Algebra and Discrete Mathematics
Volume **22** (2016). Number 1, pp. 48–81 SURVEY ARTICLE

Generalized norms of groups

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Communicated by I. Ya. Subbotin

Abstract. In this survey paper the authors specify all the known findings related to the norms of the group and their generalizations. Special attention is paid to the analysis of their own study of different generalized norms, particularly the norm of non-cyclic subgroups, the norm of Abelian non-cyclic subgroups, the norm of infinite subgroups, the norm of infinite Abelian subgroups and the norm of other systems of Abelian subgroups.

Introduction

In group theory findings related to the study of characteristic subgroups (in particular, the center, the derived subgroup, Frattini subgroup, etc.) and the impact of properties of these subgroups on the structure of the group are in the focus. Nowadays the list of such characteristic subgroups can be broaden by means of different Σ-norms of a group.

Let Σ be the system of all subgroups of the group which have some theoretical group property. For example, Σ can consist of all subgroups of the group, of all cyclic, all non-cyclic, all Abelian, all non-Abelian, all subnormal, all maximal, all infinite subgroups of the group. The intersection $N_\Sigma(G)$ of the normalizers of all subgroups of the group which

2010 MSC: 20D25, 20E28.

Key words and phrases: norm of group, generalized norm, Wielandt subgroup, norm of non-cyclic subgroups, norm of Abelian non-cyclic subgroups, norm of infinite subgroups, norm of infinite Abelian subgroups, norm of decomposable subgroups.

belong to the system Σ is called Σ-norm of a group G. In the case $\Sigma=\varnothing$ we assume that $G=N_{\Sigma}(G)$.

By the definition of the Σ-norm it follows that it is a characteristic subgroup of the group and contains the center of the group. Also, $N_{\Sigma}(G)$ is the maximal subgroup of the group that normalizes all Σ-subgroups of the group. Therefore, all subgroups of the Σ-norm, which belong to the system Σ, are normal in $N_{\Sigma}(G)$ (although these subgroups may not exist).

Considering the Σ-norm, there are several problems related to the study of the group properties with the given system Σ of subgroups and some restrictions, which the norm satisfies. Many algebraists solved the similar problems but the choice of a system Σ and properties of the Σ-norm varied.

Knowing the structure of Σ-norms and the nature of its attachment to the group, the properties of the group can be characterized in many cases. For example, if the Σ-norm coincides with the group and $\Sigma \neq \varnothing$, then all subgroups of the system Σ are normal in the group. First non-Abelian groups with this property were considered in the XIX century by R. Dedekind [1], who gave a complete description of finite non-Abelian groups, all subgroups of which are normal, and called them Hamiltonian groups. Infinite Hamiltonian groups were described in 1933 by R. Baer [2]. Sets of Abelian and Hamiltonian groups combined are called the set of Dedekind groups.

However, the study of groups with other systems Σ of normal subgroups were continued only in the second part of the XX century, that slowed down the study of Σ-norms. The findings of S. M. Chernikov and his disciples are from the very field of the research. Thus nowadays the structure of groups that coinside with the norm $N_{\Sigma}(G)$ is known for many systems of subgroup. So the question on the study of the properties of groups, in which the Σ-norm is a proper subgroup, arises naturally.

1. The norm of group and subgroups close to it

For the first time the problem of the study of the properties of groups, which differ from the Σ-norm, was formulated by R. Baer in 30s of the previous century. In [3] he introduced a subgroup $N(G)$, which is the intersection of normalizers of all subgroups of a group, and called it the norm of the group G. It is clear that the norm $N(G)$ is the Σ-norm of the group for the system Σ, which consists of all subgroups of the group. The norm $N(G)$ is contained in all other Σ-norms and they can be considered as its generalizations. It is also clear that Dedekind groups coincide with their norms, so the index of the norm in a group can serve as a certain "degree of Dedekindness" of a group.

The norm of a group was studied by R. Baer [3–10] and several other authors [11–28]. R. Baer noticed that the restrictions that are imposed on the norm of the group influence the structure of the group in a certain way. Thus, there is a proposition.

Proposition 1.1 ([3])**.** *If a norm $N(G)$ of a group G is Hamiltonian, then the following propositions take place:*

1) *G is a periodic group;*
2) *G contains no elements which orders are divisible by 8;*
3) *all elements of a group G, which have orders multiple of 4, can be represented in the form of za, where $a \in N(G)$, $|a| = 4$, and $z \in G$, more over z is permutable with each element of a norm $N(G)$;*
4) *any element, which order is not divisible by 4, is permutable with every element of the subgroup $N(G)$.*

Studying the relations between the norm and the center of the group R. Baer showed that the norm coincides with the center of the group, if it contains elements of infinite order [3]. Another important result, which specifies the influence of the center of the group on its norm, was offered in [10].

Proposition 1.2 ([10])**.** *The norm of a group G is identity if and only if G is a group with an identity center.*

Developing the study of the properties of the norm of a group L. Wos [11] found out that the norm $N(G)$ is contained in the third hypercentre of the group, and the group of automorphisms, which are induced on the subgroup *N(G)* by G, is nilpotent of class at most 2. In addition, it was proved that the norm of the group is contained in the second hypercentre if and only if the group of automorphisms induced on $N(G)$ by the group G is Abelian. This result was substantially refined by E. Shenkman in [12].

Proposition 1.3 ([12])**.** *The norm $N(G)$ of a group G is contained in the second hypercentre of G. The derived subgroup G' is a subgroup of centralizer of a norm $N(G)$ in G.*

So the group of automorphisms induced on $N(G)$ by G is Abelian. Let's note that $N(G) = E$ in groups with the identity center by Proposition 1.3.

In [6,8] the properties of periodic groups with an Abelian norm quotient group were considered. In particular, in [6] it was proved that a periodic group G, which quotient group $G/N(G)$ is Abelian and $N(G) \neq Z(G)$, is a direct product of its primary components, and its norm $N(G)$ is a direct product of norms of these components.

In this regard, let's note that unlike some other characteristic subgroups (the center of the group, the derived subgroup, Fitting subgroup and others) the norm of the direct product of arbitrary subgroups is not equal to the direct product of the norms of the correlative components in general case.

Example 1.1. Let $G = Q \times B$, where B is a non-periodic Abelian group of rank 1, Q is a quaternion group of order 8. In this group

$$N(G) = N(Q \times B) = Q^2 \times B \neq N(Q) \times N(B) = G.$$

The problem of finding the norms of direct products of groups was studied by J. Evan in [13].

Let's also regard the following finding of R. Baer, which characterizes the properties of p-groups with an Abelian norm quotient group.

Proposition 1.4 ([8])**.** *If G is a p-group ($p \neq 2, p \neq 3$) that has an Abelian quotient group for the norm $N(G)$, more over $N(G) \neq Z(G)$ and p^r is the exponent of the group $C_G(N(G))$, then:*

1) *G is a group of finite exponent;*
2) *$N(G)/Z(G)$ is a cyclic group, the order of which is equal to the exponent of the group of automorphisms induced on $N(G)$ by G;*
3) *centralizer $C_G(N(G))$ consists of those and only those elements $x \in G$ for which $x^{p^r} = 1$.*

The restrictions $p \neq 2, p \neq 3$ in Proposition 1.4 are significant, as it is illustrated by the examples of the respective groups (see [6]).

Nowadays the interest to the norm $N(G)$ of a group is not reduced, as research works [13–31], devoted to the study of its properties, are still numerous. Thus, in [17, 18] R. Bryce and J. Cossey considered series of norms

$$1 = N_0(G) \subseteq N_1(G) \subseteq \ldots \subseteq N_i(G) \subseteq \ldots,$$

where $N_i(G)/N_{i-1}(G) = N(G/N_{i-1}(G))$ for $i \geqslant 1$.

It was proved that in the class of 2-groups from the fact that quotient group $N_{i+1}(G)/N_i(G)$ is Hamiltonian it follows that $N_{i+1}(G) = G$. Moreover a finite 2-group, in which the quotient group $G/N(G)$ is Hamiltonian, but any quotient group $N_i(G)/N_{i-1}(G)$ is not Hamiltonian, has order 2^7 and is uniquely determined up to isomorphism [18].

Starting from R. Baer, L. Wos and E. Schenkman studies of the norm $N(G)$ focus on its relation to the centre of the group. In particular, in [19] J. Beidleman, H. Heineken and M. Newell have shown that in an arbitrary p-group G either quotient group $G/Z(G)$ or group $[G, N(G)]$ is cyclic. In this article the problem of the influence of properties of a norm of a group and its center on the capability of a group G is considered.

A group G is called capable, if it is a group of inner automorphisms of some group H that is $G \cong H/Z(H)$. R. Baer [29] studied such groups for the first time. He described capable finitely generated Abelian groups

$$G = Z_{n_1} \oplus Z_{n_2} \oplus \ldots \oplus Z_{n_k},$$

where $n_{i+1} \vdots n_i$, $n_i \in N \bigcup \{0\}$ and $Z_{n_i} = Z$ is an infinite cyclic group for $n_i = 0$. It was found out that the group G is capable if and only if $k \geqslant 2$ and $n_{k-1} = n_k$. The Baer's characterization remains the only complete one for a certain class of capable groups today.

Developing studies of the norm of a group in capable groups, X. Guo and X. Zhang [20] in 2012 established necessary and sufficient conditions for the coinsiding of the norm of the group with its centre, and also dwelled upon the properties of the norm $N(G)$ in the class of nilpotent groups with a cyclic derived subgroup.

In 2005 N. Gavioli, L. Legarreta, S. Sica, M. Tota [22] considered the relations between the centre $Z(G)$, the norm $N(G)$ and the second hypercentre $Z_2(G)$ depending on the number $v(G)$ of conjugacy classes of non-normal subgroups and the number $w(G)$ of conjugacy classes of subgroups, which are normalizers of some subgroups, in finite p-groups $(p \neq 2)$ of nilpotency class c.

In 2008 F. Russo [23] studied the relations between the centre $Z(G)$, the norm $N(G)$, the quazicenter $Q(G)$ and the hyperquazicenter $Q^*(G)$ of quazicentral-by-finite groups. Let's regard that the quazicenter $Q(G)$ of a group G is the subgroup, generated by all elements x of a group G, such that the subgroup $\langle x \rangle$ is permutable in a group G (with other subgroups). Accordingly, the hyperquazicenter $Q^*(G)$ of a group G is the largest term of the chain of normal subgroups

$$E = Q_0(G) \leqslant Q_1(G) = Q(G) \leqslant \ldots \leqslant Q_\alpha \leqslant Q_{\alpha+1} \leqslant \ldots,$$

where α is an ordinal and $Q_{\alpha+1}(G)/Q_\alpha(G) = Q(G/Q_\alpha(G))$.

Proposition 1.5 ([23], Proposition 3.2)**.** *Let G be a quasicentral-by-finite group, $Q(G)$ be the quasicenter of G, $N(G)$ be the norm of G, then*

1) *if $Q(G)$ contains only elements of prime or infinite order and $Q(G) = N'$, where N is the subgroup generated by the quasicentral elements of infinite order, then G is finite;*
2) *if there is an element $x \in N(G)$ such that the index $|Q(G) : \langle x\rangle_G|$ is finite, then G is central-by-finite;*
3) *G is central-by-finite if and only if the index $|Q(G) : N(G)|$ is finite.*

The relations between the norm $N(G)$ and the center $Z(G)$ in the class of finite groups have also been studied by I. V. Lemeshev in [24]. His findings add much to Baer's results related to finite groups.

The study of finite groups, in which Baer norm has a certain index, is very effective. In particular, in [25] J. Wang and X. Guo studied finite p-groups, in which the norm has a prime index, in [26] they studied finite groups, in which the norm is a subgroup of index p or pq, where p and q are different prime numbers. J. Smith [27] studied groups in which each subgroup of the norm is normal in the group.

Subgroups of an arbitrary group can be considered as elements of some subgroups lattice $L(G)$ relative to the operations of union and intersection, ordered by inclusion. In this sense, the norm $N(G)$ of a group can be defined as following [28]:

$$N(G) = \bigcap_{X \in L(G)} N_G(X).$$

In this context, in [28] the relation between the non-cyclicness of the norm $N(G)$ on the one hand and the subgroup lattice $L(G)$ of the group G and generalized degree of commutativity of the group G on the other hand is under the analysis.

A question naturally arises why this characteristic subgroup, in contrast to the center and the derived subgroup, did not get adequate attention in the early development of group theory in view of the simple definition of the norm and its usefullness in the study of groups. G. Miller [31] explains that at that time other problems were posed in algebra and the main focus of group theory has been directed to the study of solutions of algebraic equations (in this theory simple groups play a fundamental role, while the norm of a simple group of composite order is identity). The norm was also not of high importance in the study of permutation groups of low degrees, which were used in the theory of algebraic equations at that time. The smallest degree of permutation group, which has the norm of prime index, is equal to 8, moreover only one of 200 groups of this order has the norm of prime index. And perhaps R.Baer drew attention to this characteristic subgroup only in 1934 for these reasons.

Considering the intersection of normalizers of subgroups of the group, we can get subgroups associated with Baer norm. These are the intersection of normalizers of all subgroups contained in the given subgroup [32–34] or conversely the intersection of the normalizers which contain the given subgroup [35]. In particular, the concept of invariator $I_G(A)$ of the subgroup A in the group G, which was introduced by I. Ya. Subbotin, is the closest to the concept of the norm $N(G)$ of the group G.

Invariator $I_G(A)$ of subgroup A in the group G [32] (*quazicenralizer* [34]) is the intersection of normalizers of all subgroups of the group A in G. This subgroup can also be called the norm of the subgroup A in the group G [36]. In the case when the subgroup A coincides with the whole group G the invariator $I_G(G)$ is exactly the norm $N(G)$ of the group G.

In 2001 M. De Falco, F. de Giovanni, C. Musella [35] introduced the concept of H-norm of the group G for some subgroup H of the group G. H-norm of a group G is called a subgroup $\ker(G : H)$ that consists of all elements which normalize every subgroup of X in G containing H. Obviously, if $H \leqslant K \leqslant G$, then $H \leqslant \ker(G : H) \leqslant N_G(H)$, $\ker(G : H) \leqslant \ker(G : K)$. Let's note that the E-norm, where E is the identity subgroup of a group G, coincides with the norm $N(G)$ of the group G.

It is clear that the norm $N(G)$ can be defined as the subgroup of a group G consisting of all elements of this group, which normalize every subgroup in G. Replacing the condition of normality to pronormality we get some analogue of the norm of a group for pronormality. It is called *pronorm* $P(G)$.

Let's regard that an element x of the group G *pronormalizes subgroup* H of a group G, if subgroups H and H^x are conjugate in $\langle H, H^x \rangle$. Accordingly, the *pronorm* $P(G)$ of a group G is the set of all elements of a group G which pronormalize every subgroup of a group. For the first time the concept of pronorm $P(G)$ group was introduced by F. de Giovanni, S. Vincenzi [37] in 2000.

In contrast to the norm of a group the pronorm is not always a subgroup of a group. In [38] some classes of groups, in which the set of all elements of a group G that pronormalize every subgroup of a group, forms a subgroup, were studied.

Proposition 1.6 ([37]). *If G is polycyclic group, then its pronorm $P(G)$ is a subgroup.*

In this work a similar statement for the class of locally soluble groups was proved.

Subgroups generated by normalizers of given subgroups are considered in some researches about groups with restrictions on normalizers of given systems of subgroups. In this context, let's consider the research of J. Smith [39], who studied the subgroup $R = R(G)$ generated by all proper normalizers, and called it *conorm of a group*. If the group G has not proper normalizers, then the group G is Dedekind and $R(G) = E$.

In 1990 H. Bell, F. Guzman, L.-Ch. Kappe [40] studied so-called Baer-kernel, which is a ring analogue of the norm of the group. Baer-kernel of the ring K is defined as the set

$$B(K) = \{a \in K | \forall y \in K, \exists r, s \in N(ay = y^r a \wedge ya = ay^s)\}.$$

In 2010 year M. R. Dixon, L. A. Kurdachenko, D. Otal used the so-called norm of subspace in linear groups in the research of linear groups with finite dimensional orbits [41].

Let A be a vector space over a field F, $GL(F, A)$ be a group of all automorphisms of a space A, G be a subgroup of a group $GL(F, A)$, B be a subspace of a space A. The norm of the subspace B in the group G is the intersection of normalizers of all F-subspases in B:

$$Norm_G(B) = \bigcap_{b \in B} N_G(bF).$$

It is known when the group G coincides with the norm $Norm_G(B)$, then the group G is isomorphic to a subgroup of the multiplicative group $U(F)$. If the group G has finite dimensional orbits over A, then A contains a FG-submodule D of finite dimension $\dim_F(D)$. If $K = C_G(D)$, then $K \leqslant Norm_G(A/D)$. When G-orbits of every subspace from A are finite, then A contains a FG-submodule B such that $\dim_F(A/B)$ and $|G : Norm_G(B)|$ are also finite.

Therefore, the research devoted to the study of the norms of the group and related subgroups is a very important and interesting direction in the group theory. At the same time, there are still many questions regarding the structural characteristics of the group depending on the structure of its norms, conditions of coinsiding of the norm of the group and its center, etc. left.

2. Generalized norms of some systems of maximal and subnormal subgroups

As noted above, the norm $N(G)$ is the Σ-norm of the group, in which the system Σ is a system of all subgroups of this group. Narrowing the system of all subgroups, for example, to the system of all Abelian or all maximal subgroups of a group, we will get new Σ-norms, which can be considered as generalizations of the norm $N(G)$.

The first generalizations of this kind were made in the 50-th of the XX century. In particular, in 1953 R. Baer [42] considered the intersection $H(G)$ of normalizers of all Sylow subgroups of a group G and called this intersection as *hypercenter of a group* G. It is clear that hypercenter $H(G)$ is the Σ-norm, where the system Σ consists of all Sylow subgroups of the group. R. Baer proved that $H(G)$ coincides with the intersection of all maximal nilpotent subgroups, and the quotient group $G/H(G)$ is a group with an identity center. Moreover, it was found out that the normal subgroup belongs to a hypercenter if and only if its elements of order p^n generate cyclic subgroups of index p^n.

In 1968 B. Huppert [43] generalized the concept of a hypercenter introducing the concept of $\Im$-*hypercenter*. Let $\Im$ be a class of finite groups which can be represented as direct products of their Hall π-subgroups with respect to some partition of non-empty set π of all primes. This class is a local formation. The chief factor H/K of a group G is called $\Im$-central [44], if $H/K \lambda (G/C_G(H/K)) \in \Im$. The product of all normal subgroups of G which G-chief factors are $\Im$-central in G is called $\Im$-*hypercenter* $Z_{\Im}(G)$ of a group G [45]. In 2013 V. I. Murashka [46] studied the properties of $\Im$-hypercenter and got some Baer's results on the norm of the group as corollaries in some cases.

One of the mentioned generalizations of the norm of the group is a so-called A-norm $N_A(G)$ of the group G. It is the intersection of normalizers of all maximal Abelian subgroups. This norm was introduced by W. Kappe [47] in 1961. As it turned out (see [47]) in finite group A-norm is a subgroup, each element of which is permutable with its conjugate (such groups were studied, in particular, by F. Levi [48]). In addition, it was found that the A-norm is close to a subgroup of right Engel elements of length 2, that allowed to use it in the study of Engel groups.

Let's regard (see e.g. [49]) that the element $x \in G$ is called the *right Engel element of length 2*, if for any element $g \in G$ there is a relation $[[x,\ g],\ g] = 1$.

Let $R(G)$ denote the subgroup of a group G generated by all right Engel elements of length 2 of a group G. The following propositions take place.

Proposition 2.1 ([47]). *A-norm $N_A(G)$ of a group G contains the second hypercenter of a group G and is contained in the subgroup $R(G)$. Moreover the quotient group $R(G)/N_A(G)$ is elementary Abelian group of exponent not exeeding 2.*

Proposition 2.2 ([47]). *For an element $x \in G$ which order is not divisible by 2, the following statements are equivalent:*

1) *$x \in N_A(G)$;*
2) *x is right Engel element of length 2 in G;*
3) *if $\langle x\rangle \lhd G$ and U is the group of automorphisms induced on $\langle x\rangle$ by G, then x belongs to A-norm of the group $\langle x\rangle U$;*
4) *for any elements $g, h \in G$ the equality $[[x, g], h] = [[x, h], g]^{-1}$ takes place.*

The following proposition on a A-norm is a generalization of Wos' [11] and Schenkman's results [12] related to the norm $N(G)$ of the group.

Proposition 2.3 ([47]). *Group G induces on the subgroup $N_A(G)$ a nilpotent group of automorphisms. Its class of nilpotency does not exceed 2.*

Later on W. Kappe [50–52] generalized the concept of the A-norm of the group and introduced a so-called E-norm, which was defined as the intersection of normalizers of all maximal subgroups of the group with the given theoretical group property E. Clearly, E-norm $N_E(G)$ contains the norm $N(G)$. The intersection of an arbitrary subgroup of a group G and the E-norm of the group is contained in the E-norm of this subgroup. Besides $N_E(N_E(G)) = N_E(G)$.

A subgroup $\Delta(G)$ is related to the concept of the E-norm. It was studied by W. Gashutz [53] and was defined as the intersection of normalizers of all maximal subgroups of the group. It is clear that Gashutz subgroup $\Delta(G)$ can be considered as Σ-norm of a group for the system Σ that consists of non-normal in G maximal subgroups. In [53] it was found out that $\Delta(G)$ is nilpotent and $\Delta(G)/\Phi(G) = Z(G/\Phi(G))$, where $\Phi(G)$ is Frattini subgroup.

In 1958 H. Wielandt [54] studied the properties of normalizers of subnormal subgroups and introduced the subgroup $W(G)$. It is the intersection of normalizers of all subnormal subgroups of a group. It is clear that *Wielandt subgroup* $W(G)$ is the *norm of subnormal subgroups* of a group.

It is obvious that a subnormal norm coincides with the norm $N(G)$ in a nilpotent group. In addition, the condition $G = W(G)$ is equivalent to the fact that all subnormal subgroups of a group are normal. By Theorem 13.3.7 [55] Wielandt subgroup $W(G)$ contains every simple non-Abelian subnormal subgroup of G and every minimal normal subgroup of G which satisfies the minimal condition for subnormal subgroups. Therefore, the subgroup $W(G)$ is not identity in a finite group G [54].

D. Robinson [56] and J. Roseblade [57] independently from each other got similar results for some classes of infinite groups.

Proposition 2.4 ([56, 57])**.** *If a group G satises the minimal condition for subnormal subgroups, then the quotient group $G/W(G)$ is finite.*

These results were summarized by J. Cossey [58] for polycyclic groups. It was found out that these groups have a finite quotient group $G/C_G(W(G))$.

Wieland subgroup and its generalizations were studied intensively by O. Kegel [59], J. Cossey, R. Bryce [60–62], R. Brandl, F. Giovanni, S. Franciosi [63]. A. Camina [64], C. Casolo [65, 66], E. Ormerod [67], C. Wetherell [68, 69], X. Zhang and X. Guo [70, 71].

In [60] it was proved that the subnormal norm $W(G)$ is contained in the FC-centre in a finitely generated soluble-by-finite group of a finite rank. Furthermore, if the norm $W(G)$ coincides with the whole group, then all subnormal subgroups are normal in this group, that is, the normality is transitive relation. Groups with such a property were studied by D. Robinson in [72] and were called T-groups. If G is a finite soluble T-group and G/L is the unique maximal nilpotent quotient group of group G, then the quotient group G/L is Abelian or Hamiltonian and L is Abelian.

In 1989 J. Cossey, R. Bryce [60] introduced *local Wielandt subgroup* $W^p(G)$ that is the intersection of normalizers of all p'-perfect subnormal subgroup of a group G. Let's regard that the p'-perfect group is a group that has no non-identity quotient groups of order coprime with p.

In 1992 C. Casolo [66] studied a special subgroup of a group $W(G)$, which was called *strong Wielandt subgroup* $\overline{W}(G)$, and defined as the intersection of the centralizers of nilpotent subnormal quotient groups of the group G:

$$\overline{W}(G) = \left\{ g \in G | [S, g] \leqslant S^R \ \text{ for all } \ S << G \right\},$$

where S^R is nilpotent residual of the subgroup S or the smallest normal subgroup N of S such that the quotient group S/N is nilpotent. C. Casolo proved that strong Wielandt subgroup $\overline{W}(G)$ is non-identity in a finite group. Note that this subgroup was also studied by C. Wetherell [68, 69].

In 1990 R. Bryce [62] introduced one more generalization of Wielandt subgroup, so-called *m-Wielandt subgroup* $U_m(G)$ of a group G that is the intersection of normalizers of all subnormal subgroups of a group G with a defect at most m for integer $m \geqslant 1$. He studied a polynilpotent lattice of finite soluble groups in terms of Wielandt m-length. The concept of m-series of Wielandt group is widely used. It is defined as following: for each natural $m \geqslant 1, U_{m,0}(G) = E$; if $i \geqslant 1$, then $U_{m,i}(G)$ is determined from the condition

$$U_{m,i}(G)/U_{m,i-1}(G) = U_m(G/U_{m,i-1}(G)).$$

If $U_{m,n}(G) = G$ for some integer n, then such a minimal number n is called Wielandt m-length. R. Bryce proved that there are limits of commutator length and Fitting length of finite soluble groups in terms of Wielandt m-length ($m \geqslant 2$), and identified the best such a restriction. Properties of *Wielandt m-subgroup* $U_m(G)$ have also been studied by C. Franchi [76, 77].

In 1995 J. Biedleman, M. Dixon, D. Robinson [73, 74] considered one more Σ-norm of a group – *generalized Wielandt subgroup* $IW(G)$ which is the intersection of normalizers of all infinite subnormal subgroups of a group. It is clear that $IW(G)$ is a characteristic subgroup and contains a subnormal norm $W(G)$. If $G = IW(G)$, then all infinite subnormal subgroups are normal in the group. Such groups have been studied by F. Giovanni, S. Franciosi [75] and were called *IT-groups.* In [73] the structure of the group G with the property $IW(G) \neq W(G)$ and the structure of the quotient group $IW(G)/W(G)$ were studied.

In [78] F. Mari, F. Giovanni introduced a new Σ-norm, in which system Σ consists of all nonsubnormal subgroups of a group. This *norm of nonsubnormal subgroups* was denoted by $W^*(G)$. It is clear that if $W^*(G) = G$, then all subgroups are subnormal in a group. Moreover, if a group G is a group with a finite number of normalizers of subnormal subgroups, then the quotient group $G/W^*(G)$ is finite [78].

Let's also mention the research [79], in which so-called *generalized N-Wielandt subgroup* $W_N(G)$ was introduced. It consists of all elements of the group G, which normalize all subnormal subgroups of N. It is a normal subgroup and, in general, may differ from N.

It is clear that $W(G) \subseteq W_N(G)$, in particular, $W(G) = W_N(G)$, if $N = G$, or $N = W(G)$, or N is the unique maximal normal subgroup. If G is a T-group and N is a normal subgroup of G, then $W_N(G) = G$. The following example proves that the converse is not true.

Example 2.1 ([79])**.** Let $G = D_8 = \langle x, y \rangle$, $x^8 = y^2 = (xy)^2 = 1$, $N_1 = \langle x^2 \rangle$, $N_2 = \langle x \rangle$, then $W_{N_1}(G) = W_{N_2}(G) = G$, but G is not a T-group.

3. Generalized norms of characteristic subgroups of a group

Nowadays algebraists direct their attention to a generalization of the norm when the system Σ is selected as a system of some characteristic subgroups. In this context Sh. Lia and Zh. Shen [80, 81] considered the Σ-norm $D(G)$ of a finite group, where the system Σ is chosen as a system of derived subgroups of all subgroups of the group. The authors proved that in the case when $D(G)$ contains all the elements of prime order, the group G is solvable of Fitting length at most 3. In the case when $G = D(G)$, derived subgroup G' is nilpotent and G'' has nilpotency class at most 2.

Recently a number of researches concern the norms of different systems of residuals. In particular, Zh. Shen, W. Shi and G. Qian [82] studied the *norm $S(G)$ of nilpotent residuals of all subgroups of prime order*. It was proved that if all elements of prime order of a finite group G are contained in the norm $S(G)$, then the group G is solvable. L. Gong and X. Guo [83] studied the norm of nilpotent residuals of all subgroups of a finite group. N. Su and Ya. Wang [84] considered the norm $D^{\mathfrak{F}}(G)$ of *$\mathfrak{F}$-residual* $G^{\mathfrak{F}}$ of all subgroups of the group G and the norm $D_p^{\mathfrak{F}}(G)\mathfrak{H}^{\mathfrak{F}}O_{p'}(G)$ of all subgroups H of a finite group G, where $\mathfrak{F}$ is the formation. Recall that $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of a group G is the smallest normal subgroup N of G such that $G/N \in \mathfrak{F}$.

X. Chen and W. Guo [85] introduced the $\mathfrak{h}\mathfrak{F}$-norm $N_{\mathfrak{h}\mathfrak{F}}(G)$ of a group G. It is the intersection of normalizers of products of $\mathfrak{F}$-residuals of all subgroups of a group G and $\mathfrak{h}$-radical of a group G

$$N_{\mathfrak{h},\mathfrak{F}}(G) = \bigcap_{H \leqslant G} N_G(H^{\mathfrak{F}} G_{\mathfrak{h}}),$$

where $\mathfrak{h}$ is Fitting class, $\mathfrak{F}$ is formation. Let's regard that $\mathfrak{h}$-radical $G_{\mathfrak{h}}$ of a group G is maximal normal $\mathfrak{h}$-subgroup of a group G.

If $\mathfrak{h} = 1$, then the subgroup $N_{1,\mathfrak{F}}(G)$ is called $\mathfrak{F}$-norm $N_{\mathfrak{F}}(G)$ of a group G and defined as

$$N_{\mathfrak{F}}(G) = \bigcap_{H \leqslant G} N_G(H^{\mathfrak{F}}).$$

If $\mathfrak{h} = \mathfrak{G}_\pi$, where $\mathfrak{G}_\pi$ is the class of finite π-solvable groups, then the subgroup $N_{\mathfrak{G}_\pi,\mathfrak{F}}(G)$ is called $\pi\mathfrak{F}$-norm $N_{\pi\mathfrak{F}}(G)$ of a group G and defined as

$$N_{\pi\mathfrak{F}}(G) = \bigcap_{H \leqslant G} N_G(H^{\mathfrak{F}} O_\pi(G)).$$

X. Chen and W. Guo studied the properties of $\mathfrak{h}\mathfrak{F}$-norm, in particular, $\pi\mathfrak{F}$-norm of a finite group G and the relations between $\pi'\mathfrak{F}$-norm and $\pi\mathfrak{F}$-hypercentre of a group G.

In 2014 A. Ballester-Bolinches, J. Cossey, L. Zhang [86] proposed to generalize the structure of Σ-norms which had appeared recently. The authors defined the *C-norm $kC(G)$* of a finite group G as the intersection of the normalizers of all subgroups of the group G which *do not belong to the class C*

$$k_C(G) = \bigcap_{H \notin C} N_G(H)$$

provided that $k_C(G) = G$, if $G \in C$. With this approach Baer norm $N(G)$ can be considered as the norm $k_C(G)$, where C is the class of groups of order 1. Groups with $k_C(G) = G$ are called C-Dedekind. In [86] the structure of non-nilpotent C-Dedekind groups for the class of nilpotent groups is described. It is also shown that the groups, which C-norm is not hypercentral, have a very restricted structure. The authors gave the classification of nilpotent classes closed under subgroups, quotient groups and direct products of groups of coprime orders, and showed that the known classifications can be deduced from this one.

Proposition 3.1 ([86]). *If $k_C(G)$ contains a non-central chief factor of G, then $k_C(G)$ contains exactly one non-central chief factor (in any chief series through $k_C(G)$ of a group G) and if p is a prime divisor of the order of this chief factor, then Hall p'-subgroup of G is C-group and G has nilpotency class at most 3.*

Consider also R. Laue's research [87]. He dealt with a subgroup close to the Σ-norm

$$A(\Sigma) = \bigcap_{X \in \Sigma} N_{Aut(G)}(X),$$

which consists of automorphisms that normalize every Σ-subgroup of a group G.

4. Generalized norms of different systems of Abelian and non-cyclic subgroups

The narrowing of a system Σ of all subgroups of the group G to the system of all Abelian and all cyclic subgroups does not lead to extension of the concept of the norm $N(G)$. However, when the system Σ is the system of all non-cyclic subgroups (provided that such subgroups exist in the group), then such Σ-norm (let's call it the *norm of non-cyclic subgroups*) differs from the norm $N(G)$ in a general case. The opportunity to study the norm of non-cyclic subgroups was provided by F. M. Lyman's research [88–90]. He received a description of some classes of non-Abelian groups in which all non-cyclic subgroups are normal. These groups were called $\overline{H}$-*groups* ($\overline{H_p}$-*groups* in the case of p-groups).

The concept of the *non-cyclic norm N_G* of a group as the intersection of the normalizers of all non-cyclic subgroups of the group was introduced by F. M. Lyman in 1997 [91], where he studied infinite groups, in which a non-cyclic norm is locally-graded and has a finite index.

Proposition 4.1 ([91]). *In the group G a non-cyclic norm is locally-graded and has a finite index if and only if the group G is central-by-finite.*

In addition, it was proved that for the condition $1 < |G/N_G| < \infty$ in the class of infinite locally finite groups the non-cyclic norm N_G is Dedekind, and in the class of non-periodic locally soluble-by-finite groups it is Abelian [91].

The study of the non-cyclic norm was continued by F. M. Lyman and T. D. Lukashova in [92–96], where the authors characterized the structure of wide classes of groups, which non-cyclic norm is non-Dedekind. Since O. Yu. Olshansky infinite groups [97] exist, periodic groups were considered by the authors provided their local finiteness. O. Yu. Olshansky infinite groups are groups, all subgroups of which are cyclic and which are the norms of their non-cyclic subgroups. Thus, in [98] it was proved that the class of infinite locally finite p-groups ($p \neq 2$), in which a non-cyclic norm N_G is non-Abelian, coincides with the class of non-Abelian p-groups, all non-cyclic subgroups of which are normal. At the same time, there are infinite locally finite 2-groups which have a proper non-Dedekind norm of non-cyclic subgroups. The structure of locally finite p-groups (p is prime), which non-cyclic norm is non-Dedekind, is described in [92–94].

Proposition 4.2 ([92]). *Locally finite p-groups ($p \neq 2$), which have non-Abelian non-cyclic norm N_G, are groups of the following types:*

1) *G is an $\overline{H}_p$-group, $N_G = G$;*
2) $G = (\langle x\rangle \times \langle b\rangle)\lambda\langle c\rangle$, $|x| = p^n$, $n > 1$, $|b| = |c| = p$, $[b, c] = x^{p^{n-1}}$, $[x, c] = x^{p^{n-1}\alpha} b^{\beta}$, $(\beta, p) = 1$; $N_G = (\langle x^p\rangle \times \langle b\rangle)\lambda\langle c\rangle$;
3) $G = \langle x\rangle\langle b\rangle, |x| = p^k, |b| = p^m, m > 1, k \geqslant m + r, Z(G) = \langle x^{p^{r+1}}\rangle \times \langle b^{p^{r+1}}\rangle, 1 \leqslant r \leqslant m - 1, [x, b] = x^{p^{k-r-1}s} b^{p^{m-1}t}, (s, p) = 1, N_G = \langle x^{p^r}\rangle\lambda\langle b\rangle$.

Proposition 4.3 ([93,94]). *Locally finite 2-groups G with a non-Dedekind non-cyclic norm N_G are groups of the following types:*

1) *G is a non-Hamiltonian $\bar{H}_2$-group, $G = N_G$;*
2) $G = (A \times \langle b\rangle)\lambda\langle c\rangle\lambda\langle d\rangle$, *$A$ is a quasicyclic 2-group,* $[A, \langle c\rangle] = 1$, $|b| = |c| = |d| = 2$, $d^{-1}ad = a^{-1}$ *for any element* $a \in A$, $[b, c] = [d, b] = [d, c] = a_1$, $a_1 \in A, |a_1| = 2$; $N_G = (\langle a\rangle \times \langle b\rangle)\lambda\langle c\rangle$, *where* $a \in A, |a| = 4$;
3) $G = (A \times H)\langle d\rangle$ *A is a quasicyclic 2-group,* $d^2 = a_1 \in A, |a_1| = 2, d^{-1}ad = a^{-1}$ *for any element* $a \in A$, $H = \langle h_1, h_2\rangle, |h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1, h_2]$, $[d, h_1] = a_1, [d, h_2] = 1$; $N_G = \langle h_2\rangle\lambda\langle h_1 a\rangle, |a| = 4, a \in A, |a| = 4$;
4) $G = (\langle x\rangle \times \langle b\rangle)\lambda\langle c\rangle\lambda\langle d\rangle, |x| = 2^n, n > 2, |b| = |c| = |d| = 2, [x, c] = 1, d^{-1}xd = x^{-1}$, $[b, c] = [d, b] = [d, c] = x^{2^{n-1}}$; $N_G = (\langle x^{2^{n-2}}\rangle \times \langle b\rangle)\lambda\langle c\rangle$;
5) $G = (\langle x\rangle\lambda\langle b\rangle)\lambda\langle c\rangle$, $|x| = 2^n$, $n > 3$, $|b| = |c| = 2$, $[x, c] = x^{\pm 2^{n-2}} b$, $[x, b] = x^{2^{n-1}}$; $N_G = (\langle x^2\rangle \times \langle b\rangle)\lambda\langle c\rangle$;

6) $G = \langle x\rangle\lambda H, |x| = 2^n, n > 2, H = \langle h_1, h_2\rangle, |h_1| = |h_1| = 4, h_1^2 = h_2^2 = [h_1, h_2], [\langle x\rangle, H] = \langle x^{2^{n-1}}\rangle; N_G = \langle x^2\rangle \times H$;
7) $G = (\langle x\rangle \times H)\langle y\rangle, |x| = 2^n, n \geqslant 2, H = \langle h_1, h_2\rangle, |h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1, h_2], y^2 = x^{2^{n-1}}, [y, h_2] = 1, [y, h_1] = y^2, y^{-1}xy = x^{-1}$; $N_G = \langle h_2\rangle\lambda\langle h_1 x^{2^{n-2}}\rangle$;
8) $G = \langle x\rangle\langle b\rangle, |x| = 2^k, |b| = 2^m, m \geqslant 1$; *if* $m = 1$, *then* $k = 3$, $[x, b] = x^2$ *and* $N_G = \langle x^2\rangle\lambda\langle b\rangle$; *if* $m > 1$, *then* $k \geqslant m + r, 1 \leqslant r \leqslant m - 1, Z(G) = \langle x^{2^{r+1}}\rangle \times \langle b^{2^{r+1}}\rangle, [x, b] = x^{2^{k-r-1}s} b^{2^{m-1}t}$, $0 < s < 2, 0 \leqslant t < 2$, ($k > 3$ *and* $t = 0$ *if* $m = 2$); $N_G = \langle x^{2^r}\rangle\lambda\langle b\rangle$.

Non-primary locally finite groups with a non-Dedekind non-cyclic norm were studied in [93, 105]. It was found out that infinite locally finite non-primary groups with such restrictions on the norm N_G are locally nilpotent.

Proposition 4.4 ([93]). *Infinite locally finite non-primary groups with a non-Dedekind non-cyclic norm N_G, are locally nilpotent and are groups of the following types:*

1) *G is an infinite non-primary non-Hamiltonian $\bar{H}$–group, $G = N_G$;*
2) *$G = G_2 \times \langle y\rangle, G_2$ is a group of one type 2) or 3) of proposition 4.3, $(|y|, 2) = 1; N_G = N_{G_2} \times \langle y\rangle$.*

Thus, a locally finite group, which non-cyclic norm N_G is non-nilpotent, is finite.

Developing the study of locally finite groups with a non-Dedekind non-cyclic norm, in [95] it was proved that finite nilpotent groups with such restrictions are groups of the type

$$G = G_p \times \langle y\rangle,$$

where G_p is a Sylow p-subgroup of a group G and a finite group with a non-trivial norm N_{G_p}, $(|y|, p) = 1$. In addition, if the non-cyclic norm N_G is non-nilpotent in the class of locally finite groups, then all non-cyclic subgroups in a group are normal.

Non-periodic locally soluble-by-finite groups with a non-Dedekind non-cyclic norm are considered in [96].

Proposition 4.5 ([96]). *Any non-periodic locally soluble-by-finite group G that has a non-Dedekind non-cyclic norm N_G is $\overline{H}$-group and $G = N_G$.*

Note that locally finite or non-periodic locally soluble-by-finite groups with a non-Dedekind norm of non-cyclic subgroups are soluble and their degree of solvability does not exceed 3 according to the results of [92–96].

Zh. Shen, W. Shi, J. Zhang [98, 99] studied the properties of the norm N_G of non-cyclic subgroups in the class of finite groups and its influence on the group. The authors proved that the norm of non-cyclic subgroups of a finite group is soluble. Note that this proposition is a direct corollary from the description of finite $\overline{H}$-groups (see [88–90]). It was also proved that a finite group is soluble if all its elements of prime order are contained in norm N_G of non-cyclic subgroups. In addition, it was found out that the derived subgroup is nilpotent if all elements of prime order or of order 4 of a group are contained in N_G [98].

Proposition 4.6 ([98]). *A finite group has a nilpotent derived subgroup if and only if a derived subgroup of a quotient group through norm N_G is also nilpotent.*

The study of infinite groups with given restrictions on normalizers of different systems Σ of infinite subgroups have been the subject matter of many theoretical-group researches for a long time. Therefore, when considering infinite groups with restrictions on Σ-norm, it is naturally to choose one of systems of infinite subgroups as a system Σ.

In this context, in the study of Σ-norms of infinite groups F. M. Lyman and T. D. Lukashova [96, 100–102] considered systems of all infinite, all infinite Abelian and all infinite cyclic subgroups, provided that these systems are non-empty. These Σ-norms were denoted as follows: $N_G(\infty)$ is *the norm of infinite subgroups of a group G*; $N_G(A_\infty)$ is *the norm of infinite Abelian subgroups of a group G*; $N_G(C_\infty)$ *is the norm of infinite cyclic subgroups of a group G*.

If the group G coincides with one of these Σ-norms, then all Σ-subgroups are normal in it. Infinite non-Abelian groups with the property $N_G(\infty) = G$ and $N_G(A_\infty) = G$ (if such subgroups exist in them) were studied by S. M. Chernikov [103, 104] and called *INH-groups* and *IH-groups* respectively.

Restrictions, which these Σ-norms satisfied, were non-Dedekindness of Σ-norm or finiteness of its index in the group. The following proposition gives sufficient conditions of Dedekindness of each of these norms.

Proposition 4.7 ([100]). *In non-periodic groups the norm $N_G(\infty)$ of infinite subgroups, the norm $N_G(A_\infty)$ of Abelian infinite subgroups, the norm $N_G(C_\infty)$ of infinite cyclic subgroups are Dedekind in each of the following cases:*

1) *G is a torsion free group or a mixed group without involution;*

2) *the center of a group G contains elements of infinite order;*
3) *G is central-by-finite;*
4) *these norms are finite;*
5) *G contains a subgroup M from the system Σ such that $M \bigcap N_G(\Sigma) = E$.*

The problem of the relations between these norms in non-periodic groups is quite interesting. The following relation is derived from the above definitions

$$Z(G) \subseteq N(G) \subseteq N_G(\infty) \subseteq N_G(A_\infty) \subseteq N_G(C_\infty).$$

So the natural question is: under what conditions do these norms coincide? The following proposition gives the answer (in terms of sufficient conditions).

Proposition 4.8 ([100]). *In a non-periodic group G the equality takes place*

$$N(G) = N_G(\infty) = N_G(A_\infty) = N_G(C_\infty)$$

provided that at least one of the statements takes place:
1) *the center of a group G contains elements of infinite order;*
2) *G is a torsion free group;*
3) *G is central-by-finite.*

Infinite groups with restrictions on the norm $N_G(\infty)$ of infinite subgroups were studied in [100]. It turned out that non-periodic groups, which norm $N_G(\infty)$ has a finite index, are mixed and are finite extensions of their centres.

It was also proved that the norm $N_G(\infty)$ of infinite subgroups of the non-periodic group is Abelian and coincides with the center of the group, if it contains elements of infinite order. This result generalizes Baer's theorem [10] on the coincidence of the norm $N(G)$ of the group and its center in the case of a non-periodic norm $N(G)$. Infinite locally finite groups, which norm $N_G(\infty)$ is non-Dedekind, are a finite extension of a quasicyclic subgroup, which is a divisible part of the norm $N_G(\infty)$ [101].

The structure of non-periodic groups, which norm $N_G(A_\infty)$ of infinite Abelian subgroups is IH-group, are characterized by the following proposition.

Proposition 4.9 ([96]). *A non-periodic group G has non-Abelian norm $N_G(A_\infty)$ of infinite Abelian subgroups, if and only if all elements of infinite* *order of the group G generate Abelian normal subgroup D that contains every infinite Abelian subgroup of a group G and there is an element b of order 2 or 4, such that $b^{-1}db = d^{-1}$ for an arbitrary element $d \in D$. Moreover $N_G(A_\infty) = D\langle b\rangle$.*

A natural generalization of Baer norm for non-periodic groups is the norm $N_G(C_\infty)$ of infinite cyclic subgroups. The study of this norm and its influence on properties of the group was started by F. M. Lyman and T. D. Lukashova in [102]. It was proved that the norm $N_G(C_\infty)$ coincides with the center of the group in torsion free groups, and any finite over the norm $N_G(C_\infty)$ torsion free group is Abelian. The following proposition characterizes the properties of the group that has non-Abelian norm $N_G(C_\infty)$.

Proposition 4.10 ([102]). *A non-periodic group G has non-Abelian norm $N_G(C_\infty)$, if and only if all elements of infinite order of the group G generate an Abelian normal subgroup A and there is an element b of order 2 or 4, such that $b^{-1}ab = a^{-1}$ for an arbitrary element $a \in A$. Moreover $N_G(C_\infty) = A\langle b\rangle$.*

Let's note, if the norm $N_G(A_\infty)$ is non-Abelian in a non-periodic group, then the norm $N_G(C_\infty)$ of infinite cyclic subgroups is non-Abelian. Moreover, in this case $N_G(C_\infty) = N_G(A_\infty)$. The following example shows that non-periodic groups, which norm $N_G(C_\infty)$ is non-Abelian and norm $N_G(A_\infty)$ is Abelian, exist.

Example 4.1. $G = (\langle a\rangle \lambda \langle b\rangle) \times C, |a| = \infty, |b| = 2, C$ is an infinite elementary Abelian 2-group, $b^{-1}ab = a^{-1}$.

It is easy to prove that $N_G(A_\infty) = C$ is Abelian, $N_G(C_\infty) = G$ and $N_G(A_\infty) \neq N_G(C_\infty)$.

The following proposition characterizes the conditions when the norm $N_G(A_\infty)$ coinsides with the norm $N_G(C_\infty)$ in a non-periodic group G (provided that the subgroup $N_G(C_\infty)$ is non-Abelian).

Proposition 4.11. *Let G be a non-periodic group, which norm $N_G(C_\infty)$ of infinite cyclic subgroups is non-Abelian. Subgroups $N_G(C_\infty)$ and $N_G(A_\infty)$ coincide, if and only if $N_G(C_\infty)$ is central-by-finite and contains every infinite Abelian subgroup of G.*

In connection with the existence of O. Yu. Olshansky groups, periodic groups with non-Dedekind norm of infinite Abelian subgroups were studied under the condition of their local finiteness. In [100] it was proved that such groups satisfy the minimal condition for subgroups, if and only if subgroup $N_G(A_\infty)$ satisfies this condition. Moreover, if $N_G(A_\infty)$ is a group with minimal condition for subgroups, then G is a finite extension of its divisible part and therefore $[G : N_G(A_\infty)] < \infty$.

Note that the norm $N(G)$ can be considered as the intersection of the normalizers of all cyclic subgroups. In this connection it is naturally to consider Σ-norm, where Σ consists of all cyclic subgroups of non-prime order of this group. Such a norm was studied by T. D. Lukashova and M. G. Drushlyak [105] in the class of non-periodic groups and was called *the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order of the group G.*

It is clear that all cyclic subgroups of compound or infinite order are normal in non-periodic group G, which coincides with the norm $N_G(C_{\bar{p}})$. Such non-Dedekind groups were studied by T. G. Lelechenko, F. M. Lyman [106] and were called *almost Dedekind groups.*

Since the norm $N_G(C_{\bar{p}})$ normalizes each infinite cyclic subgroup of a group G, $N_G(C_{\bar{p}}) \subseteq N_G(C_\infty)$ in non-periodic groups. It turns out that these norms coincide, if the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order is non-Abelian.

Proposition 4.12 ([105]). *Any non-periodic group G that has non-Abelian norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order is almost Dedekind and coincides with this norm. Moreover $G = A\lambda\langle b\rangle$, where A is a normal Abelian subgroup which contains all elements of prime order of group G, $|b| = 2$, $b^{-1}ab = a^{-1}$ for an arbitrary element $a \in A$.*

In 2004 in [107,108] F. M. Lyman and T. D. Lukashova introduced one more Σ-norm, where Σ is a system of all Abelian non-cyclic subgroups of a group. This Σ-norm was called *the norm of Abelian non-cyclic subgroups* of a group G and denoted by N_G^A. It is clear if the group G coincides with the norm N_G^A then all Abelian non-cyclic subgroups are normal in it (assuming the existence of at least one of such a subgroup). Non-Abelian groups with this property were fully described in [107, 109, 110] and called $\overline{HA}$-groups ($\overline{HA}_p$-groups in the case of p-groups).

In [92,107,114] infinite locally finite $p-$groups (p is an arbitrary prime), which norm N_G^A is non-Dedekind, are considered. The authors obtained a complete description of such groups and proved that if the norm N_G^A is infinite and non-Dedekind, then all Abelian non-cyclic subgroups are normal in a group, that is in this case $G = N_G^A$. It was also proved that locally finite p-groups with non-Dedekind norm N_G^A are finite extensions of a quazicyclic group. In particular, the following propositions take place.

Proposition 4.13 ([107]). *Infinite 2-groups with non-Dedekind norm N_G^A of Abelian non-cyclic subgroups are groups of one of the following types:*

1) *G is an infinite $\overline{HA}_2$-group, $N_G^A = G$;*
2) *$G = (A \times \langle b\rangle)\lambda\langle c\rangle\lambda\langle d\rangle$, where A is a quasicyclic 2-group,$|b| = |c| = |d| = 2, [A, \langle c\rangle] = 1, [b, c] = [b, d] = [c, d] = a_1 \in A, |a_1| = 2, d^{-1}ad = a^{-1}$ for any element $a \in A$; $N_G^A = N_G = (\langle a_2\rangle \times \langle b\rangle)\lambda\langle c\rangle, a_2 \in A, |a_2| = 4$;*
3) *$G = A\langle y\rangle H$, where A is a quasicyclic 2-group, $[A, H] = E, H = \langle h_1, h_2\rangle, |h_1| = 4, h_1^2 = h_2^2 = [h_1, h_2]$, $|y| = 4, y^2 = a_1 \in A, y^{-1}ay = a^{-1}$ for any element $a \in A, [\langle y\rangle, H] \subseteq \langle a_1\rangle \times \langle h^2\rangle$; $N_G^A = \langle a_2\rangle \times H, a_2 \in A, |a_2| = 4, N_G = \langle h_2\rangle\lambda\langle h_1a_2\rangle$.*

Proposition 4.14 ([108]). *Infinite locally finite p-groups ($p \neq 2$), which norm N_G^A of Abelian non-cyclic subgroups is non-Dedekind, are $\overline{HA}_p$-groups and $G = N_G^A = N_G$.*

Let's note that Proposition 4.14 fails in the case of infinite locally finite 2-groups: there are infinite 2-groups with finite non-Dedekind norm N_G^A of Abelian non-cyclic subgroups, which may not coincide with the norm N_G.

The study of finite p-groups for certain restrictions on the norm N_G^A of Abelian non-cyclic subgroups were continued by M. G. Drushlyak, T. D. Lukashova and F. M. Lyman in [112,113]. In particular, in [112] the structure of finite p-groups ($p \neq 2$) with a non-Abelian norm of Abelian non-cyclic subgroups was completely described, in [113] the structure of finite 2-groups with a non-cyclic centre and non-Dedekind norm N_G^A of Abelian non-cyclic subgroups was described. It is also proved that an arbitrary 2-group with a non-cyclic centre and a non-Dedekind norm N_G^A does not contain a quaternion subgroup, if and only if the norm N_G^A does not contain such a subgroup. In this case the norm N_G^A coinsides with the norm N_G [113].

The following proposition clarifies the result of [92] on the coincidence of norms N_G^A and N_G for infinite locally finite p-groups ($p \neq 2$) under the condition that the subgroup N_G is non-Abelian.

Proposition 4.15 ([112]). *If either norm N_G^A or N_G is non-Abelian, then $N_G = N_G^A$ in the class of locally finite p-groups ($p \neq 2$).*

The Proposition 4.15 leads to the conclusion that any finite p-group ($p \neq 2$) with non-Abelian norm N_G^A is a group of one of the types of 1)-3) of Proposition 4.2.

Proposition 4.16 ([113]). *Finite 2-groups with a non-cyclic centre and a non-Dedekind norm N_G^A of Abelian non-cyclic subgroups is a group of the following types:*

1) *G is a non-Dedekind non-metacyclic $\overline{HA_2}$–group with a non-cyclic center, $G = N_G^A$;*
2) *$G = H \cdot Q$ is a product of a quaternion group of order 8 and a generalized quaternion group; $H = \langle h_1, h_2 \rangle$, $|h_1| = |h_2| = 4$, $[h_1, h_2] = h_1^2 = h_2^2$, $Q = \langle y, x \rangle$, $|y| = 2^n$, $n \geqslant 3, |x| = 4, y^{2^{n-1}} = x^2, x^{-1}yx = y^{-1}, [Q, H] \subseteq \langle x^2, h_1^2 \rangle; N_G^A = \langle y^{2^{n-2}} \rangle \times H;$*
3) *$G = \langle x \rangle \langle b \rangle, |x| = 2^k$, $|b| = 2^m, m > 2, k \geqslant m + r, 1 \leqslant r < m - 1, Z(G) = \langle x^{2^{r+1}} \rangle \times \langle b^{2^{r+1}} \rangle, [x, b] = x^{2^{k-r-1}s} b^{2^{m-1}t}, (s, 2) = 1, 0 \leqslant t < 2; N_G^A = N_G = \langle x^{2^{m-1}} \rangle \lambda \langle b \rangle.$*

Developing the study of finite 2-groups T. D. Lukashova, F. M. Lyman and M. G. Drushlyak obtained a structural description of groups with a cyclic center and a non-metacyclic non-Dedekind norm N_G^A.

Proposition 4.17. *Finite 2-groups with a non-metacyclic non-Dedekind norm N_G^A of Abelian non-cyclic subgroups and a cyclic centre are groups of the following types:*

1) *G is a non-metacyclic non-Hamiltonian $\overline{HA_2}$-group with a cyclic center, $G = N_G^A$;*
2) *$G = (\langle x \rangle \lambda \langle c \rangle) \lambda \langle b \rangle$, $|x| = 2^n$, $n > 3$, $|b| = |c| = 2$, $[x, b] = x^{\pm 2^{n-2}} c$, $[b, c] = [x, c] = x^{2^{n-1}}$, $N_G^A = N_G = (\langle x^2 \rangle \times \langle c \rangle) \lambda \langle b \rangle$;*
3) *$G = (\langle x \rangle \times \langle b \rangle) \lambda \langle c \rangle \lambda \langle d \rangle$, $|x| = 2^n, n > 2, |b| = |c| = |d| = 2$, $[x, c] = [x, b] = 1$, $[b, c] = [c, d] = [b, d] = x^{2^{n-1}}$, $d^{-1}xd = x^{-1}$, $N_G^A = N_G = (\langle x^{2^{n-2}} \rangle \times \langle b \rangle) \lambda \langle c \rangle$;*
4) *$G = (\langle c \rangle \lambda H) \langle y \rangle$, $H = \langle h_1, h_2 \rangle, |h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1, h_2]$, $|c| = 4, [c, h_2] = 1$, $[c, h_1] = c^2$, $y^2 = h_1, [y, h_2] = c^2 h_1^2, [y, c] = h_2^{\pm 1}; N_G^A = \langle c \rangle \lambda H$.*

The question of the structure of finite 2-groups with a cyclic center, in which the norm N_G^A is a metacyclic non-Dedekind group, is still open.

Study of the influence of the properties of the norm of Abelian non-cyclic subgroups on the properties of the group was continued in [114], where infinite periodic groups, which norm N_G^A is non-Dedekind and locally nilpotent, were considered. It was proved that such groups satisfy the minimal condition for Abelian subgroups and are Chernikov groups.

Proposition 4.18 ([114]). *An infinite periodic locally nilpotent group G has a non-Dedekind norm of Abelian non-cyclic subgroups, if and only if*

$$G = G_p \times G_{p'},$$

where G_p is an infinite Sylow p-subgroup of a group G with a non-Dedekind norm $N_{G_p}^A$ of Abelian non-cyclic subgroups (where $p \in \pi(G)$) and $G_{p'}$ is a finite cyclic or finite Hamiltonian p'-subgroup, all Abelian subgroups of which are cyclic, and $N_G^A = N_{G_p}^A \times G_{p'}$.

If G is a locally finite, not locally nilpotent group which has an infinite locally nilpotent non-Dedekind norm N_G^A, then $G = G_p \leftthreetimes H$, where G_p is an infinite $\overline{HA}_p$-group, which coincides with a Sylow p-subgroup of a norm N_G^A, and H is a finite group, all Abelian subgroups of which are cyclic, $(|H|, p) = 1$. In addition, the structure of infinite locally finite non-nilpotent groups, which norm N_G^A is finite non-Dedekind nilpotent subgroup, was described.

Study of the norm N_G^A of Abelian non-cyclic subgroups in the class of non-periodic groups were continued by M. G Drushlyak and F. M. Lyman. In [115, 116] non-periodic groups with non-Dedekind norm of Abelian non-cyclic subgroups depending on the presence [115] or the absence [116] of a free Abelian subgroup of rank 2 were considered.

Proposition 4.19 ([115]). *If a non-periodic group G contains a free Abelian subgroup of rank 2, its norm N_G^A of Abelian non-cyclic subgroups is non-Dedekind, contains an Abelian non-cyclic subgroup and a finite Abelian, normal in G, subgroup F and the centralizer $C_G(F)$ contains all elements of infinite order of a group, then $N_G^A = N_G(C_\infty) = B\langle d\rangle$, where B is the Abelian subgroup generated by all elements of infinite order of the group G, $|d| = 2$ or $|d| = 4$, $d^2 \in B$, d^2 is a unique involution in G and $d^{-1}bd = b^{-1}$ for an arbitrary element $b \in B$.*

It was also proved that the non-periodic group G does not contain free Abelian subgroups of rank 2, if its norm N_G^A is non-Hamiltonian $\overline{HA}$-group and does not contain such subgroups.

In 2015 F. M. Lyman and T. D. Lukashova [117] considered one more generalization of the concept of the norm of the group – *the norm N_G^d of decomposable subgroups*, which is defined as the intersection of the normalizers of all decomposable subgroups of the group. In the case when the group does not contain any decomposable subgroups, we can assume that $N_G^d = G$. The structure of locally soluble groups, in which a system of decomposable subgroups is empty, as well as groups, in which each decomposable subgroup is normal (groups with the condition $N_G^d = G$), was described in [118].

It is clear that the group contains decomposable subgroups, if and only if it contains decomposable Abelian subgroups. Therefore, the study of the norm N_G^d of decomposable subgroups was conducted, regarding on the existence of systems of decomposable Abelian subgroups in the group. Thus the norm N_G^d of decomposable subgroups is closely related to the norm N_G^A of Abelian non-cyclic subgroups. In particular, in [117] it was proved that these norms coinside in the class of locally finite p-groups. The inclusion $N_G^A \supseteq N_G^d$ takes place and the case $N_G^A \neq N_G^d$ is achieved in classes of finite non-primary groups, as well as in classes of infinite periodic locally nilpotent non-primary groups.

Proposition 4.20 ([117]). *A periodic locally nilpotent group G which contains an Abelian non-cyclic subgroup has a non-Dedekind norm N_G^d of decomposable subgroups, if and only if G is a locally finite p-group with a non-Dedekind norm N_G^A of Abelian non-cyclic subgroups.*

Study of the influence of the norm N_G^d of decomposable subgroups on the properties of the group were extended by the authors in the class of non-periodic groups. In particular, in [119] the following was established.

Proposition 4.21 ([119]). *Let G be a non-periodic group that has a non-Dedekind norm N_G^d of decomposable subgroups. Then the following propositions take place:*

1) *G does not contain decomposable subgroups if and only if the norm N_G^d of the group does not contain them;*
2) *G contains a free Abelian subgroup of rank $r \geqslant 2$, if and only if the norm N_G^d contains a free Abelian subgroup of such a rank;*
3) *G contains a non-primary Abelian subgroup, if and only if the norm N_G^d of the group contains subgroups with this property;*
4) *any decomposable Abelian subgroup of a group G is mixed, if and only if any decomposable Abelian subgroup of its norm N_G^d is mixed.*

It was also proved that in the class of non-periodic locally soluble groups only one of the inclusions $N_G^A \supseteq N_G^d$ or $N_G^A \subseteq N_G^d$ takes place, provided that at least one of these norms is non-Dedekind and the norm

N_G^d is infinite. The following examples confirm that the condition of the infiniteness of the norm N_G^d is essential.

Example 4.2 ([119])**.** $G = (\langle a\rangle \leftthreetimes B)\leftthreetimes\langle c\rangle$, where $|a| = p, p$ is a prime $(p \neq 2)$, B is a group isomorphic to an additive group of q-adic numbers, $q \notin \{2, p\}, B = B_1\langle x\rangle, x^2 \in B_1,\ x^{-1}ax = a^{-1}, [B_1, \langle a\rangle] = E, |c| = 2, [c, a] = 1, c^{-1}bc = b^{-1}$ for any element $b \in B$.

In this group all periodic decomposable subgroups are of order $2p$ and are groups of the type $\langle a^m cb_1^k\rangle$, where $b_1 \in B_1,\ k \in \{0, 1\}, (m, p) = 1$. Accordingly, all non-periodic decomposable subgroups are mixed and contained in the group $B_1 \times \langle a\rangle$ and therefore are normal in G. Since $N_G(\langle a^m cb_1^k\rangle) = \langle a^m cb_1^k\rangle$, $N_G^d = \langle a\rangle$.

On the other hand, G does not contain periodic Abelian non-cyclic subgroups and all mixed Abelian subgroups contain $\langle a\rangle$ and are subgroups of the group $(B_1 \times\langle a\rangle)$ and therefore are normal in G. Moreover, all Abelian non-cyclic subgroups of rank 1 are contained either in the subgroup B, or in subgroups conjugated with it $g^{-1}Bg, g \in G$, or in the group $(B_1 \times \langle a\rangle)$.

Let's consider an infinite sequence of subgroups in B_1:

$$\langle b_1\rangle \subset \langle b_2\rangle \subset \cdots \subset \langle b_n\rangle \subset \cdots ,$$

$|b_1| = \infty, b_{n+1}^{\alpha_{n+1}} = b_n,\ \alpha_{n+1} \in \mathbb{N}, (\alpha_{n+1}, p) = 1$ for $n = 1, 2, \ldots$ Since the isolator A of the subgroup $\langle ab_1\rangle$ is non-cyclic (because the root of an arbitrary degree coprime with p can be taken from the element a), $N_G(A) = \langle a, B_1\rangle$. $N_G^A = B_1$ and $N_G^d \cap N_G^A = E$ by $N_G(B) = B \leftthreetimes \langle c\rangle$.

Example 4.3 ([119])**.** $G = (\langle a\rangle \leftthreetimes B)\leftthreetimes\langle c\rangle$, where $|a| = p, p$ is a prime $(p \neq 2)$, B is a group isomorphic to an additive group of p-adic numbers, $B = B_1\langle x\rangle, x^2 \in B_1, x^{-1}ax = a^{-1}, [B_1, \langle a\rangle] = E, |c| = 2, [c, a] = 1, c^{-1}bc = b^{-1}$ for any element $b \in B$.

As in Example 4.2 in this group the norm of decomposable subgroups is $N_G^d = \langle a\rangle$. However, the norm of Abelian non-cyclic subgroups is $N_G^A = (B_1 \leftthreetimes \langle c\rangle)$. This follows from the fact that for any non-identity element $y_1 \in B_1$ the isolator of a subgroup $\langle ay_1\rangle$ is cyclic, and therefore the element c normalizes each Abelian non-cyclic subgroup of a group G. In this case, the norm N_G^A of Abelian non-cyclic subgroups is non-Dedekind and $N_G^d \cap N_G^A = E$.

In 2005 F. Mari, F. de Giovanni [78] considered the concept of *the non-Abelian norm* $N^*(G)$ that is the intersection of normalizers of all non-Abelian subgroups of the group. If $N^*(G) = G$, then all non-Abelian subgroups are normal in the group. These groups were studied by G. M. Romalis and N. F. Sesekin [120–122] and were called *metahamiltonian*. Further metahamiltonian groups were studied by V. T. Nagrebezkiy [123], O. A. Makhnev [124], S. M. Chernikov [125], M. M. Semko and M. F. Kuzennyi [126].

In [78] the results that generalize Schur theorem [127] on finiteness of derived subgroups in central-by-finite groups were offered.

Proposition 4.22 ([78])**.** *If G is a locally graded group and the quotient group $G/N^*(G)$ is finite, then a derived subgroup G' is finite.*

Conclusion

The authors make a conclusion that the study of different Σ-norms and properties of groups with respect on properties of their Σ-norms is a very important field in the group theory. Nowadays the research of groups that differ from their Σ-norms as well as groups that have a non-Dedekind Σ-norm becomes possible, because the structure of groups that coincide with Σ-norms is well known in many cases. Therefore it will give the opportunity to extend the known classes of generalized Dedekind groups and will allow to study groups with restrictions on the normalizers of different systems of subgroups more effectively.

There are still a number of problems in the study of groups with generalized norms:

– the study of groups that coincide with their Σ-norms;

– the study of groups that have identity Σ-norms or their Σ-norms coinside with the center;

– the study of groups that have non-central Dedekind Σ-norms;

– the study of groups that have non-Dedekind Σ-norms;

– the study of infinite groups that have Σ-norms of finite index.

The solution of these problems will significantly expand the base of the modern group theory.

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Received by the editors: 22.01.2016
and in final form 24.02.2016.