) show that Theorem 2 is proved for every $s$, $0 < s \leqslant \frac{1}{8}$. □

## Acknowledgment

The authors would like to thank the referee for useful remarks and suggestions.

## References

[1] D. Burgess, *The Distribution of quadratic residues and non-residues*, Mathematika, N.**4**, 1957, pp.106–112.

[2] E. Hecke, *Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen I-II*, Math.Z., T.**1**, 1918, ss.357-376; T.**6**, 1920, pp.11–51.

[3] J. Kubilius, *On one problem of multidimensional analytic number theory*, Proc. Vilnius Univ., N.**4**, 1955, pp.5–41 (in Lithuanian).

[4] H. Montgomery, R. Vaughan, *Exponential sums with multiplicative coefficients*, Inventiones math., N.**43**, 1977, pp.69–82.

[5] S. Sergeev, P. Varbanets, *Exponential sums over norm group*, Siauliai Math.Seminar, N.**9(17)**, 2014, pp.83–92.

[6] Elias M. Stein, Guido Weiss, *Introduction to Fourier analysis on Euclidian spaces*, Princeton, New Jersey, Princeton University Press, 1971, P.312.

[7] P. Varbanets, *Problem of circle in aritmetic progression*, Mat. Zametki, N.**8(6)**, 1970, pp.787–798 (in Russian).

[8] P. Varbanec, P. Zarzycki, *Divisors of the Gaussian integers in an Arithmetic Progression*, Journal of Number Theory, V.**33**, N.**2**, 1989, pp.152–169.

[9] S. Varbanets, *General Klosterman sums over ring of Gaussian integers*, Ukr. Math. J., N.**59(9)**, 2007, pp.1179–2000.

[10] I.M. Vinogradov, *Selectas*, Academic Press USSR, Moscow, 1952 (in Russian).

[11] A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A., **34**, 1948, pp. 204–207.

## CONTACT INFORMATION

**L. Balyas,**
**P. Varbanets**

Department of Computer Algebra and Discrete Mathematics, I. I. Mechnikov Odessa National University, Dvoryanskaya 2 65026 Odessa, Ukraine
*E-Mail(s)*: `balyas@ukr.net`, `varb@sana.od.ua`
*Web-page(s)*: `onu.edu.ua`

Received by the editors: 29.01.2016
and in final form 29.12.2016.