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RESEARCH ARTICLE

A note on Hall S-permutably embedded subgroups of finite groups

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Abstract. Let G be a finite group. Recall that a subgroup A of G is said to *permute* with a subgroup B if $AB = BA$. A subgroup A of G is said to be S-*quasinormal* or S-*permutable* in G if A permutes with all Sylow subgroups of G. Recall also that H^{sG} is the S-*permutable closure* of H in G, that is, the intersection of all such S-permutable subgroups of G which contain H. We say that H *is Hall* S-*permutably embedded in* G if H is a Hall subgroup of the S-permutable closure H^{sG} of H in G.

We prove that the following conditions are equivalent: (1) every subgroup of G is Hall S-permutably embedded in G; (2) the nilpotent residual $G^{\mathfrak{N}}$ of G is a Hall cyclic of square-free order subgroup of G; (3) $G = D \rtimes M$ is a split extension of a cyclic subgroup D of square-free order by a nilpotent group M, where M and D are both Hall subgroups of G.

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. The symbol $G^{\mathfrak{N}}$ denotes the *nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with nilpotent quotient G/N.

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Recall that a subgroup A of G is said to *permute* with a subgroup B if $AB = BA$. A subgroup A of G is said to be *S-quasinormal* or *S-permutable* in G if A permutes with all Sylow subgroups of G.

The S-permutable subgroups possess a series of interesting properties. For instance, the S-permutable subgroups of G form a sublattice of the lattice of all subnormal subgroups of G (Kegel [1]). This important property of S-permutable subgroups allows to introduce the concept of the S-permutable closure of a subgroup. The intersection of all such S-permutable subgroups of G which contain a subgroup H of G is called the *S-permutable closure of H in G* and denoted by H^{sG} (see Guo and Skiba [2]).

Recall also that a subgroup H of G is said to be a *Hall normally embedded subgroup* of G [3] if H is a Hall subgroup of the normal closure H^G of H in G. By analogy with it, we say that a subgroup H of G is called a *Hall S-permutably embedded subgroup* of G if H is a Hall subgroup of the S-permutable closure H^{sG} of H in G.

In the paper [4], Shirong Li and Jianjun Liu described groups G such that every subgroup of G is Hall normally embedded in G. Our main goal here is to prove the following generalization of this result.

Theorem 1. *The following conditions are equivalent:*

(1) *every subgroup of G is Hall S-permutably embedded in G;*

(2) *the nilpotent residual $G^{\mathfrak{N}}$ of G is a Hall cyclic of square-free order subgroup of G;*

(3) *$G = D \rtimes M$ is a split extension of a cyclic subgroup D of square-free order by a nilpotent group M, where M and D are both Hall subgroups of G.*

Corollary 1 (Shirong Li and Jianjun Liu [4, Theorem 3.4])**.** *Every subgroup of G is Hall normally embedded in G if and only if $G = D \rtimes M$ is a split extension of a cyclic subgroup D of square-free order by a Dedekind group M, where M and D are both Hall subgroups of G.*

Proofs of Theorem 1 and Corollary 1

We will need a few facts about S-permutable subgroups.

Lemma 1 (see Kegel [1] or [5, Theorem 1.2.14])**.** *Let $H \leqslant K \leqslant G$.*

(1) *If H is S-permutable in G, then H is S-permutable in K.*

(2) *Suppose that H is normal in G. Then K/H is S-permutable in G/H if and only if K is S-permutable in G.*

(3) *If H is S-permutable in G, then H is subnormal in G.*

We write $H^{\cdot\cdot G}$ to denote the *subnormal closure* of H in G, that is, the intersection of all subnormal subgroups of G which contain H (see [6, A, Definition 14.13]).

A subgroup H of G is called a *Hall subnormally embedded subgroup* of G [4, Definition 1.4] if H is a Hall subgroup of the subnormal closure $H^{\cdot\cdot G}$ of H in G. We need also some properties of Hall subnormally embedded subgroups (see [4, Theorem 3.3]).

Lemma 2. *If every subgroup of G is Hall subnormally embedded in G, then the following statements hold:*
(1) *$G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is the nilpotent residual of G;*
(2) *D and M are Hall subgroups of G;*
(3) *M acts irreducibly on each Sylow subgroup of D.*

Lemma 3. (1) *If H is a Hall S-permutably embedded subgroup of G, then H is a Hall subnormally embedded subgroup of G.*
(2) *If H is a Hall normally embedded subgroup of G, then H is a Hall S-permutably embedded subgroup of G.*

Proof. (1) Since every S-permutable subgroup of G is a subnormal subgroup of G by Lemma 1(3), $H^{\cdot\cdot G} \leqslant H^{sG}$. Moreover, H is a Hall subgroup of H^{sG} by hypothesis, so H is a Hall subgroup of $H^{\cdot\cdot G}$.

(2) See the proof of (1). □

Lemma 4 (see Deskins [7] or [5, Theorem 1.2.14])**.** *If the subgroup H of G is S-permutable in G, then H/H_G is nilpotent.*

Lemma 5 (see [8, Lemma 2.4])**.** *Let H be a Hall S-permutably embedded subgroup of G. Then the following statements hold:*
(1) *if $H \leqslant K \leqslant G$, then H is Hall S-permutably embedded in K;*
(2) *if $N \lhd G$, then HN/N is Hall S-permutably embedded in G/N.*

Lemma 6. *Let $G = D \rtimes M$, where D is a Hall cyclic of square-free order subgroup of G and M is a nilpotent (respectively Dedekind) subgroup of G. Then every subgroup of G is Hall S-permutably embedded (respectively Hall normally embedded) in G.*

Proof. Let H be a subgroup of G. Let $D_1 = H \cap D$. Clearly, D_1 is a Hall subgroup of D and D_1 has a complement D_2 in D.

Since $M \simeq G/D$ is nilpotent (respectively Dedekind), all subgroups of G/D are S-permutable (respectively normal) in G/D. Then DH/D is

S-permutable (respectively normal) in G/D. Hence by Lemma 1(2), DH is S-permutable (respectively normal) in G. Therefore $H \leqslant H^{sG} \leqslant DH$ (respectively $H \leqslant H^G \leqslant DH$).

Now we show that H is a Hall subgroup of H^{sG} (respectively of H^G). Since

$$|DH : H| = \frac{|D_1D_2H|}{|H|} = \frac{|D_2H|}{|H|} = \frac{|D_2||H|}{|D_2 \cap H||H|} = |D_2|,$$

$(|H|, |DH : H|) = 1$. Thus H is a Hall subgroup of DH, therefore H is a Hall subgroup of H^{sG} (respectively of H^G). Hence H is Hall S-permutable embedded (respectively Hall normally embedded) in G. □

Lemma 7 (see [5, Theorem 1.2.16]). *Let H be a p-subgroup of G, where p is a prime. Then H is S-permutable in G if and only if*

$$O^p(G) \leqslant N_G(H).$$

Now we are in position to proof the main result.

Proof. Let $D = G^{\mathfrak{N}}$.

(1) ⇒ (2) Assume that this is false and let G be a counterexample of minimal order.

(a) *If N is a minimal normal subgroup of G, then the hypothesis holds for G/N and so Condition (2) is true for G/N.*

Let H/N be any subgroup of G/N. Then H is Hall S-permutably embedded in G by hypothesis. Hence H/N is Hall S-permutably embedded in G/N by Lemma 5(2). Therefore the hypothesis holds for G/N. In view of

$$|G/N| < |G|,$$

the choice of G implies that Condition (2) is true for G/N.

(b) *G is soluble.*

Assume that this is false. Claim (a) implies that G/N is soluble for every minimal normal subgroup N of G, so N is the unique minimal normal subgroup of G, N is non-abelian and $N \nleqslant \Phi(G)$. Let X be a maximal subgroup of G such that $N \nleqslant X$. Then $G = NX$.

Let p be a prime dividing the order of G. Then there exist a Sylow p-subgroup N_p of N and a Sylow p-subgroup X_p of X such that $P = N_pX_p$ is a Sylow p-subgroup of G. We have that either $P = X_p$ and then X contains a Sylow p-subgroup of G or there exists a maximal subgroup K of P such that X_p is contained in K. Suppose the second possibility is true. By hypothesis, K is Hall S-permutably embedded in G. Hence we can find a subgroup B of G such that B is S-permutable in G and K is a Sylow p-subgroup of B. Assume that $B_G \neq 1$. Then N is contained in B and so $P = X_pN_p$ is a Sylow p-subgroup of B. Hence $|K| = |P|$, a contradiction. Hence $B_G = 1$. Then B is a nilpotent group by Lemma 4. Moreover, B is subnormal in G by Lemma 1(3). This implies that B is contained in $F(G)$. But $F(G) = 1$ because N is non-abelian. Hence $K = 1$ and so $N_p = P$ is a cyclic group of order p.

Thus we have that if p divides the order of G, then either X contains a Sylow p-subgroup of G or N contains a Sylow p-subgroup of G. In the second case, this Sylow p-subgroup should be cyclic of order p.

Denote $\pi = \pi(N)$. Since G/N is supersoluble by Claim (a), it follows that

$$G/N = XN/N \cong X/(N \cap X)$$

is supersoluble. In particular, $X/(N \cap X)$ is soluble. Let H be a subgroup of G such that $H/(N \cap X)$ is a Hall π-subgroup of $X/N \cap X$. Suppose that NH is a proper subgroup of G. Then by Lemma 5(1) the hypothesis of the theorem holds for NH. By the minimal choice of G, it follows that NH is supersoluble, a contradiction. Hence we have $G = NH$ and G is a π-group. Suppose that for each prime $p \in \pi$, the Sylow p-subgroups of N are Sylow p-subgroups of G. Then $G = N$ and, by the above argument, every Sylow subgroup of G is cyclic. By [9, IV, Theorem 2.9], G is soluble, a contradiction. Therefore there is $q \in \pi$ such that a Sylow q-subgroup N_q of N is not a Sylow q-subgroup of G. Then arguing as above we get a contradiction. Thus G is soluble.

(c) *G is supersoluble.*

Assume that this is false. Then, since the class of all supersoluble groups is a saturated formation, Claim (a) implies that G has a unique minimal normal subgroup, say N, and $N \nleq \Phi(G)$. Moreover, since G is soluble by Claim (b),

$$N = O_p(G) = C_G(N)$$

is a non-cyclic abelian p-group for some prime p by [6, A, Theorem 15.2]. Let P be a Sylow p-subgroup of G containing N. Let A be a maximal subgroup of P not containing N. Since A is Hall S-permutably embedded in G, we can find an S-permutable subgroup of G, say B, such that A is a Sylow p-subgroup of B. From the fact that N is not contained in A, we have $B_G = 1$. Hence, by Lemma 4, B is nilpotent and so B is contained in $F(G) = N$ in view of subnormality of B in G. Therefore $A = B$ and

$P = N$. Since A is S-permutable in G, $O^p(G) \leqslant N_G(A)$ by Lemma 7. Hence A is normal in G. But $P = N$ is a minimal normal subgroup of G. Thus $A = 1$ and $|P| = p$. This contradiction shows that G is supersoluble.

Final contradiction. Since G is supersoluble, D is nilpotent. Moreover, by Lemmas 2(1)(2) and 3(1), D is a Hall subgroup of G.

Since D is nilpotent, each Sylow subgroup of D is normal in D and therefore each Sylow subgroup of D is characteristic in D. Hence each Sylow subgroup of D is normal in G. Let $V \neq 1$ be a Sylow subgroup of D and let R be a minimal normal subgroup of G contained in V. Since M acts irreducible on each Sylow subgroup of D by Lemmas 2(3) and 3(1), $R = V$. Therefore, since G is supersoluble, $|R| = |V|$ is a prime. Hence D is a cyclic group of square-free order.

(2) $\Rightarrow$ (3) Since D is a Hall subgroup of G, D has a complement M in G by the Schur-Zassenhaus theorem. Finally, since

$$M \simeq G/D = G/G^{\mathfrak{N}},$$

M is a Hall nilpotent subgroup of G.

(3) $\Rightarrow$ (1) This directly follows from Lemma 6.

The theorem is proved. □

Finally, we prove Corollary 1.

Proof. Necessity. In view of Lemma 3(2), Theorem 1 and [5, Theorem 1.4], it is enough to show that G is a T-group. Let H be a subnormal subgroup of G. Then H is subnormal in H^G by [6, A, Theorem 14.8]. Then, since H is a Hall subgroup of H^G by hypothesis, H is characteristic in H^G. Hence H is a normal subgroup of G, so G is a T-group.

Sufficiency. This directly follows from Lemma 6.

The corollary is proved. □

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