

On dimension of product of groups

Alexander Dranishnikov*

Communicated by R. I. Grigorchuk

ABSTRACT. We prove the product formula $\text{cd}(\Gamma \times \Gamma) = 2 \text{cd} \Gamma$ for cohomological dimension of geometrically finite groups.

1. Introduction

We recall that the cohomological dimension of a group Γ is defined as

$$\text{cd} \Gamma = \max\{n \mid H^n(\Gamma, M) \neq 0\}$$

where the maximum is taken over all $\mathbb{Z}\Gamma$ -modules M . If there is no such maximum we write $\text{cd} \Gamma = \infty$. By the Eilenberg-Ganea theorem [Br] the cohomological dimension $\text{cd} \Gamma$ coincides with the geometric dimension of Γ whenever $\text{cd} \Gamma \neq 2$. The geometric dimension of Γ is the minimal dimension of CW-complexes representing the Eilenberg-MacLane space $K(\Gamma, 1)$.

Bestvina and Mess made a connection between the cohomological dimension of a hyperbolic torsion free group and topological dimension of its boundary [BM],[Be]:

$$\text{cd} \Gamma = \dim \partial\Gamma - 1.$$

In the 30s Pontryagin constructed compact subsets $X_p \subset \mathbb{R}^5$ for all prime p , now called Pontryagin surfaces, such that $\dim(X_p \times X_q) = 3$ whenever

*The author was supported by the Simons Foundation.

2010 MSC: 20F99.

Key words and phrases: product of groups, cohomological dimension.

$p \neq q$ and $\dim(X_p \times X_p) = 4$ for all p . Then in the late 40s Boltyanskii constructed a 2-dimensional compact subset $B \subset \mathbb{R}^5$ with $\dim(B \times B) = 3$. In [Dr1] the right-angled Coxeter groups with boundaries Pontryagin surfaces X_p were constructed for all p . This brought an example of a family of hyperbolic groups Γ_p with $\text{cd} \Gamma_p = 3$ for all p and $\text{cd}(\Gamma_p \times \Gamma_q) = 5$ for $p \neq q$ [Dr2]. Also in [Dr1] it was proven that Boltyanskii compactum cannot be a boundary of a Coxeter group. This result left some hope for the logarithmic law

$$\text{cd}(\Gamma \times \Gamma) = 2 \text{cd} \Gamma$$

for all geometrically finite groups. The above equality is the main result of this paper. We recall that a group Γ is called geometrically finite if there is a finite Eilenberg-MacLane complex $K(\Gamma, 1)$. Generally, for groups which are not geometrically finite the above equality does not hold: For example, for the group of rationals $\text{cd} \mathbb{Q} = 2$ and $\text{cd}(\mathbb{Q} \times \mathbb{Q}) = 3$.

The author is thankful to Alex Margolis for valuable remarks.

2. Dimension of the square of a group

2.1. Cohomological dimension of group with respect to a ring

Let R be a commutative ring with unit and let Γ be a group. By $R\Gamma$ we denote the group ring. Recall that $R\Gamma$ is the ring of all functions $f : \Gamma \rightarrow R$ with finite support and the convolution as the product. The cohomological dimension of a group Γ with respect to the ring R is defined as follows [Br],[Bi],

$$\text{cd}_R \Gamma = \max_M \{n \mid H^n(\Gamma, M) \neq 0\}$$

where the maximum is taken over all $R\Gamma$ -modules. In the case $R = \mathbb{Z}$ we use the notation $\text{cd} \Gamma = \text{cd}_{\mathbb{Z}} \Gamma$ and call the number $\text{cd} \Gamma$ the *cohomological dimension* of Γ .

For a Γ -module M we use notations $H^*(\Gamma, M)$ for cohomology groups of a discrete group Γ and $H^*(X; M)$ for cohomology groups of a topological space X with the fundamental group $\pi_1(X) = \Gamma$. Note that a Γ -module defines a locally trivial sheaf \mathcal{M} on X and $H^*(X; M)$ can be treated as cohomology with coefficients in the sheaf [Bre], $H^*(X; M) = H^*(X; \mathcal{M})$. Note that $H^*(\Gamma, M) = H^*(B\Gamma; M)$.

For abelian groups A and B we are using the notation $A * B = \text{Tor}(A, B)$. A group Γ is a group of finite type if there is a classifying CW complex $B\Gamma$ with finite skeletons $B\Gamma^{(n)}$ for all n .

We recall the Cohomology Universal Coefficient Formula (UCF) for twisted coefficients

UCF: For a group Γ of finite type, a $\mathbb{Z}\Gamma$ -module M , and an abelian group R there the Universal Coefficient Formula [Bre]:

$$H^n(\Gamma, M \otimes R) = H^n(\Gamma, M) \otimes R \bigoplus H^{n+1}(\Gamma, M) * R.$$

Note that $R\Gamma = \mathbb{Z}\Gamma \otimes R$ for any commutative ring R . Then the UCF turns into

$$H^n(\Gamma, R\Gamma) = H^n(\Gamma, \mathbb{Z}\Gamma) \otimes R \bigoplus H^{n+1}(\Gamma, \mathbb{Z}\Gamma) * R.$$

We recall the following facts about cohomological dimension of groups [Bi], [Br]:

- (1) $\text{cd}_R \Gamma$ equals the minimal length of projective resolution of R as a trivial $R\Gamma$ -module;
- (2) For geometrically finite groups $\text{cd}_R \Gamma = \max\{n \mid H^n(\Gamma, R\Gamma) \neq 0\}$.

Proposition 2.1. For any group Γ of finite type and any ring R , $\text{cd}_R \Gamma \leq \text{cd} \Gamma$.

Proof. We may assume that $\text{cd} \Gamma < \infty$. By the UCF if $H^n(\Gamma, R\Gamma) \neq 0$, then either $H^n(\Gamma, \mathbb{Z}\Gamma) \neq 0$ or $H^{n+1}(\Gamma, \mathbb{Z}\Gamma) \neq 0$. In either case $\text{cd}(\Gamma) \geq n$. \square

The main technical result of the paper is the following theorem which will be proven at the end of the section.

Theorem 2.2. For any geometrically finite group Γ there is a field \mathbf{k} such that $\text{cd} \Gamma = \text{cd}_{\mathbf{k}} \Gamma$.

The Kunneth Formula [Bre]: For groups Γ, Γ' of finite type, a $\mathbb{Z}\Gamma$ -module M , and a $\mathbb{Z}\Gamma'$ module M' with $M * M' = 0$ as abelian groups, there is the equality

$$\begin{aligned} H^n(\Gamma \times \Gamma', M \otimes M') \\ = \bigoplus_k H^k(\Gamma, M) \otimes H^{n-k}(\Gamma', M') \oplus \bigoplus_k H^{k+1}(\Gamma, M) * H^{n-k}(\Gamma', M'). \end{aligned}$$

Usually the Kunneth Formula as well as the Universal Coefficient Formula are stated as short exact sequences. This happens because the splitting of those short exact sequences is not natural. Since the naturality is not used in this paper we state both as formulas.

There is a similar formula for $R\Gamma$ and $R\Gamma'$ modules M and M' for any principal ideal domain R in which \otimes and $*$ are taken over R . When $R = k$ is a field, the Kunneth Formula turns into the following

$$H^n(\Gamma \times \Gamma', M \otimes_k M') = \bigoplus_{k=0}^n H^k(\Gamma, M) \otimes_k H^{n-k}(\Gamma', M').$$

Proposition 2.3. *For any geometrically finite group Γ and any field \mathbf{k} ,*

$$\text{cd}_{\mathbf{k}}(\Gamma \times \Gamma) = 2 \text{cd}_{\mathbf{k}} \Gamma.$$

Proof. Let $\text{cd}_{\mathbf{k}} \Gamma = n$. Then the vector space $H^n(\Gamma, \mathbf{k}\Gamma) \neq 0$ is nonzero. By the Kunneth Formula over a field,

$$H^{2n}(\Gamma \times \Gamma, \mathbf{k}\Gamma \otimes_{\mathbf{k}} \mathbf{k}\Gamma) = H^n(\Gamma, \mathbf{k}\Gamma) \otimes_{\mathbf{k}} H^n(\Gamma, \mathbf{k}\Gamma) \neq 0. \quad \square$$

Here is our main result.

Theorem 2.4. *For any geometrically finite group Γ ,*

$$\text{cd}(\Gamma \times \Gamma) = 2 \text{cd} \Gamma.$$

Proof. Let a field \mathbf{k} be as in Theorem 2.2. Then by Proposition 2.3 and Proposition 2.1

$$2 \text{cd} \Gamma = 2 \text{cd}_{\mathbf{k}} \Gamma = \text{cd}_{\mathbf{k}}(\Gamma \times \Gamma) \leq \text{cd}(\Gamma \times \Gamma).$$

The result follows in view of an obvious inequality $2 \text{cd} \Gamma \geq \text{cd}(\Gamma \times \Gamma)$. \square

Corollary 2.5. *For any geometrically finite group Γ ,*

$$\text{cd}(\Gamma^n) = n \text{cd} \Gamma.$$

Proof. In view of the obvious inequality $\text{cd}(\Gamma^n) \leq n \text{cd} \Gamma$ it suffices to show the inequality $\text{cd}(\Gamma^n) \geq n \text{cd} \Gamma$.

By induction on k we obtain the equality $\text{cd}(\Gamma^{2^k}) = 2^k \text{cd} \Gamma$. For $n < 2^k$ the inequality

$$2^k \text{cd} \Gamma = \text{cd}(\Gamma^{2^k}) \leq \text{cd}(\Gamma^n) + \text{cd}(\Gamma^{2^k-n}) \leq \text{cd}(\Gamma^n) + (2^k - n) \text{cd} \Gamma$$

implies that $n \text{cd} \Gamma \leq \text{cd}(\Gamma^n)$. \square

Corollary 2.6. *Boltyaskii's compactum cannot be a Z -boundary of a geometrically finite group.*

Proof. Suppose that Boltyanskii's compactum B is a Z -boundary of a geometrically finite group Γ . Then the join product $B * B$ is a Z -boundary of $\Gamma \times \Gamma$. In view of Bestvina's formula

$$\dim_{\mathbb{Z}} \partial G + 1 = \text{cd } G.$$

for a group G with a Z -boundary ∂G [Be] we obtain a contradiction:

$$5 = \dim_{\mathbb{Z}}(B * B) + 1 = \text{cd}(\Gamma \times \Gamma) = 2 \text{cd } \Gamma = 2(\dim_{\mathbb{Z}} B + 1) = 6. \quad \square$$

We use the notations $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for the cyclic group of order p , $\mathbb{Z}_{p^\infty} = \lim_{\rightarrow} \{\mathbb{Z}_{p^k}\}$ for the quasi-cyclic group, and $\mathbb{Z}_{(p)}$ for the localization of \mathbb{Z} at p .

Proposition 2.7. *For any geometrically finite group Γ there is a prime number p such that $\text{cd } \Gamma = \text{cd}_{\mathbb{Z}_{(p)}} \Gamma$.*

Proof. Let $\text{cd } \Gamma = n$. Let $T(A)$ denote the torsion subgroup of $A = H^n(\Gamma, \mathbb{Z}\Gamma)$. If $A/T(A) \neq 0$, then $A/T(A) \otimes \mathbb{Z}_{(p)} \neq 0$ and, hence, $A \otimes \mathbb{Z}_{(p)} \neq 0$ for all p . By the UCF, $H^n(\Gamma, \mathbb{Z}_{(p)}\Gamma) \neq 0$. Hence $\text{cd}_{\mathbb{Z}_{(p)}} \Gamma \geq n$. Proposition 2.1 implies that $\text{cd } \Gamma = \text{cd}_{\mathbb{Z}_{(p)}} \Gamma$.

Let $T(A) = A$. Every torsion abelian group A contains either \mathbb{Z}_{p^k} or \mathbb{Z}_{p^∞} for some p as a direct summand [Fu]. Since $\mathbb{Z}_{p^k} \otimes \mathbb{Z}_{(p)} \neq 0$ as well as $\mathbb{Z}_{p^\infty} \otimes \mathbb{Z}_{(p)} \neq 0$, we obtain $H^n(\Gamma, \mathbb{Z}\Gamma) \otimes \mathbb{Z}_{(p)} \neq 0$. Hence $H^n(\Gamma, \mathbb{Z}_{(p)}\Gamma) \neq 0$ and therefore, $\text{cd } \Gamma = \text{cd}_{\mathbb{Z}_{(p)}} \Gamma$. \square

Proposition 2.8. *Suppose that $J \subset \bigoplus^m R\Gamma$ is a $R\Gamma$ -submodule with the quotient module $M = (\bigoplus^m R\Gamma)/J$ for which $M \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ where $R = \mathbb{Z}_{(p)}$ for some prime p . Then $p^n M = 0$ for some $n \in \mathbb{N}$.*

Proof. We apply induction on m . In the case $m = 1$ we may assume that J does not contain $e \in \Gamma$. Since $(R\Gamma/J) \otimes \mathbb{Q} = 0$ there is r such that $p^r e \in J$. Then $p^r R\Gamma \subset J$. This implies that $p^r M = 0$.

Assume that the statement of Proposition holds for m . Let $J \subset \bigoplus^{m+1} R\Gamma$ and $M \otimes \mathbb{Q} = 0$ where $M = (\bigoplus^{m+1} R\Gamma)/J$. Let $\pi : \bigoplus^m R\Gamma \oplus R\Gamma \rightarrow \bigoplus^m R\Gamma$ denote the projection. Let $M' = (\bigoplus_{i=1}^k R\Gamma)/\pi(J)$. We consider

the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & J \cap R\Gamma & \longrightarrow & J & \longrightarrow & \pi(J) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & R\Gamma & \longrightarrow & \oplus^{m+1} R\Gamma & \xrightarrow{\pi} & \oplus^m R\Gamma \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & R\Gamma/(J \cap R\Gamma) & \longrightarrow & M & \xrightarrow{\bar{\pi}} & M' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

with exact columns and rows.

In view of surjectivity of $M \otimes \mathbb{Q} \rightarrow M' \otimes \mathbb{Q}$ and the assumption of the Proposition we obtain $M' \otimes \mathbb{Q} = 0$. By induction assumption $p^s M' = 0$ for some s . Tensoring the bottom row with \mathbb{Q} implies that $R\Gamma/(J \cap R\Gamma) \otimes \mathbb{Q} = 0$. By induction assumption $p^r(R\Gamma/(J \cap R\Gamma)) = 0$ for some r . Then $p^{r+s} M = 0$. Indeed, for any $x \in M$ we have $\bar{\pi}(p^s x) = p^s \bar{\pi}(x) = 0$ and, hence, $p^s x \in R\Gamma/(J \cap R\Gamma)$. Thus, $p^r p^s x = 0$. \square

*Proof of Theorem 2.2.*¹ Let $\text{cd } \Gamma = n$. Then $H^n(\Gamma, \mathbb{Z}\Gamma) \neq 0$ [Br]. Let p be as in Proposition 2.7. Then $H^n(\Gamma, \mathbb{Z}_{(p)}\Gamma) \neq 0$. Since $B\Gamma$ is a finite complex, the chain complex for $E\Gamma$ consists of finitely generated free $\mathbb{Z}\Gamma$ modules:

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0.$$

Then the cohomology group $H^n(\Gamma, \mathbb{Z}_{(p)}\Gamma)$ is defined by the cochain complex

$$0 \longleftarrow \text{Hom}_{\mathbb{Z}\Gamma}(C_n, \mathbb{Z}_{(p)}\Gamma) \xleftarrow{\delta} \text{Hom}_{\mathbb{Z}\Gamma}(C_{n-1}, \mathbb{Z}_{(p)}\Gamma) \longleftarrow \dots$$

Note that $\text{Hom}_{\mathbb{Z}\Gamma}(C_n, \mathbb{Z}_{(p)}\Gamma) \cong \oplus^m \mathbb{Z}_{(p)}\Gamma$ for some $m \in \mathbb{N}$. We denote by $F^r = \oplus^r \mathbb{Z}_{(p)}\Gamma$ and by $J = \text{im}(\delta : F^\ell \rightarrow F^m)$ the image of δ . Thus, $F^m/J \neq 0$. If $(F^m/J) \otimes \mathbb{Q} \neq 0$, then $H^n(\Gamma, \mathbb{Q}\Gamma) \neq 0$ and we take $\mathbf{k} = \mathbb{Q}$. If $(F^m/J) \otimes \mathbb{Q} = 0$ we apply Proposition 2.8 to obtain $H^n(\Gamma, \mathbb{Z}_p\Gamma) \neq 0$ for some prime number p . Then $\mathbf{k} = \mathbb{Z}_p$.

¹After the paper was accepted, the author learned about a shorter proof of Theorem 2.2 based on some basic facts on the structure of finitely presented modules.

3. Cohomological dimension over fields

We recall the excision theorem for cohomology with coefficients in a sheaf [Bre], Proposition 12.3.

Theorem 3.1. *If $A \subset X$ is a closed subset of compact space, then for any sheaf \mathcal{M} on X there is a natural isomorphism*

$$H^p(X, A; \mathcal{M}) \cong H_c^p(X \setminus A; \mathcal{M}).$$

The excision theorem and the long exact sequence of pair imply the following

Proposition 3.2. *For a finite complex X and any local coefficient system,*

$$H^m(X; \mathcal{M}) = H^m(X^{(m+1)}; \mathcal{M})$$

for all m .

Theorem 3.3. *For geometrically finite groups $\text{cd}_{\mathbb{Z}(p)} \Gamma = \text{cd}_{\mathbb{Q}} \Gamma$ for all but finitely many prime numbers p .*

Proof. Let $B\Gamma$ be a finite complex of $\dim B\Gamma = n$. Let $m > d = \text{cd}_{\mathbb{Q}} \Gamma$. Then $H^n(\Gamma, \mathbb{Q}\Gamma) = 0$ and in view of Proposition 3.2, $H^m(B\Gamma^{(m+1)}; \mathbb{Q}\Gamma) = 0$. Let $\{C_i, \partial_i\}$ denote the cellular chain complex for $E\Gamma$. The cohomology groups of $B\Gamma^{(m+1)}$ are defined by means of the truncated chain complex

$$0 \rightarrow C_{m+1} \rightarrow C_m \rightarrow \dots \rightarrow C_1 \rightarrow C_0.$$

Since C_{m+1} is a free $\mathbb{Z}\Gamma$ -module, $\text{Ext}_{\mathbb{Z}\Gamma}^1(C_{m+1}, \mathbb{Q}\Gamma) = 0$. Then the exact sequence

$$\leftarrow \text{Ext}_{\mathbb{Z}\Gamma}^1(C_{m+1}, \mathbb{Q}\Gamma) \leftarrow \text{Hom}_{\mathbb{Z}\Gamma}(C_m/C_{m+1}, \mathbb{Q}\Gamma) \leftarrow \text{Hom}_{\mathbb{Z}\Gamma}(C_m, \mathbb{Q}\Gamma) \leftarrow$$

implies that $\text{Hom}_{\mathbb{Z}\Gamma}(C_m/C_{m+1}, \mathbb{Q}\Gamma)$ is a finitely generated module.

The condition $H^m(B\Gamma^{(m+1)}, \mathbb{Q}\Gamma) = 0$ means that the homomorphism

$$\delta : \text{Hom}_{\mathbb{Z}\Gamma}(C_{m-1}, \mathbb{Q}\Gamma) \rightarrow \text{Hom}_{\mathbb{Z}\Gamma}(C_m/C_{m+1}, \mathbb{Q}\Gamma)$$

defined by means of $\partial : C_m \rightarrow C_{m-1}$ is onto. Let ϕ_1, \dots, ϕ_k be a generating set for the $\mathbb{Z}\Gamma$ -module $\text{Hom}_{\mathbb{Z}\Gamma}(C_m/C_{m+1}, \mathbb{Q}\Gamma)$. Let $\{\alpha_1, \dots, \alpha_k\} \in \text{Hom}_{\mathbb{Z}\Gamma}(C_{m-1}, \mathbb{Q}\Gamma)$ be such that $\delta(\alpha_j) = \phi_j$. For all $(m-1)$ -cells in $B\Gamma$, $\kappa_1, \dots, \kappa_\ell$, we fix lifts $\bar{\kappa}_1, \dots, \bar{\kappa}_\ell$. Let

$$\alpha_j(\bar{\kappa}_i) = \sum \lambda_k^{i,j} g_k^{i,j} \quad \text{where } \lambda_k^{i,j} \in \mathbb{Q} \text{ and } g_k^{i,j} \in \Gamma.$$

Clearly, there is a finite set of primes $\mathcal{P}_m = \{p_1, \dots, p_{s_m}\}$ such that $\lambda_k^{i,j} \in \mathbb{Z}[\frac{1}{p_1 \cdots p_s}]$. Then the homomorphism

$$\delta_p : \text{Hom}_\Gamma(\bar{C}_{m-1}, \mathbb{Z}_{(p)}\Gamma) \rightarrow \text{Hom}_\Gamma((\bar{C}_m/\bar{C}_{m+1}), \mathbb{Z}_{(p)}\Gamma)$$

is surjective for all $p \notin \mathcal{P}_m$. Therefore, $H^m(B\Gamma^{(m+1)}; \mathbb{Z}_{(p)}\Gamma) = 0$ for all $p \notin \mathcal{P}_m$. By Proposition 3.2, $H^m(B\Gamma; \mathbb{Z}_{(p)}\Gamma) = 0$ for all $p \notin \mathcal{P}_m$. Hence, $\text{cd}_{\mathbb{Z}_{(p)}\Gamma} \Gamma \leq d$ for all $p \notin \bigcup_{d < m \leq n} \mathcal{P}_m$.

The inequality $\text{cd}_{\mathbb{Q}} \Gamma \leq \text{cd}_{\mathbb{Z}_{(p)}} \Gamma$ completes the proof. □

Proposition 3.4. *For all groups for every prime p and any $k \in \mathbb{N}$,*

$$\text{cd}_{\mathbb{Z}_{p^k}} \Gamma = \text{cd}_{\mathbb{Z}_p} \Gamma.$$

Proof. We apply induction on k . The short exact sequence $0 \rightarrow \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_{p^{k+1}} \rightarrow \mathbb{Z}_p \rightarrow 0$ produces the short exact sequence of Γ -modules

$$0 \rightarrow \mathbb{Z}_{p^k}\Gamma \rightarrow \mathbb{Z}_{p^{k+1}}\Gamma \rightarrow \mathbb{Z}_p\Gamma \rightarrow 0.$$

Let $\text{cd}_{\mathbb{Z}_{p^{k+1}}\Gamma} \Gamma = n$. Then the coefficient long exact sequence

$$\dots \rightarrow H^n(\Gamma, \mathbb{Z}_{p^k}\Gamma) \rightarrow H^n(\Gamma, \mathbb{Z}_{p^{k+1}}\Gamma) \rightarrow H^n(\Gamma, \mathbb{Z}_p\Gamma) \rightarrow \dots$$

and the induction assumption imply that $\text{cd}_{\mathbb{Z}_p} \Gamma \geq n$. The same sequence at dimension $n + 1$ together with the induction assumption implies that $\text{cd}_{\mathbb{Z}_p} \Gamma < n + 1$. □

Problem 3.5. *Does the equality $\text{cd}_{\mathbb{Z}_p} \Gamma = \text{cd}_{\mathbb{Z}_{(p)}} \Gamma$ for a geometrically finite group Γ hold true for all prime numbers p ?*

4. Some remarks

4.1. Finitely generated torsion modules over $\mathbb{Z}\Gamma$

An algebraic reason for the main result of the paper is the fact that for a geometrically finite group Γ any finitely generated torsion $\mathbb{Z}\Gamma$ -module does not contain \mathbb{Z}_{p^∞} for any prime p :

Proposition 4.1. *Let M be a finitely generated $\mathbb{Z}\Gamma$ -module for a geometrically finite group Γ where the underlying abelian group M is a torsion group. Then for any prime p the subgroup of p -torsions $M[p] \subset M$ is bounded, i.e., there is $r \in \mathbb{N}$ such that $p^r x = 0$ for all $x \in M[p]$.*

Proof. Let $\xi : \oplus^m \mathbb{Z}\Gamma \rightarrow M$ be an epimorphism of $\mathbb{Z}\Gamma$ -modules. Then the tensor product with $\mathbb{Z}_{(p)}$ defines an epimorphism

$$\xi \otimes 1_{\mathbb{Z}_{(p)}} : (\oplus^m \mathbb{Z}\Gamma) \otimes \mathbb{Z}_{(p)} = \oplus^m \mathbb{Z}_{(p)}\Gamma \rightarrow M[p] = M \otimes \mathbb{Z}_{(p)}$$

with the kernel $J = \ker(\xi)$. Proposition 2.8 implies that $M[p]$ has bounded order. \square

Remark 4.2. The examples of right-angled Coxeter groups Γ in [Dr1] mentioned in the introduction have geometrically finite subgroups Γ' that admit infinite finitely generated $\mathbb{Z}\Gamma'$ -modules M which are torsions as the additive groups.

4.2. Application to topological complexity

Motivated by Topological Robotics, Michael Farber introduced the notion of topological complexity $TC(X)$ of the configuration space X [F] as the minimal number k such that $X \times X$ can be covered by $k + 1$ open sets U_0, \dots, U_k for each of which there is a continuous map $\phi_i : U_i \rightarrow C([0, 1], X)$ satisfying the condition $\phi_i(x, y)(0) = x$ and $\phi_i(x, y)(1) = y$ for all $(x, y) \in U_i$. Since $TC(X)$ is a homotopy invariant, the topological complexity can be defined for groups Γ as $TC(B\Gamma)$. Computation of topological complexity of groups presents a great challenge. The main theorem of this paper allows to complete the computation of topological complexity of hyperbolic groups [Dr3]

$$TC(\Gamma) = 2 \operatorname{cd} \Gamma$$

originated by Farber and Mescher [FM]. Also in the case of geometrically finite groups the main theorem simplifies the formula for topological complexity of the free product of groups [DS] to the following

$$TC(\Gamma * \Gamma) = 2 \operatorname{cd} \Gamma.$$

References

- [Be] M. Bestvina, Local homology properties of boundaries of groups. *Michigan Math. J.* 43 (1996), no. 1, 123-139.
- [BM] M. Bestvina, G. Mess, The boundary of negatively curved groups. *J. Amer. Math. Soc.* 4 (1991), no. 3, 469-481.
- [Bi] R. Bieri, Homological dimension of discrete groups, Quinn Mary College, 191.
- [Bo] V. Boltyanskii, *An example of a two-dimensional compactum whose topological square is three-dimensional.* (Russian) *Doklady Akad. Nauk SSSR (N.S.)* 67, (1949). 597-599 (English translation in *Amer. Math. Soc. Translation* 1951, (1951). no. 48, 3-6).

- [Br] K. Brown, *Cohomology of groups*, Springer, New York-Heidelberg-Berlin, 1982.
- [Bre] G. Bredon, *Sheaf Theory. Graduate Text in Mathematics*, **170**, Springer, New York Heidelberg Berlin, 1997.
- [Dr1] A. Dranishnikov, On the virtual cohomological dimensions of Coxeter groups. Proc. Amer. Math. Soc. 125 (1997), no. 7, 1885-1891.
- [Dr2] A. Dranishnikov, Boundaries of Coxeter groups and simplicial complexes with given links. J. Pure Appl. Algebra 137 (1999), no. 2, 139-151.
- [Dr3] A. Dranishnikov, On topological complexity of hyperbolic groups. Preprint arXiv:1904.06720.
- [DS] A. Dranishnikov, R. Sadykov, The topological complexity of the free product, *Mathematische Zeitschrift*, 293(1), (2019), 407-416.
- [F] M. Farber, *Invitation to topological robotics. Zurich Lectures in Advanced Mathematics*. European Mathematical Society (EMS), Zürich, 2008.
- [FM] M. Farber, S. Mescher, On the topological complexity of aspherical spaces, *Journal of Topology and Analysis*, to appear
- [Fu] L. Fuchs, *Infinite abelian groups*, v. 1, Academic Press, 1970.
- [P] L. S. Pontryagin, *Sur une hypothese fondamentale de la theorie de la dimension*. C. R. Acad. Sci. Paris 190 (1930), 1105-1107.

CONTACT INFORMATION

A. Dranishnikov Department of Mathematics, University of
Florida, 358 Little Hall, Gainesville, FL
32611-8105, USA
E-Mail(s): `dranish@math.ufl.edu`

Received by the editors: 19.11.2019.