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A study on dual square free modules

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Abstract. Let M be an H-supplemented coatomic module with FIEP. Then we prove that M is dual square free if and only if every maximal submodule of M is fully invariant. Let $M = \bigoplus_{i \in I} M_i$ be a direct sum, such that M is coatomic. Then we prove that M is dual square free if and only if each M_i is dual square free for all $i \in I$ and, M_i and $\bigoplus_{j \neq i} M_j$ are dual orthogonal. Finally we study the endomorphism rings of dual square free modules. Let M be a quasi-projective module. If $\mathrm{End}_R(M)$ is right dual square free, then M is dual square free. In addition, if M is finitely generated, then $\mathrm{End}_R(M)$ is right dual square free whenever M is dual square free. We give several examples illustrating our hypotheses.

Introduction

We consider associative rings R with identity and all modules considered are unitary right R-modules. The notations $\mathrm{Rad}(M)$ and $\mathrm{End}_R(M)$ denote the radical and the endomorphism ring of any module M, respectively.

A module M is said to be *dual square free* or briefly *DSF* if whenever its factor module is isomorphic to $N^2 = N \oplus N$ for some module N, then $N = 0$. Note that any factor module of a DSF module is also DSF. A ring R is said to be *right (resp. left) dual square free* if it is dual square free as a right (resp. left) R-module. This concept was introduced first in [6]. We

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note that R is right DSF if and only if every cyclic right R-module is DSF. We also know that a module M is DSF if and only if M has no proper submodules A and B with $M = A + B$ and $M/A \cong M/B$ (see [12]).

Recall that a module M is *coatomic* if every proper submodule of M is contained in a maximal submodule. It is not difficult to see that M is coatomic if and only if every nonzero factor module of M has a maximal submodule. Let M and N be two right R-modules. M and N are called *dual orthogonal* if, no nonzero factor module of M is isomorphic to a factor module of N (it is called *factor-orthogonal* in [10]).

Let $\{M_i \mid i \in I\}$ be a family of modules. Recall that the direct sum decomposition $M = \bigoplus_I M_i$ is said to be *exchangeable* if, for any direct summand X of M, there exist $M_i' \subseteq M_i$ $(i \in I)$ such that $M = X \oplus (\oplus_I M_i')$. A module M is said to have the *(finite) internal exchange property* (or briefly, *(F)IEP*) if, any (finite) direct sum decomposition $M = \oplus_I M_i$ is exchangeable.

The organization of our paper is as follows:

In the first section, we investigate some properties of DSF modules. We also prove that for an H-supplemented coatomic module M with FIEP, M is DSF if and only if every maximal submodule of M is fully invariant. We illustrate our hypotheses in this section, as well.

In the second section, we work on direct sums of DSF modules. Let $M = \bigoplus_{i\in I} M_i$ be a direct sum, such that M is coatomic. Then M is DSF if and only if each M_i is DSF for all $i \in I$ and, M_i and $\bigoplus_{j\neq i} M_j$ are dual orthogonal. As a corollary we obtain that if $M = A \oplus B$ where A is a finitely generated DSF module and $B = \bigoplus_{i\in I} S_i$ is a direct sum of non-isomorphic simple modules, then M is a DSF module if and only if A and B are dual orthogonal.

In the last section, we investigate the endomorphism rings of DSF modules. In [10, Example 2.5], they prove that a strongly regular ring is a DSF ring. In [15], it is presented a module-theoretic version of strongly regular rings called *abelian endoregular modules*. As a generalization of [10, Example 2.5] we prove that if M is an endoregular quasi-projective module, then M is abelian if and only if M is a DSF. It sounds interesting to know when the endomorphism ring of a DSF module is a DSF ring and the converse. In this vein we prove that for any quasi-projective module M, if $\mathrm{End}_R(M)$ is right DSF, then M is DSF. In addition, if M is finitely generated, then $\mathrm{End}_R(M)$ is right DSF whenever M is DSF. Again we give examples illustrating our hypotheses in this section.

For undefined notions we refer to [12] and [13].

1. Dual square free modules

We start with the following result which can be established using the same arguments in [10, Proposition 2.13 and Proposition 2.15]. We just point out that, in Theorem 1, the implication (1) $\Rightarrow$ (2) is the proof of (1) $\Rightarrow$ (2) in [12, Lemma 2.6] and for the implication (2) $\Rightarrow$ (3), if $\mathrm{Rad}(M) = M$, then trivially M satisfies (3).

Theorem 1. (compare with [10, Theorem 2.16]) *Consider the following conditions for a module M:*

1) *M is DSF,*
2) *For any simple module S and every nonzero homomorphisms f, g from M to S, $\mathrm{Ker} f = \mathrm{Ker} g$,*
3) *Every maximal submodule of M is fully invariant.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*. If M is coatomic, then* (2) $\Rightarrow$ (1)*. In addition, if M is quasi-projective, then* (3) $\Rightarrow$ (1)*.*

The following examples illustrate that coatomic and quasi-projective hypotheses on Theorem 1 are not superfluous.

Example 1. Let $A = B = \mathbb{Z}(p^\infty)$ and $C = \mathbb{Z}_q$, where p and q are primes. Put $G_{\mathbb{Z}} = A \oplus B \oplus C$. Note that G is not coatomic and not quasi-projective. Also G is not DSF since it has the part $A \oplus B$.

Since $\mathrm{Rad}(A \oplus B) = A \oplus B$, $A \oplus B$ does not have a maximal submodule. On the other hand, $A \oplus B$ is the unique maximal submodule of G. Say $X = A \oplus B$.

Now let $f : G \to G$ be any endomorphism of G. If $f(X) \nsubseteq X$, then $G = X + f(X)$. Hence $C \cong G/X \cong f(X)/(X \cap f(X))$. By considering the epimorphism $X \to f(X)/(X \cap f(X))$ we have the epimorphism $\alpha : X \to C$. Then $\mathrm{Ker}\alpha$ is a maximal submodule of X, a contradiction. Therefore $f(X) \subseteq X$.

Example 2. Consider the above example. The $\mathbb{Z}$-module G is not coatomic and not DSF. Since it has a unique maximal submodule, condition (2) in Theorem 1 is satisfied.

Example 3. We take the next example from [3, Example 1.12]. Let $R = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ be the trivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This ring can be described as

$$R \cong \left\{ \begin{pmatrix} a & 0 & 0 \\ x & a & 0 \\ y & 0 & a \end{pmatrix} \mid a, x, y \in \mathbb{Z}_2 \right\}$$

with the usual operations of matrices. This ring is a finite local ring, hence there is only one simple right R-module up to isomorphism, say S. Consider an injective hull $E(S)$ of S. The right R-module $E(S)$ can be seen as the abelian group $M_{1\times 3}(\mathbb{Z}_2)$ with action the product of vectors by matrices. In [4, Section 3, Example 4] it is proved that every submodule of $E(S)$ is fully invariant. Thus, $E(S)$ satisfies (3) in Theorem 1 and is coatomic. Consider the lattice of submodules $\{0, S, K, L, N, E(S)\}$ of $E(S)$, which is drawn in [3, Example 1.12]. The module $E(S)$ is not DSF because $E(S)/S \cong S \oplus S$. On the other hand, assume $E(S)$ is quasi-projective. Consider the following diagram

$$\begin{array}{ccc} & & E(S) \\ & \swarrow^{\alpha} & \downarrow{\scriptstyle f} \\ E(S) & \xrightarrow[\pi]{} & S \oplus S \end{array}$$

where π is the composition $E(S) \twoheadrightarrow E(S)/S \cong S \oplus S$ and f is the composition $E(S) \twoheadrightarrow E(S)/N \cong S \hookrightarrow S \oplus S$. Suppose there exists $\alpha : E(S) \to E(S)$ such that $f = \pi\alpha$. Note that α cannot be the zero homomorphism neither an isomorphism. We have that every proper factor module of $E(S)$ is semisimple, hence $\alpha(E(S)) \subseteq S$. This implies that $0 = \pi\alpha = f$, which is a contradiction. Thus $E(S)$ is not quasi-projective.

Example 4. Let $\mathbb{Z}_{\widehat{p}}$ be the ring of p-adic integers and $\mathbb{Q}_{\widehat{p}}$ its quotient field. It is known that $\mathbb{Z}_{\widehat{p}}$ is a Dedekind domain which is a complete discrete valuation ring. Note that $\mathbb{Q}_{\widehat{p}}$ is a nonsingular injective $\mathbb{Z}_{\widehat{p}}$-module. By [16, Lemma 5.1], $\mathbb{Q}_{\widehat{p}}$ is quasi-projective. Let $M = \mathbb{Q}_{\widehat{p}} \oplus \mathbb{Q}_{\widehat{p}}$ be a right $\mathbb{Z}_{\widehat{p}}$-module. It follows from the fact that M is an injective module over a PID that M has no maximal submodules. Hence, we have that M is quasi-projective and every maximal submodule of M is fully invariant. Note that M is not coatomic neither DSF.

Recall that a module A is said to be *weakly generalized* (*epi-*)B-*projective* if, for any homomorphism (epimorphism) $f : A \to X$ and any epimorphism $g : B \to X$, there exist a small epimorphism $\rho : X \to Y$ for some module Y, decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism (an epimorphism) $h_1 : A_1 \to B_1$ and an epimorphism $h_2 : B_2 \to A_2$ such that $\rho(f|_{A_1}) = \rho g h_1$ and $\rho(g|_{B_2}) = \rho f h_2$.

Proposition 1. *Let M be an H-supplemented coatomic module with FIEP. Then M is DSF if and only if every maximal submodule of M is fully invariant.*

Proof. "Only if" part: By Theorem 1.

"If" part: Suppose that M is not DSF. Then, by the assumption, there exists an epimorphism $\rho : M \to S^2$ with S a simple module. Let $p_i : S^2 = S_1 \oplus S_2 \to S_i$ $(i = 1, 2)$ be the projections. Since M is H-supplemented, there exist a decomposition $M = M_1 \oplus M_2$ and a submodule K_1 of M such that K_1/M_1 is small in M/M_1 and $K_1/\mathrm{Ker}p_1\rho$ is small in $M/\mathrm{Ker}p_1\rho$. Then $M = \mathrm{Ker}p_1\rho + M_2$. As $\mathrm{Ker}p_1\rho$ is a maximal submodule of M, $\mathrm{Ker}p_1\rho = K_1$. Clearly, we have $M = \mathrm{Ker}p_1\rho + \mathrm{Ker}p_2\rho = M_1 + \mathrm{Ker}p_2\rho$. Let $p : M = M_1 \oplus M_2 \to M_2$ be the projection. Now $M_2 = p(\mathrm{Ker}p_2\rho)$ and since every maximal submodule of M is fully invariant, $M_2 \subseteq \mathrm{Ker}p_2\rho$.

Let $\pi_i : M \to M/\mathrm{Ker}p_i\rho$ $(i = 1, 2)$ be the natural epimorphisms. Then $\pi_i|_{M_j} : M_j \to M/\mathrm{Ker}p_i\rho$ $(i \neq j)$ is onto. By [13, Proposition 2.4], M_1 is weakly generalized epi-M_2-projective. Hence there exist decompositions $M_i = M_i' \oplus M_i''$ and epimorphisms $h_i : M_i' \to M_j''$ $(i, j = 1, 2,\ i \neq j)$ such that $(\pi_1|_{M_2})h_1 = \alpha(\pi_2|_{M_1'})$ and $\alpha(\pi_2|_{M_1})h_2 = \pi_1|_{M_2'}$, where $\alpha : M/\mathrm{Ker}p_2\rho \to S_1 \to S_2 \to M/\mathrm{Ker}p_1\rho$ is the natural isomorphism.

In the case that M_1' is not contained in $\mathrm{Ker}p_2\rho$, then $M = M_1' + \mathrm{Ker}p_2\rho$. Define $\varphi : M = M_1' \oplus M_1'' \oplus M_2 \to M_2''$ by $\varphi(m_1' + m_1'' + m_2) = h_1(m_1')$, where $m_1' \in M_1'$, $m_1'' \in M_1''$ and $m_2 \in M_2$. Since $\varphi(\mathrm{Ker}p_1\rho) \subseteq \mathrm{Ker}p_1\rho$ (every maximal submodule is fully invariant), for any $m_1' \in M_1' - \mathrm{Ker}p_2\rho$, $0 \neq \alpha(\pi_2|_{M_1})(m_1') = (\pi_1|_{M_2})h_1(m_1') = (\pi_1|_{M_2})\varphi(m_1') \in \pi_1(\mathrm{Ker}p_1\rho) = 0$, a contradiction.

In the case of $M_1' \subseteq \mathrm{Ker}p_2\rho$, then $0 = \alpha\pi_2(M_1') = (\pi_1|_{M_2})h_1(M_1')$ and so $M_2'' = h_1(M_1') \subseteq \mathrm{Ker}p_1\rho$. Hence M_2' is not contained in $\mathrm{Ker}p_1\rho$. Define $\psi : M = M_1 \oplus M_2' \oplus M_2'' \to M_1''$ by $\psi(m_1 + m_2' + m_2'') = h_2(m_2')$, where $m_1 \in M_1$, $m_2' \in M_2'$ and $m_2'' \in M_2''$. Since $\psi(\mathrm{Ker}p_2\rho) \subseteq \mathrm{Ker}p_2\rho$, for any $m_2' \in M_2' - \mathrm{Ker}p_1\rho$, $0 \neq (\pi_1|_{M_2})(m_2') = \alpha\pi_2 h_2(m_2') = \alpha\pi_2\psi(m_2') \in \alpha\pi_2(\mathrm{Ker}p_2\rho) = 0$, a contradiction.

Therefore M is DSF. □

Example 5. Following the notation in Example 3, set $M = E(S)$. Then every maximal submodule of M is fully invariant, M is coatomic and satisfies FIEP because it is uniform. Consider $K \leqslant M$ in Example 3. Note that the unique submodule X^* of M satisfying $X^*/K \ll M/K$ is $K = X^*$, and there is no direct summand A of M such that $K/A \ll M/A$. Thus, M is not H-supplemented neither DSF.

Example 6. Let $A = B = \mathbb{Z}(p^\infty)$ and put $M = A \oplus B$. Let $f : A \to X$ be a nonzero homomorphism and $g : B \to X$ be an epimorphism for some module X. If f is onto, then $\mathrm{Ker}f \subseteq \mathrm{Ker}g$ or $\mathrm{Ker}g \subseteq \mathrm{Ker}f$ since $\mathbb{Z}(p^\infty)$

is a uniserial module. Hence there exists an epimorphism $h : A \to B$ such that $gh = f$ or an epimorphism $h : B \to A$ such that $fh = g$ by Ker$f$ and Ker$g$ is small in $\mathbb{Z}(p^\infty)$. If f is not onto, then $f(A)$ is small in X since X is hollow. Let $\rho : X \to X/f(A)$ be the natural epimorphism and let $h' = 0$. Then ρ is a small epimorphism and $\rho f = 0 = \rho g h'$. Thus A is weakly generalized B-projective. By [13, Theorem 2.7], M is H-supplemented. In addition, since M has no maximal submodules, it satisfies that every maximal submodule of M is fully invariant. Moreover, M is injective and so it satisfies the exchange property. Thus M is H-supplemented with the exchange property. But it is not coatomic, not DSF.

Let U_R be a module. U is called *quasi-small* if given a family of modules $\{U_\alpha : \alpha \in \Gamma\}$ such that U is isomorphic to a direct summand of $\oplus_{\alpha\in\Gamma} U_\alpha$, there exists a finite subset $F \subseteq \Gamma$ such that U is isomorphic to a direct summand of $\oplus_{\alpha\in F} U_\alpha$. Suppose that U is a uniserial module. Then the endomorphism ring $E = \mathrm{End}_R(U)$ has two (two sided) ideals $L = \{f \in E : \text{f is not injective}\}$ and $K = \{f \in E : \text{f is not surjective}\}$ such that every proper right ideal of E is contained either in L or in K (see [11, Proposition 3.7] and [7, Theorem 1.2]).

Now we give the following example which is important in terms of existing of an H-supplemented module that does not satisfy FIEP.

Example 7. Let U be a uniserial right R-module which is not quasi-small with $E = \mathrm{End}_R(U)$. Let $K = \{f \in E : f \text{ is not surjective}\}$ and $L = \{f \in E : f \text{ is not injective}\}$ be two two-sided ideals of E. Consider the subset $\{f_n : n \in \mathbb{N}^*\}$ of E such that $f_{n+1} f_n = f_n$ for all $n \in \mathbb{N}^*$ and $K = \sum_{n=1}^{\infty} f_n E$. Fix $m \in \mathbb{N}^*$ such that $f_m \notin KJ(E)$. By [12, Example 3.3], the right E-module $E/f_m E$ is radical projective hollow and $\mathrm{End}_E(E/f_m E)$ is not local. By [13, Theorem 2.7] and [1, Proposition 12.10], $E/f_m E \oplus E/f_m E$ is an H-supplemented E-module which does not satisfy FIEP.

In [9, Lemma 2.2] it is proved that a module M is distributive if and only if every submodule of M is DSF. The next proposition adds new equivalent conditions to that lemma and makes a connection with the general distributivity presented in [2]. For, we introduce some terminology from [2]. Given a cardinal ω, we write ω^+ for the smallest cardinal larger that ω. By $\mathrm{crs}(R)$ we denote the cardinality of all (non-isomorphic) simple left R-modules and if M is a semisimple module, then dim$M$ denotes the cardinal number of simple summands of M. Let ω be a cardinal. It is said that a module M is *ω-thick* provided dim$S < \omega$ for any semisimple subfactor S of M.

Proposition 2. *Let M be a module. Then the following are equivalent:*

1) *M is distributive.*
2) *$X/\mathrm{Rad}(X)$ is DSF for any submodule X of M.*
3) *M is crs$(R)^+$-thick. (In particular M is crs$(R)^+$-distributive)*

Proof. (1)⇒(2) By Lemma 2.2 in [9], M is distributive if and only if any factor module of M is square free if and only if any submodule of M is DSF. Hence it is clear that, if M is distributive, then $X/\mathrm{Rad}(X)$ is DSF for any submodule X of M.

(2)⇒(3) We claim that $\mathrm{Soc}(M/X)$ is square free for any submodule X of M. Suppose that $\mathrm{Soc}(M/X)$ is not square free. Then there exists a submodule N of $\mathrm{Soc}(M/X)$ such that $N = S_1 \oplus S_2$, where $S_1 \cong S_2$ is simple. Let $\pi : M \to M/X$ be the natural epimorphism and put $T = \pi^{-1}(N)$, $f = \pi\mid_T$. Then $\mathrm{Ker}p_i f$ is maximal in T, where $p_i : N = S_1 \oplus S_2 \to S_i$ is the projection. By $\mathrm{Rad}(T) \subseteq \mathrm{Ker}p_1 f \cap \mathrm{Ker}p_2 f = \mathrm{Ker}f$, there exists an epimorphism from $T/\mathrm{Rad}(T)$ to $T/\mathrm{Ker}f \cong N = S_1 \oplus S_2$. Since $T/\mathrm{Rad}(T)$ is DSF, $N = 0$, a contradiction. Thus $\mathrm{Soc}(M/X)$ is square free for any submodule X of M. Now, let S be a semisimple subfactor of M, that is, $S = N/X$ for submodules N and X of M. Since S is semisimple, $S \leqslant \mathrm{Soc}(M/X)$. By the claim above, the homogeneous components of S have size one. This implies that $\dim S \leqslant \mathrm{crs}(R) < \mathrm{crs}(R)^+$. Thus, M is $\mathrm{crs}(R)^+$-thick.

(3)⇒(1) Let N be a submodule of M. Suppose that there is an epimorphism $\rho : N \to L \oplus L$ for some module L. If L is nonzero, there is a semisimple subfactor $S \oplus S$ of N for some simple module S. It follows from [2, Corollary 4.5(c)] that $S \oplus 0$ is fully invariant in $S \oplus S$ which cannot be. Thus, $L = 0$. This implies that every submodule of M is DSF. Hence, M is distributive by Lemma 2.2 in [9]. □

2. Direct sums of dual square modules

Proposition 3. *Let $M = \bigoplus_{i\in I} M_i$ be a direct sum, such that M is coatomic. Then M is DSF if and only if each M_i is DSF for all $i \in I$ and, M_i and $\bigoplus_{j\neq i} M_j$ are dual orthogonal.*

Proof. (⇒) It is clear.

(⇐) Let $f : M \to S_1 \oplus S_2$ be an epimorphism, with $0 \neq S_1 \cong S_2$. Since any nonzero factor module of M has a maximal submodule, we can assume S_1 is simple. Put $N = S_1 \oplus S_2$.

Since $f \neq 0$, there is $i \in I$ such that $f(M_i) \neq 0$. If $f(M_i) \cap S_j \neq 0$ for each $j = 1, 2$, then $f(M_i) = N = S_1 \oplus S_2$, a contradiction. Then, suppose $f(M_i) \cap S_2 = 0$. Then $f(M_i) \oplus S_2 = f(M_i) + f\left(\bigoplus_{j\neq i} M_j\right)$. If this sum is not direct, then $f(M_i)$ is a direct summand of $f\left(\bigoplus_{j\neq i} M_j\right)$ or $f\left(\bigoplus_{j\neq i} M_j\right)$ is a direct summand of $f(M_i)$, which is a contradiction because M_i and $\bigoplus_{j\neq i} M_j$ are dual orthogonal. Therefore $f(M_i) \oplus S_2 = f(M_i) \oplus f\left(\bigoplus_{j\neq i} M_j\right)$. This implies that $f(M_i) \cong S_1 \cong S_2 \cong f\left(\bigoplus_{j\neq i} M_j\right)$, a contradiction. Thus $S_1 = 0 = S_2$. □

The proof of next lemma can be found in [8, Corollary 5], we write the statement here for the convenience of the reader.

Lemma 1. *Let M and N be two coatomic modules. Then $M \oplus N$ is coatomic.*

Corollary 1. *Let $M_1, ..., M_n$ be coatomic modules and $M = \bigoplus_{i=1}^{n} M_i$. Then M is DSF if and only if each M_i is DSF for all $1 \leqslant i \leqslant n$ and, M_i and $\bigoplus_{j\neq i} M_j$ are dual orthogonal.*

The following corollary should be compared with Proposition 2.8 in [10].

Corollary 2. *Let $M = A \oplus B$ where A is a finitely generated DSF module and $B = \bigoplus_{i\in I} S_i$ is a direct sum of non-isomorphic simple modules. Then M is a DSF module if and only if A and B are dual orthogonal.*

3. Endomorphism rings of dual square free modules

Next proposition is a generalization of [10, Example 2.5].

Proposition 4. *If M is an endoregular quasi-projective module, then M is abelian if and only if M is DSF.*

Proof. ($\Rightarrow$) Let A and B proper submodules of M such that $M = A+B$ and $M/A \cong M/B$. Since M is quasi-projective, $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, A) + \mathrm{Hom}_R(M, B)$. This implies that $M = \mathrm{Hom}_R(M, A)M + \mathrm{Hom}_R(M, B)M$. Set $A' = \mathrm{Hom}_R(M, A)M$ and $B' = \mathrm{Hom}_R(M, B)M$. Then $M = A' + B'$ and there are epimorphisms $\rho_1 : M/A' \to M/A$ and $\rho_2 : M/B' \to M/B$. By [15, Proposition 2.9], A' and B' are fully invariant submodules of M. Hence $A' \cap B'$ is fully invariant. It follows that $M/A' \cap B'$ is quasi-projective and by [15, Proposition 2.8], it is an abelian endoregular module. Moreover,

$M/A' \cap B' = (A'/A' \cap B') \oplus (B'/A' \cap B')$ with $A'/A' \cap B' \cong M/B'$ and $B'/A' \cap B' \cong M/A'$. By [15, Proposition 2.11(d)],

$$0 = \mathrm{Hom}_R\left(A'/A' \cap B', B'/A' \cap B'\right) = \mathrm{Hom}_R(M/B', M/A').$$

Since $M/A' \cap B'$ is quasi-projective, M/B' is M/A'-projective. This implies that the following diagram can be completed commutatively only with the zero homomorphism

$$\begin{array}{ccc} M/B' & \overset{0}{\dashrightarrow} & M/A' \\ {\scriptstyle \rho_2}\downarrow & & \downarrow{\scriptstyle \rho_1} \\ M/B & \underset{\cong}{\longrightarrow} & M/A. \end{array}$$

Thus $\rho_2 = 0$, that is, $M = B$. Analogously $M = A$. Therefore, M is DSF.

($\Leftarrow$) Let A and B be M-generated submodules of M such that $A \cap B = 0$ and $A \cong B$. There exists a nonzero homomorphism $\alpha : M \to A$. Then there exists $B' \leqslant B$ such that $\alpha(M) \cong B'$ and $\alpha(M) \cap B' = 0$. Hence, without loss of generality we can take $A = \alpha(M)$ and $B = B'$. Since M is endoregular, A and B are direct summands of M. The module M satisfies SSP (sum of any two direct summands of M is again a direct summand of M) by [14, Theorem 2.4]. Then $A \oplus B$ is a direct summand of M, that is, there exists, $D \leqslant M$ such that $M = A \oplus B \oplus D$. It follows that $M/(B \oplus D) \cong A \cong B \cong M/(A \oplus D)$. Since M is DSF, $A \oplus D = B \oplus D$. In particular, $A \subseteq B \oplus D$. Therefore, $0 = A \cap (B \oplus D) = A$. Analogously $B = 0$. Hence M is abelian endoregular by [15, Proposition 2.11(c)]. $\square$

Corollary 3. ([10, Example 2.5]) *Let R be a ring. Then, R is strongly regular if and only if R is a regular and left (right) DSF module.*

Remark 1. Any DSF module M with SSP and SIP (intersection of any two direct summands of M is again a direct summand of M) satisfies the internal cancellation property. For, let M be a DSF module with SSP and SIP. Let $M = A \oplus C = B \oplus D$ and let $f : A \to B$ be an isomorphism. Since M satisfies SIP, there exist decompositions $A = (A \cap D) \oplus A'$ and $D = (A \cap D) \oplus D'$. Then $B = f(A \cap D) \oplus f(A')$. Hence $M = B \oplus D = f(A \cap D) \oplus (A \cap D) \oplus f(A') \oplus D'$. Since M is DSF with SSP, $A \cap D = 0$ and $A \oplus D$ is a direct summand of M. Put $M = (A \oplus D) \oplus K$. By $M = B \oplus D$, there exists a decomposition $B = B' \oplus B''$ such that $K \cong B'$. Hence $M = A \oplus D \oplus K = f^{-1}(B') \oplus f^{-1}(B'') \oplus D \oplus K$ and $f^{-1}(B') \cong B' \cong K$.

Since M is DSF, $B' = 0$ and hence $K = 0$. Thus we see $M = A \oplus D$ and so $C \cong D$. This means that M satisfies the internal cancellation property.

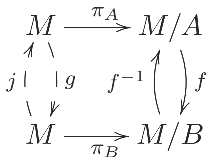
Now by [14, Theorem 2.4], any endoregular module satisfies the SSP and SIP properties. Hence any DSF endoregular module satisfies the internal cancellation property.

Theorem 2. *Let M be a quasi-projective module. Consider the following conditions:*

1) *$S = \mathrm{End}_R(M)$ is a right DSF ring.*
2) *M is a DSF module.*

Then $(1) \Rightarrow (2)$. *In addition, if M is finitely generated, then* $(2) \Rightarrow (1)$.

Proof. $(1) \Rightarrow (2)$: Let A and B submodules of M such that $M = A + B$ with $M/A \cong M/B$. Since M is quasi-projective, $S = \mathrm{Hom}_R(M, A) + \mathrm{Hom}_R(M, B)$. Let $f : M/A \to M/B$ be an isomorphism, and let $\pi_A : M \to M/A$ and $\pi_B : M \to M/B$ be the natural epimorphisms.



Since M is quasi-projective, there exists $g \in S$ such that $\pi_B g = f\pi_A$. It follows that $g(A) \subseteq B$. We claim that $\theta : S/\mathrm{Hom}_R(M, A) \to S/\mathrm{Hom}_R(M, B)$ given by $\theta(1 + \mathrm{Hom}_R(M, A)) = g + \mathrm{Hom}_R(M, B)$ is an isomorphism of right S-modules. Let $h \in \mathrm{Hom}_R(M, A)$. Then $gh(M) \subseteq g(A) \subseteq B$. Thus $gh \in \mathrm{Hom}_R(M, B)$ and so θ is well defined. It is clear that θ is an S-homomorphism of right S-modules. Now, if $h + \mathrm{Hom}_R(M, A)$ is such that $\theta(h + \mathrm{Hom}_R(M, A)) = 0$, that is, $gh \in \mathrm{Hom}_R(M, B)$, then $0 = \pi_B gh = f\pi_A h$. Since f is an isomorphism, $\pi_A h = 0$. This implies that $h(M) \subseteq A$ and hence $h \in \mathrm{Hom}_R(M, A)$. Thus, θ is a monomorphism. If we take now $f^{-1} : M/B \to M/A$, there exists $j \in S$ such that $\pi_A j = f^{-1}\pi_B$ because M is quasi-projective. Note that $\pi_B = ff^{-1}\pi_B = f\pi_A j = \pi_B gj$. Let $l + \mathrm{Hom}_R(M, B) \in S/\mathrm{Hom}_R(M, B)$. Then,

$$\pi_B(gjl - l) = \pi_B gjl - \pi_B l = \pi_B l - \pi_B l = 0.$$

It follows that $gjl - l \in \mathrm{Hom}_R(M, B)$. Therefore, $\theta(jl + \mathrm{Hom}_R(M, A)) = gjl + \mathrm{Hom}_R(M, B) = l + \mathrm{Hom}_R(M, B)$. Hence θ is an epimorphism and so an isomorphism. Since S is right DSF, $\mathrm{Hom}_R(M, A) = S = \mathrm{Hom}_R(M, B)$. Hence $M = SM = \mathrm{Hom}_R(M, A)M \subseteq A$. Analogously, $B = M$. Thus, M is DSF.

Now, assume that M is a finitely generated quasi-projective module. (2) $\Rightarrow$ (1): Let I and J be right ideals of S such that $S = I + J$ and $S/I \cong S/J$. It follows that $M = IM + JM$. Let $\theta : S/I \to S/J$ be an isomorphism of right S-modules. Write $h + J = \theta(1 + I)$. Let $f : M/IM \to M/JM$ be given by $f(m + IM) = h(m) + JM$. Suppose $m - n \in IM$. Therefore $m - n = \sum_{i=1}^{\ell} g_i(k_i)$ with $g_i \in I$ and $k_i \in M$. Then, $0 = \theta(g_i + I) = \theta(1 + I)g_i = (h + J)g_i = hg_i + J$, for all $1 \leqslant i \leqslant \ell$. Thus, $hg_i \in J$ for all $1 \leqslant i \leqslant \ell$. Hence,

$$h(m - n) = h(\sum g_i(k_i)) = \sum hg_i(k_i) \in JM.$$

It is clear that f is an R-homomorphism. Analogously, if $\theta^{-1}(1+J) = h'+I$ then we have an R-homomorphism $g : M/JM \to M/IM$ given by $g(m + JM) = h'(m)+IM$. Note that $1+I = \theta^{-1}\theta(1+I) = \theta^{-1}(h+J) = h'h+I$, hence $1 - h'h \in I$. Analogously, $1 - hh' \in J$. Now, let $m + IM \in M/IM$. Then

$$gf(m + IM) = g(h(m) + JM) = h'h(m) + IM = m + IM,$$

and

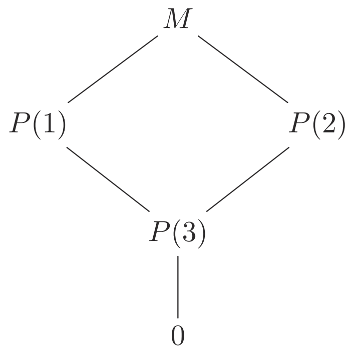
$$fg(m + JM) = f(h'(m) + IM) = hh'(m) + JM = m + JM.$$

It follows that $M/IM \cong M/JM$. Since M is DSF, $IM = M = JM$ and so $S = \mathrm{Hom}_R(M, IM) = \mathrm{Hom}_R(M, JM)$. We have that $I = \mathrm{Hom}_R(M, IM)$ and $J = \mathrm{Hom}_R(M, JM)$ because M is finitely generated and quasi-projective [17, 18.4]. Thus, $I = S = J$, that is, S is right DSF. □

Example 8. Let K be a field and A a hereditary K-algebra. Let P be an indecomposable projective A-module. By [15, Example 2.2(v)] we know that, $\mathrm{End}_A(P) \cong K$. Therefore $\mathrm{End}_A(P)$ is a right (and left) DSF ring. Then from the above Theorem, P is a DSF A-module.

Example 9. (see [5, Example 2.3]) Let K be a field and let A be the hereditary K-algebra given by the quiver $\underset{1}{\bullet} \longrightarrow \underset{3}{\bullet} \longleftarrow \underset{2}{\bullet}$. Let M be the indecomposable injective left A-module $I(3) = \begin{smallmatrix} 1 & & 2 \\ & 3 & \end{smallmatrix}$. Here the indecomposable projective non-isomorphic left A-modules are $P(1) = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$,

$P(2) = \begin{matrix} 2 \\ 3 \end{matrix}$ and $P(3) = 3$. Then the lattice of submodules of M is



Note that the projective left A-module $P = P(1) \oplus P(2)$ is the projective cover of M. Since M is finitely generated, P is finitely generated. On the other hand, $\mathrm{End}_A(P) \cong K \oplus K$. Therefore $\mathrm{End}_A(P)$ is not a left (and right) DSF ring. Hence P is not DSF left A-module by Theorem 2. But from Example 8, each $P(1)$ and $P(2)$ is DSF. Clearly, $M = P(1) + P(2)$ and $M/P(1) \ncong M/P(2)$, where $P(1)$ and $P(2)$ are the only proper submodules of M with sum M. Therefore M is clearly a DSF left A-module. Also $\mathrm{End}_A(M)$ is a left (right) DSF ring since $\mathrm{End}_A(M) \cong K$.

As we see in the following example, in Theorem 2 (1) $\Rightarrow$ (2), "quasi-projective" hypothesis is not superfluous.

Example 10. Consider the module $E(S)$ in Example 3. $E(S)$ is not quasi-projective and it is not DSF. On the other hand, since $E(S)$ is indecomposable injective, $\mathrm{End}_R(E(S))$ is local, which is a right and left DSF ring.

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