© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **30** (2020). Number 2, pp. 235–238
DOI:10.12958/adm1485

Decompositions of set-valued mappings

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On 100th anniversary of Professor V. S. Čarin[*]

Abstract. Let X be a set, B_X denotes the family of all subsets of X and $F : X \to B_X$ be a set-valued mapping such that $x \in F(x)$, $\sup_{x\in X} |F(x)| < \kappa$, $\sup_{x\in X} |F^{-1}(x)| < \kappa$ for all $x \in X$ and some infinite cardinal κ. Then there exists a family $\mathcal{F}$ of bijective selectors of F such that $|\mathcal{F}| < \kappa$ and $F(x) = \{f(x) : f \in \mathcal{F}\}$ for each $x \in X$. We apply this result to G-space representations of balleans.

1. Decompositions

For a set X, B_X denotes the family of all subsets of X. Given a set-valued mapping $F : X \to B_X$, any function $f : X \to X$ such that, for each $x \in X$, $f(x) \in F(x)$ is called a *selector* of F. We say that a selector f is *bijective* if $f : X \to X$ is a bijection. For $x \in X$, we denote $F^{-1}(x) = \{y \in X : x \in F(y)\}$.

In section 1 we prove the mail result and apply it to G-space representations of balleans in section 2.

Theorem 1. *Let $F : X \to B_X$ be a set-valued mapping such that $x \in F(x)$, $\sup_{x\in X} |F(x)| < \kappa$, $\sup_{x\in X} |F^{-1}(x)| < \kappa$ for each $x \in X$ and some infinite cardinal κ. Then there exists a family $\mathcal{F}$ of bijective selectors of X such that $|\mathcal{F}| < \kappa$ and $F(x) = \{f(x) : f \in \mathcal{F}\}$ for each $x \in X$.*

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[*]Victor Sil'vestrovich Čarin is known as the founder of topological algebra in Kyiv University, but his mathematical interests were not bounded by topological groups. He encouraged and supported the activity of students and collaborators in many areas, in particular, in combinatorics.

2010 MSC: 03E05, 54E05.

Key words and phrases: set-valued mapping, selector, ballean.

Proof. We consider two cases.

Case $\kappa = \omega$. We put $\mathcal{P} = \{F(x) : x \in X\}$ and define a graph Γ with the set of vertices $\mathcal{P}$ and the set of edges $\{\{F(x), F(y)\} : F(x) \cap F(y) \neq \varnothing\}$. We take a natural number m such that $m > \sup_{x \in X} |F(x)|$, $m > \sup |F^{-1}(x)|$ and show that the local degree of each vertices of Γ does not exceed $m^2 - 1$. Assume the contrary and choose $y \in X$ and distinct $y_1, \dots, y_{m^2} \in X$ such that $F(y) \cap F(y_i) \neq \varnothing\}$ for every $i \in \{1, \dots, m^2\}$. Then $y_i \in F^{-1}F(y)$ but, by the choice of m, we have $|F^{-1}F(y)| < m^2$.

We use the following simple fact [2]: if the local degree of each vertices of a graph Γ' does not exceed k then the chromatic number of Γ' does not exceed $k + 1$.

Hence the set $\mathcal{P}$ of vertices of Γ can be partition $\mathcal{P}_1, \dots, \mathcal{P}_{m^2}$ so that any two vertices from each $\mathcal{P}_i$ are not incident.

To construct the family $\mathcal{F}$, we enumerate $\mathcal{P}_i = \{F(y_\alpha) : \alpha < \gamma\}$. Let $M = \sup_{x \in X} |F(x)|$. Then we enumerate each $F(y_\alpha)$ (with repetitions, if necessary) $F(y_\alpha) = \{y_{\alpha j}) : j < M\}$, $y_{\alpha_0} = y_\alpha$. For each $j < M$, we define a bijective function f_j such that f_j acts as a transposition of y_α and $y_{\alpha j}$ at each $F(y_\alpha)$ and identically at all other elements of X. We put $\mathcal{F}_i = \{f_j : j < M\}$ and note that $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{m^2}$ is the desired family of selectors of F.

Case $\kappa > \omega$. We take an infinite cardinal σ such that $\sigma < \kappa$ and $|F(x)| \leqslant \sigma$, $|F^{-1}(x)| \leqslant \sigma$ for each $x \in X$. Then we define a partition $\mathcal{P}$ of X such that each $P \in \mathcal{P}$ is the minimal by inclusion subset of X satisfying $F(y) \in P$, $F^{-1}(y) \in P$ for each $y \in P$. Constructively, every P can be obtained applying to $x \in P$ the sequence of operations F, $F^{-1} : F(x)$, $F^{-1}F(x)$, $FF^{-1}F(x), \dots$. Then P is the union of all numbers of this sequence.

By the choice of σ, we have $|P| \leqslant \sigma$. We enumerate $\mathcal{P} = \{P_\alpha : \alpha < \gamma\}$, $P_\alpha = \{x_{\alpha j} : j < \gamma\}$. For each $j < \sigma$, we choose a family $\mathcal{F}_j$ of bijective selectors of F such that $|F_j| \leqslant \sigma$ and $F(x_{\alpha j}) = \{f(x_{\alpha j}) : f \in \mathcal{F}_j\}$ for each $\alpha < \gamma$, see the case $\kappa = \omega$. Then $\bigcup_{j<\sigma} \mathcal{F}_j$ is the desired family $\mathcal{F}$ of bijective selectors of F. $\square$

2. Applications

Let X be a set. A family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) : x \in X\}$;
- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z((x, z) \in E, \ \ (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;

- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
- for any $x, y \in X$, there exists $E \in \mathcal{E}$ such that $(x, y) \in E$.

A subset $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ we denote $E[x] = \{y \in X : (x, y) \in E\}$, $E[A] = \cup_{a \in A} E[a]$ and say $E[x]$ and $E[A]$ are *balls of radius* E *around* x and A.

The pair $(X, \mathcal{E})$ is called a *coarse space* [6] or a *ballean* [5].

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \to X'$ is called *macro-uniform* if, for every $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}'$ such that $E[x] \subseteq E'[f(x)]$. If f is a bijection such that f, f^{-1} are macro-uniform then f is called an *asymorphism*.

Now we describe some general way of constructing balleans. Let G be a group. A family $\mathcal{I}$ of subsets of G is called an *ideal* if, for every $A, B \in \mathcal{I}$ and $A' \subseteq A$, we have $A \cup B \in \mathcal{I}$ and $A' \in \mathcal{I}$. An ideal $\mathcal{I}$ is called a *group ideal* if $F \in \mathcal{I}$ for every finite subset of G and $A, B \in \mathcal{I}$ imply $AB^{-1} \in \mathcal{I}$.

Let a group G acts transitively on a set X by the rule $(g, x) \longmapsto gx$, $g \in X$, $x \in X$. Every group ideal $\mathcal{I}$ on G defines the ballean $(X, G, \mathcal{I})$ on X with the base of entourages $\{\{(x, y) : y \in Ax\} : A \in \mathcal{I}\}$. By Theorem 1 from [3], for every ballean $(X, \mathcal{E})$, there exist a group G of permutations of X and a group ideal $\mathcal{I}$ on G such that $(X, \mathcal{E})$ is asymorphic to $(X, G, \mathcal{I})$.

Theorem 2. *Let $(X, \mathcal{E})$ be a ballean and let κ be an infinite cardinal such that, for each $E \in \mathcal{E}$, $\sup_{x \in E} |E[x]| < \kappa$. Then there exist a group G of permutations of X and a group ideal $\mathcal{I}$ on G such that $(X, \mathcal{E})$ is asymorphic to $(X, \mathcal{E}, \mathcal{I})$ and $|A| < \kappa$ for each $A \in \mathcal{I}$.*

Proof. For each $E \in \mathcal{E}$, we define a mapping $F_E : X \to B_X$ by $F_E(x) = E[x]$. By Theorem 1, there exists a family F_E of permutations of X such that $|\mathcal{F}_E| < \kappa$ and $F_E(x) = \{f(x) : f \in \mathcal{F}_E\}$ for each $x \in X$. We denote by $\mathcal{I}$ the minimal by inclusion group ideal of G such that $\mathcal{F}_E \in \mathcal{I}$ for each $E \in \mathcal{E}$. Then $(X, \mathcal{E})$ is asymorphic to $(X, G, \mathcal{I})$. □

In the case $\kappa = \omega$, Theorem 2 was proved in [4]. For its applications see Remark 3.5 in [1].

A ballean $(X, \mathcal{E})$ is called *cellular* if $\mathcal{E}$ has a base consisting of equivalence relations. By Theorem 3 from [3], every cellular ballean is asymorphic to some ballean $(X, G, \mathcal{I})$ such that $\mathcal{I}$ has a base consisting of subgroups of G.

A ballean $(X, \mathcal{E})$ is called *finitary* if, for every $E \in \mathcal{E}$ there exists a natural number m such $|E[x]| < m$ for each $x \in X$. The finitary ballean of a G space X is the ballean $(X, G, \mathcal{I})$, where $\mathcal{I}$ is the ideal of all finite subsets of G.

Theorem 3. *For every finitary cellular ballean $(X, \mathcal{E})$ there exists a locally finite group of permutations of X such that $(X, \mathcal{E})$ is asymorphic to the finitary ballean of G-space X.*

Proof. We take a base $\mathcal{E}'$ of consisting of partitions of X. For every $\mathcal{P} \in \mathcal{E}$ we pick a natural number $n_{\mathcal{P}}$ such that $|P| \leqslant n_{\mathcal{P}}$ for each $P \in \mathcal{P}$. We denote by $G_{\mathcal{P}}$ the direct product of the family of symmetric groups $\{S_m : m \leqslant n_{\mathcal{P}}\}$ and note that $G_{\mathcal{P}}$ acts on each $P \in \mathcal{P}$ so that $G_{\mathcal{P}} x = P$ for each $x \in P$. Then the group G generated by the family $\{G_{\mathcal{P}} : \mathcal{P} \in \mathcal{E}'\}$ satisfies the conclusion of Theorem 3. □

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Received by the editors: 29.10.2019.