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A new characterization of finite σ-soluble $P\sigma T$-groups

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Abstract. Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and G a finite group. G is said to be *σ-soluble* if every chief factor H/K of G is a σ_i-group for some $i = i(H/K)$. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \varnothing$. A subgroup A of G is said to be *σ-quasinormal* or *σ-permutable* in G if G has a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$. We obtain a new characterization of finite σ-soluble groups G in which σ-permutability is a transitive relation in G.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. The group G is called *π-supersoluble* provided every chief factor of G is either cyclic or a π'-group. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

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In what follows, σ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$. The symbol $\sigma(n)$ denotes the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \varnothing\}$; $\sigma(G) = \sigma(|G|)$.

The group G is said to be: *σ-primary* (A.N. Skiba [1]) if G is a σ_i-group for some $i \in I$; *σ-decomposable* (L.A. Shemetkov [2]) or *σ-nilpotent* [3, 4] if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$; *σ-soluble* [1] if every chief factor of G is σ-primary.

A set $\mathcal{H}$ of subgroups of G is a *complete Hall σ-set* of G [3, 5] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$.

Let $\tau_{\mathcal{H}}(A) = \{\sigma_i \in \sigma(G) \setminus \sigma(A) \mid \sigma(A) \cap \sigma(H^G) \neq \varnothing$ for a Hall σ_i-subgroup $H \in \mathcal{H}\}$.

Then we say, following Beidleman and Skiba [6], that a subgroup A of G is: (i) *τ_σ-permutable in G with respect to $\mathcal{H}$* if $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ such that $\sigma(H) \subseteq \tau_{\mathcal{H}}(A)$; (ii) *$\tau_\sigma$-permutable* in G if A is *τ_σ-permutable* in G with respect to some complete Hall σ-set $\mathcal{H}$ of G.

Recall also that a subgroup A of G is said to be: *σ-permutable* in G [1] if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$; *σ-semipermutable* in G [7] if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ with $\sigma(A) \cap \sigma(H) = \varnothing$; *$\sigma$-subnormal* in G [1] if there is a subgroup chain

$$A = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$.

In the classical case when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$ (we use here the notations in [3]), σ-permutable, σ-semipermutable and τ_σ-quasinormal subgroups are also called respectively *S-permutable* [8], *S-semipermutable* [9] and *τ-permutable* [10, 11], and in this case σ-subnormal subgroups are exactly subnormal subgroups of the group.

It is clear that every σ-permutable subgroup is also σ-semipermutable and every σ-semipermutable subgroup is τ_σ-permutable.

Recall also that G is said to be a *$P\sigma T$-group* [1] if σ-permutability is a transitive relation in G, that is, if H is a σ-permutable subgroup of K and K is a σ-permutable subgroup of G, then H is σ-permutable in G. In the case when $\sigma = \sigma^1$, a $P\sigma T$-group is called a *PST-group* [8].

In view of Theorem B in [1], $P\sigma T$-groups can be characterized as the groups in which every σ-subnormal subgroup is σ-permutable. Another characterizations of $P\sigma T$-groups are obtained in the papers [3, 7, 12–14].

Our main goal here is to give a characterization of $P\sigma T$-groups in the terms of τ_σ-permutable subgroups.

Theorem 1.1. *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Suppose that G possesses a complete σ-set $\mathcal{H}$ all members of which are π-supersoluble. Then G is a σ-soluble $P\sigma T$-group if and only if every σ_i-subgroup of G is τ_σ-permutable in G for all $\sigma_i \in \sigma(D)$.*

In this theorem the symbol $G^{\mathfrak{N}_\sigma}$ denotes the *σ-nilpotent residual of G*, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N; $G^{\mathfrak{N}}$ is the nilpotent residual of G.

Corollary 1.1 (see Theorem A in [7]). *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Suppose G possesses a complete Hall σ-set $\mathcal{H}$ all members of which are π-supersoluble. If every σ_i-subgroup of G is σ-semipermutable in G for all $\sigma_i \in \sigma(D)$, then G is a σ-soluble $P\sigma T$-group.*

As another application of Theorem 1.1, we give the following new characterization of soluble PST-groups.

Corollary 1.2. *G is a soluble PST-group if and only if every subgroup of every Sylow p-subgroup of G is τ-semipermutable in G for all $p \in \pi(G^{\mathfrak{N}})$.*

All unexplained notation and terminology are standard. The reader is referred to [8], [15] or [16] if necessary.

2. Preliminaries

We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups.

Lemma 2.1 (see Lemma 2.5 in [1]). *The class $\mathfrak{N}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, if E is a normal subgroup of G and $E/E \cap \Phi(G)$ is σ-nilpotent, then E is σ-nilpotent.*

In view of Proposition 2.2.8 in [16], we get from Lemma 2.1 the following

Lemma 2.2. *If N is a normal subgroup of G, then*

$$(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} N/N.$$

Lemma 2.3 (V. N. Knyagina and V. S. Monakhov [17]). *Let H, K and N be pairwise permutable subgroups of G and H a Hall subgroup of G. Then*

$$N \cap HK = (N \cap H)(N \cap K).$$

Recall that G is said to be: a D_π-group if G possesses a Hall π-subgroup E and every π-subgroup of G is contained in some conjugate of E; a *σ-full group of Sylow type* [1] if every subgroup E of G is a D_{σ_i}-group for every $\sigma_i \in \sigma(E)$; *σ-full* [5] provided G possesses a complete Hall σ-set.

In view of Theorems A and B in [5], the following fact is true.

Lemma 2.4. *If G is σ-soluble, then G is a σ-full group of Sylow type.*

Lemma 2.5 (see Lemma 3.1 in [1]). *Let H be a σ_i-subgroup of a σ-full group G. Then H is σ-permutable in G if and only if $O^{\sigma_i}(G) \leqslant N_G(H)$.*

Lemma 2.6. *Suppose that G is σ-full and $D := G^{\mathfrak{N}_\sigma}$ is a nilpotent Hall subgroup of G. If every σ_i-subgroup of G is τ_σ-permutable in G for all $\sigma_i \in \sigma(D)$, then every subgroup of D is normal in G.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. By hypothesis, G possesses a complete Hall σ-set $\{H_1, \ldots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$.

First we show that the hypothesis holds on G/N for every minimal normal subgroup N of G. First note that

$$(G/N)^{\mathfrak{N}_\sigma} = DN/N \simeq D/(D \cap N)$$

is a nilpotent Hall subgroup of G/N by Lemma 2.2.

Now let V/N be a non-identity σ_i-subgroup of G/N for some

$$\sigma_i \in \sigma((G/N)^{\mathfrak{N}_\sigma}) = \sigma(DN/N) = \sigma(D/(D \cap N)) \subseteq \sigma(D).$$

And let U be a minimal supplement to N in V. Then $U \cap N \leqslant \Phi(U)$, so U is a σ_i-subgroup of G since $V/N = UN/N \simeq U/(U \cap N)$. Therefore U is τ_σ-permutable in G by hypothesis and $\sigma(U) = \sigma(UN/N) = \{\sigma_i\}$, which implies that $V/N = UN/N$ is τ_σ-quasinormal in G/N by Lemma 2.6(1) in [6]. Hence the hypothesis holds on G/N.

Now let H be a subgroup of the Sylow p-subgroup P of D for some prime $p \in \pi$. We show that H is normal in G. For some i we have $P \leqslant O_{\sigma_i}(D) = H_i \cap D$. On the other hand, we have $D = O_{\sigma_i}(D) \times O^{\sigma_i}(D)$

since D is nilpotent. Assume that $O^{\sigma_i}(D) \neq 1$ and let N be a minimal normal subgroup of G contained in $O^{\sigma_i}(D)$. Then $HN/N \leqslant DN/N = (G/N)^{\mathfrak{N}_\sigma}$, so the choice of G implies that HN/N is normal in G/N. Hence $H = H(N \cap O_{\sigma_i}(D)) = HN \cap O_{\sigma_i}(D)$ is normal in G.

Now assume that $O^{\sigma_i}(D) = 1$, so D is a σ_i-group. Since G/D is σ-nilpotent by Lemma 2.1, H_i/D is normal in G/D and hence H_i is normal in G. Therefore all subgroups of H_i are σ-permutable in G by Lemma 2.6(3) in [6] and hypothesis. Since D is a normal Hall subgroup of H_i, it has a complement S in H_i by the Schur-Zassenhaus theorem. Lemma 2.5 implies that $D \leqslant O^{\sigma_i}(G) \leqslant N_G(S)$. Hence $H_i = D \times S$. Hence $S \leqslant N_G(H)$, so

$$G = H_i O^{\sigma_i}(G) = (SD)O^{\sigma_i}(G) = SO^{\sigma_i}(G) \leqslant N_G(H),$$

so H is normal in G. Therefore every subgroup of D is normal in G since D is nilpotent by hypothesis. The lemma is proved. □

Lemma 2.7 (see Theorem A in [3]). *Let $D = G^{\mathfrak{N}_\sigma}$. If G is a σ-soluble $P\sigma T$-group, then the following conditions hold:*

(i) *$G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order, M is σ-nilpotent and every element of G induces a power automorphism in D;*

(ii) *$O_{\sigma_i}(D)$ has a normal complement in a Hall σ_i-subgroup of G for all i.*

Conversely, if Conditions (i) *and* (ii) *hold for some subgroups D and M of G, then G is a $P\sigma T$-group.*

Lemma 2.8 (see Theorem A in [3]). *The following statements hold:*

(1) *G is a $P\sigma T$-group if and only if every σ-subnormal subgroup of G is σ-permutable in G;*

(2) *If G is a σ-soluble $P\sigma T$-group, then every quotient G/N of G is also a σ-soluble $P\sigma T$-group.*

3. Proof of Theorem 1.1

Let $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$.

First show that if every σ_i-subgroup of G is τ_σ-permutable in G for all $\sigma_i \in \sigma(D)$, then G is a σ-soluble $P\sigma T$-group. Assume that this is false and let G be a counterexample of minimal order.

(1) $G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order, M is σ-nilpotent and every element of G induces a power automorphism in D (this claim directly follows from Lemma 2.6 and Theorem 1.5 in [18]).

(2) If R is a non-identity normal subgroup of G, then the hypothesis holds for G/R, so G/R is a σ-soluble $P\sigma T$-group.

First note that

$$\mathcal{H}_0 = \{H_1N/N, \dots, H_tN/N\}$$

is a complete Hall σ-set of G/N. Moreover, every member $H_iN/N \simeq H_i/(H_i \cap N)$ of $\mathcal{H}_0$ is π-supersoluble since H_i is π-supersoluble by hypothesis. On the other hand, $(G/N)^{\mathfrak{N}_\sigma} = DN/N \simeq D/(D \cap N)$ by Lemma 2.2. Hence $\pi_0 \subseteq \pi$, where $\pi_0 = \pi((G/N)^{\mathfrak{N}_\sigma})$, so every member of $\mathcal{H}_0$ is π_0-supersoluble.

Let V/N be a non-identity σ_i-subgroup of G/N for some

$$\sigma_i \in \sigma((G/N)^{\mathfrak{N}_\sigma}) = \sigma(DN/N) = \sigma(D/D \cap N) \subseteq \sigma(D).$$

And let U be a minimal supplement to N in V. Then $U \cap N \leqslant \Phi(U)$, so U is a σ_i-subgroup of G by the isomorphism $V/N = UN/N \simeq U/U \cap N$. Therefore U is τ_σ-permutable in G by hypothesis and $\sigma(U) = \sigma(UN/N) = \{\sigma_i\}$, which implies that $V/N = UN/N$ is τ_σ-permutable in G/N by Lemma 2.6(1) in [6]. Hence the hypothesis holds on G/N, so the choice of G implies that G/N is a σ-soluble $P\sigma T$-group.

(3) $H_i = O_{\sigma_i}(D) \times S$ for some subgroup S of H_i for each $\sigma_i \in \sigma(D)$.

Since D is a nilpotent Hall subgroup of G by Claim (1), $D = L \times N$, where $L = O_{\sigma_i}(D)$ and $N = O^{\sigma_i}(D)$ are Hall subgroups of G. First assume that $N \neq 1$. Then

$$O_{\sigma_i}((G/N)^{\mathfrak{N}_\sigma}) = O_{\sigma_i}(D/N) = LN/N$$

has a normal complement V/N in $H_iN/N \simeq H_i$ by Claim (2). On the other hand, N has a complement S in V by the Schur-Zassenhaus theorem. Hence $H_i = H_i \cap LSN = LS$ and $L \cap S = 1$ since

$$(L \cap S)N/N \leqslant (LN/N) \cap (V/N) = (LN/N) \cap (SN/N) = 1.$$

It is clear that V/N is a Hall subgroup of H_iN/N, so V/N is characteristic in H_iN/N. On the other hand, H_iN/N is normal in G/N by Lemma 2.2

since $D/N \leqslant H_iN/N$. Hence V/N is normal in G/N. Thus $H_i \cap V = H_i \cap NS = S(H_i \cap N) = S$ is normal in H_i, so $H_i = O_{\sigma_i}(D) \times S$.

Now assume that $D = O_{\sigma_i}(D)$. Then H_i is normal in G, so all subgroups of H_i are σ-permutable in G by Lemma 2.6(3) in [6]. Since D is a normal Hall subgroup of H_i, it has a complement S in H_i. Lemma 2.5 implies that $D \leqslant O^{\sigma_i}(G) \leqslant N_G(S)$. Hence $H_i = D \times S = O_{\sigma_i}(D) \times S$.

Now, from Lemma 2.7 and Claims (2) and (3) it follows that G is a σ-soluble $P\sigma T$-group, contrary our assumption on the G. This completes the proof of the sufficiency of the condition of the theorem.

Now we show that if G is a σ-soluble $P\sigma T$-group, then every σ_i-subgroup of G is τ_σ-permutable in G for each $\sigma_i \in \sigma(D)$. It is enough to show that H is a σ_i-subgroup of G, then H permutes with every Hall σ_j-subgroups of G for all $j \neq i$.

Assume that this is false and let G be a counterexample of minimal order. Then $D \neq 1$ and there are σ_i and σ_j $(i \neq j)$ such that $\sigma_i \in \sigma(D)$ and $HE \neq EH$ for some σ_i-subgroup H and some Hall σ_j-subgroup E of G. Then H is not σ-subnormal in G by Lemma 2.8. Hence a Hall σ_i-subgroup H_i of G is not normal in G since otherwise we have $H \leqslant H_i$ and so H is σ-subnormal in G by Lemma 2.6(6) in [1]. Now note that $|\sigma(D)| > 1$, Indeed, if $|\sigma(D)| = 1$, then $\sigma(D) = \{\sigma_i\}$ and so $D \leqslant H_i$, which implies that H_i/D is normal in G/D since G/D is σ-nilpotent by Lemma 2.1. But then H_i is normal in G, a contradiction.

Now we show that EHN is a subgroup of G for every minimal normal subgroup N of G. First note that the hypothesis holds for G/N by Lemma 2.8. Moreover, $HN/N \simeq H/H \cap N$ is a σ_i-subgroup of G/N. Therefore, if $\sigma_i \in \sigma(DN/N) = \sigma((G/N)^{\mathfrak{N}_\sigma})$, then the choice of G implies that

$$(HN/N)(EN/N) = (EN/N)(HN/N) = EHN/N$$

is a subgroup of G/N. Hence EHN is a subgroup of G. Now assume that $\sigma_i \notin \sigma(DN/N)$. Then a Hall σ_i-subgroup H_i of G is contained in N, so $H_i = N$ since N is σ-primary. But then $H \leqslant N$ and so H is σ-subnormal in G, a contradiction. Hence EHN is a subgroup of G.

Since $|\sigma(D)| > 1$ and D is abelian by Lemma 2.7, G has at least two σ-primary minimal normal subgroups R and N such that $R, N \leqslant D$ and $\sigma(R) \neq \sigma(N)$. Then at least one of the subgroups R or N, R say, is a σ_k-group for some $k \neq j$. Moreover,

$$R \cap E(HN) = (R \cap E)(R \cap HN) = R \cap HN$$

by Lemma 2.3 and $R \cap HN \leqslant O_{\sigma_k}(HN) \leqslant V$, where V is a Hall σ_k-subgroup of H, since N is a σ_k'-group and G is a σ-full group of Sylow type by Lemma 2.1. Hence

$$\begin{aligned} EHR \cap EHN &= E(HR \cap EHN) = EH(R \cap E(HN)) \\ &= EH(R \cap HN) = EH(R \cap H) = EH \end{aligned}$$

is a subgroup of G. Hence $HE = EH$. This contradicts the fact that $HE \neq EH$. The necessity of the condition of the theorem is proved.

The theorem is proved.

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