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RESEARCH ARTICLE

Group of continuous transformations of real interval preserving tails of G_2-representation of numbers

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Abstract. In the paper, we consider a two-symbol system of encoding for real numbers with two bases having different signs $g_0 < 1$ and $g_1 = g_0 - 1$. Transformations (bijections of the set to itself) of interval $[0, g_0]$ preserving tails of this representation of numbers are studied. We prove constructively that the set of all continuous transformations from this class with respect to composition of functions forms an infinite non-abelian group such that increasing transformations form its proper subgroup. This group is a proper subgroup of the group of transformations preserving frequencies of digits of representations of numbers.

Introduction

Many two-symbol systems of encoding (representation) of fractional part of real numbers are known. They use the alphabet $A = \{0, 1\}$ and are based on expansions of numbers in series, infinite products, continued fractions, etc. These systems identify a number as a sequence of zeros and

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ones, i.e., as element of space $L = A \times A \times \ldots$ of sequences of zeros and ones. Every such a system has some advantages and restrictednesses as well as certain conveniences for solving some problems of number-theoretical, topological-metric and probabilistic kind. A system with two positive bases ($q_0 \in (0,1)$, $q_1 = 1 - q_0$) is among them. It generalizes classic binary system, has a self-similar geometry and zero redundancy. This is a system of Q_2-representation of numbers based on representation of a number by the series

$$x = \alpha_1 q_{1-\alpha_1} + \sum_{k=2}^{\infty} (\alpha_k q_{1-\alpha_k} \prod_{j=1}^{k-1} q_{\alpha_j}) \equiv \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}.$$

For $q_0 = 0{,}5$, it is a classic binary system.

In expression $\prod_{j=1}^{k} q_{\alpha_j} = q_1^{N_0(x,k)} q_1^{N_1(x,k)}$, where $N_1(x,k) = \alpha_1 + \alpha_2 + \ldots + \alpha_k$ and $N_0(x,k) = k - N_1(x,k)$, one can see that this is a system with two bases. This system of encoding of numbers has various applications in metric and probabilistic theory of numbers, function theory and measure theory, fractal analysis and fractal geometry. For this system, the left shift operator is a discontinuous function, linear and increasing on cylinders of rank 1: $\Delta^{Q_2}_0 = [0, q_0]$ and $\Delta^{Q_2}_1 = [q_0, 1]$.

In the papers [7], [8], an analogue of Q_2-representation of numbers, namely, system of encoding of numbers in interval $[0, g_0]$ with bases $g_0 \in (0,1)$ and $g_1 = g_0 - 1$ having different signs is introduced. It is based on expansion of a number in the series

$$x = \alpha_1 g_{1-\alpha_1} + \sum_{k=2}^{\infty} (\alpha_k g_{1-\alpha_k} \prod_{j=1}^{k-1} g_{\alpha_j}) \equiv \Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}. \tag{1}$$

This system also has a zero redundancy (any number has at most two representations, and set of numbers having two representations is countable).

Generally speaking, series (1) has positive as well as negative terms. So G_2-representation is not topologically equivalent to Q_2-representation and it is not a simple reencoding of Q_2-representation. At the same time metric theories of these representations are similar. A vivid peculiarity of G_2-representation is the fact that left shift operator of G_2-representation is a continuous function on a whole interval $[0, g_0]$. This is a fundamental difference between this system and previously studied systems.

Real numbers in interval $[0, g_0]$ having two G_2-representations are called G_2-*binary numbers*, and numbers having a unique representation are called G_2-*unary numbers*. It is known [10] that the set B of all G_2-binary numbers is countable and consists of numbers with the following G_2-representation: $\Delta^{G_2}_{c_1\ldots c_m 01(0)} = \Delta^{G_2}_{c_1\ldots c_m 11(0)}$.

Algorithm for comparison of numbers in terms of their G_2-representations is the following.

Theorem 1 ([10]). *Numbers* $\Delta^{G_2}_{c_1c_2\ldots c_m 1 d_1 d_2\ldots} = x_1 \neq x_2 = \Delta^{G_2}_{c_1c_2\ldots c_m 0 d'_1 d'_2\ldots}$ *are in the relation*

$$x_1 \geqslant x_2 \quad \textit{if} \quad \sigma_m \equiv c_1 + c_2 + \ldots + c_m = 2k,$$

$$x_1 \leqslant x_2 \quad \textit{if} \quad \sigma_m \equiv c_1 + c_2 + \ldots + c_m = 2k - 1.$$

In this paper, we consider continuous transformations of interval $[0, g_0]$, i.e., bijections of interval to itself, preserving "tails of G_2-representation of numbers", namely, we study group properties of the family of these transformations. Groups of transformations preserving tails of different representations of numbers were studied in the papers [2], [3], [4]. Specific properties of G_2-representation generate a peculiar corresponding transformation group having a non-trivial subgroup of increasing functions unlike other systems.

1. Tails and tail sets

Definition 1. We say that G_2-representations of numbers $x = \Delta^{G_2}_{\alpha_1\ldots\alpha_n\ldots}$ and $y = \Delta^{G_2}_{\beta_1\beta_2\ldots\beta_n\ldots}$ have the same tail if there exist positive integers k and m such that

$$\alpha_{k+j} = \beta_{m+j} \tag{2}$$

for any $j \in N$. We denote it symbolically by $x \sim y$.

Definition 2. Suppose that k and m are the smallest numbers satisfying condition (2). Then G_2-representation and corresponding number

$$z \equiv [x \wedge y] = \Delta^{G_2}_{\alpha_{k+1}\alpha_{k+2}\ldots} = \Delta^{G_2}_{\beta_{m+1}\beta_{m+2}\ldots}$$

is called a common tail of representations of numbers x and y.

It is evident that binary relation $\sim$ ("has the same tail") is an equivalence relation, i.e., it is reflexive, symmetric and transitive. Thus it provides a partition of the set Z of all G_2-representations of numbers in interval $[0, g_0]$ into the equivalence classes, and they form together a quotient set $W = Z/\sim$. Any element of the set W is called a *tail set*, this set is uniquely determined by an arbitrary its element.

Theorem 2. *Any tail set is a countable everywhere dense in interval $[0, g_0]$ set. The set W of all tail sets is a continuum set.*

Proof. Suppose K_x is a tail set containing the number x, K_1 is a set of all numbers y such that $[y \wedge x] = x$, and K_n is a set of all numbers y such that $[y \wedge \Delta^{G_2}_{\alpha_{n+1}(x)\alpha_{n+2}(x)\ldots}] = \Delta^{G_2}_{\alpha_{n+1}(x)\alpha_{n+2}(x)\ldots}$.

It is evident that every set K_n $(n \in N)$ is countable and $K_x = \bigcup\limits_n K_n$. Thus K_x is a countable set because it is a countable union of countable sets.

An arbitrary cylinder $\Delta^{G_2}_{c_1\ldots c_n}$ contains points of tail set K_x because points $\Delta^{G_2}_{c_1\ldots c_n\alpha_1(x)\alpha_2(x)\ldots}$, $\Delta^{G_2}_{c_1\ldots c_n\alpha_2(x)\alpha_3(x)\ldots}$, and $\Delta^{G_2}_{c_1\ldots c_n\alpha_k(x)\alpha_{k+1}(x)\ldots}$ belong to this cylinder. This proves that the set K_x is everywhere dense in interval $[0, g_0]$.

The set W is a continuum set. Indeed, suppose that it is a countable set. Then we see that interval $[0, g_0]$ is a countable set because it is a countable union of countable sets. This contradicts the fact that interval is a continuum set. □

Note that all G_2-binary numbers belong to the same tail set.

2. Left and right shift operators of G_2-representation of numbers

Theorem 3 ([8]). *Left shift operator ω of G_2-representation of numbers in interval $[0, g_0]$ defined by equality*

$$\omega(\Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{G_2}_{\alpha_2\alpha_3\ldots\alpha_n\ldots}$$

in space of G_2-representations, is analytically expressed in the form $\omega(x) = \dfrac{1}{g_{\alpha_1(x)}}x - \dfrac{\delta_{\alpha_1(x)}}{g_{\alpha_1(x)}}$, is a continuous well-defined function on $[0, g_0]$, is linear on every cylinder of rank 1, is increasing on Δ_0 and decreasing on Δ_1.

Numbers

$$0 = \Delta^{G_2}_{(0)} \quad \text{and} \quad \Delta^{G_2}_{(1)} = g_0 + g_0g_1 + g_0g_1^2 + \ldots = \frac{g_0}{2 + g_0}$$

are invariant points of left shift operator ω.

Remark 1. The last theorem shows an essential difference between G_2-representation and other known two-symbol representations, in particular, Q_2^*-representation, $\tilde{Q}$-representation [9] and A_2-continued fractions [3].

Let n be a positive integer that is greater than 1. Put

$$\omega^n(x) = \omega(\omega^{n-1}(x)) = \Delta^{G_2}_{\alpha_{n+1}(x)\alpha_{n+2}(x)\ldots}.$$

Since

$$x = \delta_{\alpha_1(x)} + \sum_{k=2}^{n}\left(\delta_{\alpha_k(x)}\prod_{j=1}^{k-1} g_{\alpha_j(x)}\right) + \left(\prod_{j=1}^{n} g_{\alpha_j(x)}\right)\omega^n(x),$$

we have

$$\omega^n(x) = \frac{x}{P_n(x)} - \frac{1}{P_n(x)}\left(\delta_{\alpha_1(x)} + \sum_{k=2}^{n}\left(\delta_{\alpha_k(x)}\prod_{j=1}^{k-1} g_{\alpha_j(x)}\right)\right).$$

Theorem 4 ([10])**.** *Function ω^n is well defined by equality*

$$\omega^n(x) = \omega^n(\Delta^{G_2}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}) \equiv \Delta^{G_2}_{\alpha_{n+1}(x)\alpha_{n+2}(x)\ldots},$$

is analytically expressed in the form $\omega^n(x) = \dfrac{1}{P_n}x - \dfrac{B_n}{P_n}$, *where* $P_n = \prod_{j=1}^{n} g_{\alpha_j(x)}$,
$B_n = \delta_{\alpha_1(x)} + \sum\limits_{k=2}^{n}\left(\delta_{\alpha_k(x)}\prod_{j=1}^{k-1} g_{\alpha_j(x)}\right)$, *is continuous on interval* $[0, g_0]$ *and linear on every cylinder of rank n.*

Remark 2. In terms of dynamical systems, relation "has the same tail" can be defined in the following form:

$$x = \Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_n} \sim y = \Delta^{G_2}_{\beta_1\beta_2\ldots\beta_n} \;\Leftrightarrow\; O_x \cap O_y \neq \varnothing,$$

where $O_u = \{u, \omega^1(u), \omega^2(u), \ldots\}$ is an orbit of point u under mapping ω, and $[x \wedge y] = O_x \cap O_y$.

Definition 3. Function τ_i defined on $[0, g_0]$ by equality

$$\tau_i(x) = \tau_i\left(\Delta^{G_2}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}\right) = \Delta^{G_2}_{i\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots},$$

where $i \in \{0, 1\}$, is called a *right shift operator of G_2-representation of numbers with parameter i* (in the sequel, we just say "right shift operator").

Since $\tau_i\left(\Delta^{G_2}_{c_1c_2\ldots c_m01(0)}\right) = \tau_i\left(\Delta^{G_2}_{c_1c_2\ldots c_m11(0)}\right)$, we see that function τ_i is well defined. It is evident that the set of values of function τ_i is cylinder $\Delta^{G_2}_i$. In particular,

$$\tau_0(0) = \tau\left(\Delta^{G_2}_{(0)}\right) = 0,\ \tau_1(0) = \tau_1\left(\Delta^{G_2}_{(0)}\right) = \Delta^{G_2}_{1(0)} = g_0.$$

Lemma 1. *Function τ_i is continuous at any point of interval $[0, g_0]$ and analytically expressed in the form $\tau_i(x) = \delta_i + g_i x$.*

Indeed, since

$$\tau_i\left(\Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \Delta^{G_2}_{i\alpha_1\alpha_2\ldots\alpha_n\ldots} = \delta_i + g_i\Delta^{G_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots},$$

we have $\tau_i(x) = \delta_i + g_i x$, i.e., $\tau_0(x) = g_0 x$, $\tau_1(x) = g_0 + g_1 x$. So, it is evident that function τ_i is continuous on cylinders of rank 1. From equality $\tau_i\left(\Delta^{G_2}_{c_1c_2\ldots c_m01(0)}\right) = \tau_i\left(\Delta^{G_2}_{c_1c_2\ldots c_m11(0)}\right)$ it follows that the function is continuous at G_2-binary points that are endpoints of cylinders.

Corollary 1. *Function τ_0 is increasing and function τ_1 is decreasing. Moreover, $\tau_0(g_0) = \tau_1(g_0)$.*

The following equalities are evident: $\omega(\delta_i(x)) = x$ and $\tau_{\alpha_1(x)}(\omega(x)) = x$.

Equation $\tau_i(x) = \omega(x)$ has two solutions: $x = \Delta^{G_2}_{(ji)}$, where $j \in \{0, 1\}$. Equation $\tau_i(x) = \omega^m(x)$ has 2^m solutions: $x = \Delta^{G_2}_{(j_1,j_2,\ldots,j_m i)}$, where $j_k \in \{0, 1\}$, $k = \overline{1, m}$.

Let $(i_1, i_2, \ldots, i_n)$ be a tuple of zeros and ones. Function $\tau_{i_1i_2\ldots i_n}$ defined by equality $\tau_{i_1i_2\ldots i_n}(x) = \Delta^{G_2}_{i_1i_2\ldots i_n\alpha_1(x)\alpha_2(x)\ldots}$ is called a right shift operator with parameters $(i_1, i_2, \ldots, i_n)$. By induction, from equality $\tau_{i_1i_2\ldots i_n}(x) = \tau_{i_1}(\tau_{i_2\ldots i_n}(x))$ it follows that operator $\tau_{i_1i_2\ldots i_n}(x)$ is well defined.

Operator $\tau_{i_1i_2\ldots i_n}$ is analytically expressed in the form

$$\tau_{i_1i_2\ldots i_n}(x) = \delta_{i_1} + \sum_{k=1}^{n}\left(\delta_{i_k}\prod_{j=1}^{k-1} g_{i_j}\right) + \left(\prod_{j=1}^{n} g_{i_j}\right)x,$$

is a linear function, is increasing if $P_n = \prod_{j=1}^{n} g_{i_j} > 0$ (this is equivalent to $i_1 + i_2 + \ldots + i_n$ is even number) and decreasing if $P_n < 0$ (this is equivalent to $i_1 + i_2 + \ldots + i_n$ is odd number).

For example, consider $n = 2$ and corresponding functions τ_{00}, τ_{01}, τ_{10}, τ_{11}. Functions $\tau_{00} = g_0^2 x$ and $\tau_{11} = g_0^2 x + g_0^2$ are linear increasing, but functions $\tau_{01} = g_0g_1x + g_0^2$ and $\tau_{10} = g_0g_1x + g_0$ are linear decreasing.

3. Continuous functions and transformations of interval $[0, g_0]$ preserving tails of G_2-representation of numbers

We say that function $y = f(x)$ preserves tails of G_2-representation of numbers in interval $[0, g_0]$ (or is a tail function) if any number $x \in [0, g_0]$ and its image $y = f(x)$ have the same tail.

Left and right shift operators ω^n, $\tau_{i_1 i_2 \ldots i_n}$ for any positive integer n and for any tuple $(i_1, i_2, \ldots, i_n)$ of zeros and ones are simple examples of continuous functions preserving tails of G_2-representation of numbers. Various "joinings" of these functions are the same. For example, function

$$f(x) = \begin{cases} \omega(x) & \text{if } 0 \leqslant x \leqslant x_1, \\ \tau_1(x) & \text{if } x_1 \leqslant x \leqslant x_2, \\ \omega(x) & \text{if } x_2 \leqslant x \leqslant g_0, \end{cases}$$

where x_1 and x_2 are solutions of equation $\omega(x) = \tau_1(x)$, i.e., $x_1 = \Delta^{G_2}_{(01)}$, $x_2 = \Delta^{G_2}_{(1)}$, preserves tails of G_2-representation.

But not every continuous function defined on interval $[0, g_0]$ is its transformation, i.e., bijection of the interval to itself. It is clear that above mentioned functions are not transformations.

It is clear that continuous transformations of interval $[0, g_0]$ can be only strictly monotonic (increasing and decreasing) functions such that their domain and set of values coincide with this interval.

Lemma 2. *Decreasing function*

$$f_1(x) = \begin{cases} \tau_1(x) & \textit{if } x \leqslant x_1 = \Delta^{G_2}_{(1)} = \dfrac{g_0}{2 - g_0}, \\ \omega(x) & \textit{if } x \geqslant x_1 = \Delta^{G_2}_{(1)} = \dfrac{g_0}{2 - g_0}, \end{cases}$$

is a continuous tail transformation of interval $[0, g_0]$.

Proof. Number x_1 is a solution of equation $\tau_1(x) = \omega(x)$ being equivalent to system of equations $1 = \alpha_2(x) = \alpha_4(x) = \ldots$, $\alpha_1(x) = \alpha_3(x) = \alpha_5(x) = \ldots$. So this equation has two solutions: $x = \Delta^{G_2}_{(\alpha_1 1)}$, $\alpha_1 \in \{0, 1\}$. First solution $x_0 = \Delta^{G_2}_{(01)}$ belongs to interval of decrease of function τ_1 and to interval of increase of function ω, and second solution x_1 belongs to interval of decrease of function τ_1 and to interval of decrease of function ω. Thus f_1 is a continuous and strictly decreasing function. Moreover, $f_1(0) = g_0$ and $f_1(g_0) = 0$. Hence f_1 is a continuous transformation of interval $[0, g_0]$. □

Example 1. Function

$$f_2(x) = \begin{cases} \tau_1(x) & \text{if } 0 \leqslant x \leqslant \Delta^{G_2}_{(101)}, \\ \omega^2(x) & \text{if } \Delta^{G_2}_{(101)} \leqslant x \leqslant g_0, \end{cases}$$

is a continuous decreasing tail transformation.

Indeed, τ_1 is a continuous decreasing tail function and $\Delta^{G_2}_{(101)}$ is a solution of equation $\tau_1(x) = \omega^2(x)$ belonging to the last interval of decrease of function $\omega^2(x)$.

Example 2. Decreasing function

$$f_3(x) = \begin{cases} \tau_{1\underbrace{0\ldots0}_{k}}(x) & \text{if } 0 \leqslant x \leqslant x_k \equiv \Delta^{G_2}_{i1\underbrace{0\ldots0}_{k}}, \\ \omega(x) & \text{if } x_k \leqslant x \leqslant g_0,\ i \in A, \end{cases}$$

is a continuous tail transformation of interval $[0, g_0]$.

Indeed, ω and $\tau_{10\ldots0}$ are continuous tail functions, number x_k is a solution of equation $\tau_{1\underbrace{0\ldots0}_{k}}(x) = \omega(x)$ belonging to intervals of decrease of both functions. Hence f_3 is a continuous decreasing tail function.

Theorem 5. *The set C of all continuous bijections of interval $[0, g_0]$ preserving tails of G_2-representation of numbers with respect to composition (superposition) $\circ$ forms an infinite non-abelian group such that increasing functions form its non-trivial subgroup.*

Proof. It is known that the set of all bijections of interval forms a group such that an identity transformation is its neutral element and an inverse transformation is its symmetric element. It is evident that composition of tail transformations is a tail transformation. The same is true for inverse transformation. Thus, by the subgroup test, $(C, \circ)$ is a group. From example 3 (where k is an arbitrary positive integer) it follows that this group is infinite.

To prove that group $(C, \circ)$ is non-abelian it is enough to provide two transformations in the set C that are not commute. To this end we consider function f_1 and f_2 from examples 1 and 2 and number x_0 that is less than x_1. Then we have

$$\begin{aligned} f_2(f_1(\Delta^{G_2}_{01(0)})) &= f_2(\tau_1(\Delta^{G_2}_{01(0)})) = f_2(\Delta^{G_2}_{101(0)}) \\ &= \tau_1(\Delta^{G_2}_{101(0)}) = \Delta^{G_2}_{1101(0)} \end{aligned}$$

because of $\Delta^{G_2}_{101(0)} < \Delta^{G_2}_{(101)}$;

$$\begin{aligned} f_1(f_2(\Delta^{G_2}_{01(0)})) &= f_1(\tau_1(\Delta^{G_2}_{01(0)})) = f_1(\Delta^{G_2}_{101(0)}) \\ &= \omega(\Delta^{G_2}_{101(0)}) = \Delta^{G_2}_{01(0)}, \end{aligned}$$

because of $\Delta^{G_2}_{101(0)} > \Delta^{G_2}_{(1)}$. Hence $f_2(f_1(x_0)) \neq f_1(f_2(x_0))$.

To prove that there exists a non-trivial subgroup of increasing functions it is enough to give an example of non-trivial increasing bijection $f \in C$. A such function is the following:

$$f_5(x) = \begin{cases} \omega(x) & \text{if } 0 \leqslant x \leqslant \Delta^{G_2}_{(011)}, \\ \tau_{11}(x) & \text{if } \Delta^{G_2}_{(011)} \leqslant x \leqslant \Delta^{G_2}_{(1)}, \\ x & \text{if } \Delta^{G_2}_{(1)} \leqslant x \leqslant g_0, \end{cases}$$

because functions ω, τ_{11} and $f(x) = x$ are increasing on given intervals, $\Delta^{G_2}_{(011)}$ is a solution of equation $\omega(x) = \tau_{11}(x)$, and $\Delta^{G_2}_{(1)}$ is a solution of equation $\tau_{11}(x) = x$. □

Remark 3. The group $(C, \circ)$ is a proper subgroup of the group of transformations of interval $[0, g_0]$ preserving frequencies of digits of representation.

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