Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **28** (2019). Number 2, pp. 157–170

On schurity of one-sided bimodule problems

Vyacheslav Babych and Nataliya Golovashchuk

Communicated by A. P. Petravchuk

Abstract. We consider a class of normal bimodule problems satisfying some structure, triangularity and finiteness conditions (one-sided bimodule problems). We study the structure of non-schurian bimodule problems from our class and describe explicitly the minimal non-schurian one-sided bimodule problems.

Introduction

The notion of bimodule problem arose as a formal language for so called matrix problem solving methods, i. e. the equivalence classes representatives description problem for a set of matrices with respect to some set of transformations ([16]). Transition from matrix to bimodule problem provide us with effective algorithms for obtaining the representation type of bimodule problem and for its representation category description. Other important tools closely related to this task are quadratic and bilinear forms ([8, 14]).

We develop approach to investigation of bimodule problem representation category and representation type based on results of [1, 9] for another class of bimodule problems. From the representation theory point of view, the simplest bimodule problems are those for which the dimensions of indecomposable representations are in bijection with the roots of corresponding Tits quadratic form. This observation leads to the notion of a

2010 MSC: 16D90, 16P60, 15A04, 16G20, 15A63.
Key words and phrases: bimodule problem, quasi multiplicative basis, Tits quadratic form, representation, schurity.

schurian bimodule problem for which any indecomposable representation has only scalar endomorphisms. In particular, quivers and posets with weakly positive Tits form are schurian ([4, 13]).

We consider the class of bimodule problems called one-sided bimodule problems which generalizes well known classes of quivers and posets. We impose two important restrictions on considered bimodule problems. Firstly we assume that the quadratic form of bimodule problem is weakly positive since the problem has an infinite representation type in the opposite case. The second assumption limits the number of objects in the category to at least three, otherwise the problems are considered directly. Bimodule problem from this class has a quasi multiplicative basis introduced in [2] which is a generalization of the notion of a multiplicative basis and allows to distinguish the minimal non-schurian subproblems effectively. Theorem 1 asserts that the minimal non-schurian admitted bimodule problem having more than two objects is standard. This theorem is a special case of the main result of [4] on DGC, but we give a new proof using more invariant language.

We use the definitions, notations and statements from [1–3, 16]. The considered class C of bimodule problems and the notion of quasi multiplicative basis are defined in [3]. The notions and facts from the theory of quadratic forms can be found in [6, 8, 14, 17, 18].

1. Preliminaries

Basic roots and singular vertices. Denote by WP the set of all weakly positive locally finite quadratic forms and by $\Re_q^+$ a set of all positive roots of a form q. Let $q(x_1, \dots, x_n) \in \mathrm{WP}$ be an unit form, $n \geqslant 2$. A sincere $x \in \Re_q^+$ is called a *basic* root, if there exist $i_1, i_2 \in I = \{1, \dots, n\}$ such that $2(e_{i_1}, x) = 2(e_{i_2}, x) = 1$, $x_{i_1} = x_{i_2} = 1$ and $2(e_i, x) = 0$, $i \in I \backslash \{i_1, i_2\}$. We call i_1, i_2 the *singular* vertices of x.

Lemma 1. *Let $q(x_1, \dots, x_n) \in \mathrm{WP}$ be an unit form, $n \geqslant 3$, let $x \in \Re_q^+$ be sincere non-basic, and let $i_1, i_2 \in I$, $i_1 \neq i_2$. Then there exists non-sincere $y \in \Re_q^+$ such that $y < x$, and $i_1, i_2 \in \operatorname{supp} y$.*

Proof. Since $q \in \mathrm{WP}$, $2(e_i, x) \in \{-1, 0, 1\}$ for any $i \in I$, so the equality $2 = 2q(x) = 2(x, x) = \sum_{i=1}^{n} 2(e_i, x)x_i$ implies $2(e_j, x) = 1$ for some $j \in I$. Then $z = x - e_j \in \Re_q^+$. If z is sincere, it is non-basic since $2(e_j, z) = -1$. Therefore, there exists a minimal sincere non-basic $z \in \Re_q^+$ such that $z \leqslant x$. Let $J \subset I$ be the set of all $j \in I$ such that $2(e_j, z) = 1$. Then for any $j \in J$, the root $y = z - e_j < z$ is non-sincere by the minimality of z, and hence $z_j = 1$. Since $2 = \sum_{i=1}^{n} 2(e_i, x)x_i$, $|J| \geqslant 2$. If $|J| = 2$, then z is basic. Therefore, $|J| \geqslant 3$, and there is $j \in J \backslash \{i_1, i_2\}$. □

Representation category. For a bimodule problem $\mathcal{A} = (\mathsf{K}, \mathsf{V})$, *a representation* M of $\mathcal{A}$ is a pair $M = (M_{\mathsf{K}}, M_{\mathsf{V}})$ of $M_{\mathsf{K}} \in \operatorname{Ob} \operatorname{add} \mathsf{K}$ and $M_{\mathsf{V}} \in \operatorname{add} \mathsf{V}(M_{\mathsf{K}}, M_{\mathsf{K}})$. If M, N are representations of $\mathcal{A}$, then a *morphism* $f : M \to N$ is a morphism $f \in \operatorname{add} \mathsf{K}(M_{\mathsf{K}}, N_{\mathsf{K}})$ such that $N_{\mathsf{V}} \cdot f - f \cdot M_{\mathsf{V}} = 0$. The unit morphisms and composition of morphisms in the *representation category* $\operatorname{rep} \mathcal{A}$ and in $\operatorname{add} \mathsf{K}$ coincide. All indecomposable representations form the subcategory in $\operatorname{rep} \mathcal{A}$ which we denote by $\operatorname{ind} \mathcal{A}$.

With a locally finite dimensional bimodule problem $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ we associate the $\mathbb{Z}$-lattice $\dim_{\mathcal{A}} = \bigoplus_{\operatorname{Ob} \mathsf{K}} \mathbb{Z}$ of elements $x = (x_A)_{A \in \operatorname{Ob} \mathsf{K}}$ with finite *support* $\operatorname{supp} x = \{A \in \operatorname{Ob} \mathsf{K} \mid x_A \neq 0\}$. The lattice $\dim_{\mathcal{A}}$ has the *standard basis* $\{e_A, A \in \operatorname{Ob} \mathsf{K}\}$ such that $(e_A)_A = 1$, and $(e_A)_B = 0$ for $B \in \operatorname{Ob} \mathsf{K} \backslash \{A\}$. Besides, $\dim_{\mathcal{A}}$ is endowed with the partial product order: for a vector $x \in \dim_{\mathcal{A}}$, we write $x \geqslant 0$ if and only if $x_A \geqslant 0$ for all $A \in \operatorname{Ob} \mathsf{K}$. For a representation $M \in \operatorname{rep} \mathcal{A}$ such that $M_{\mathsf{K}} \simeq \bigoplus_{A \in \operatorname{Ob} \mathsf{K}} A^{x_A}$ where almost all $x_A = 0$, a *dimension vector* of M is defined by equality $\dim M = \dim_{\mathcal{A}} M = (x_A)_{A \in \operatorname{Ob} \mathsf{K}} \in \dim_{\mathcal{A}}$. By definition, a *support* $\operatorname{supp} M$ of the representation M is $\operatorname{supp} \dim_{\mathcal{A}} M$ and is always finite.

There exists the identical on morphisms forgetful functor $\mathcal{F}(= \mathcal{F}_{\mathcal{A}}) : \operatorname{rep} \mathcal{A} \longrightarrow \operatorname{add} \mathsf{K}$ such that $\mathcal{F}(M) = M_{\mathsf{K}}$. A morphism $f : M \longrightarrow N$ in $\operatorname{rep} \mathcal{A}$ is an isomorphism if and only if $\mathcal{F}(f) : M_{\mathsf{K}} \longrightarrow N_{\mathsf{K}}$ is an isomorphism in $\operatorname{add} \mathsf{K}$. We will denote an isomorphism by $\simeq$, and for a family of representations S from $\operatorname{rep} \mathcal{A}$, we denote by $S/_{\simeq}$ the set the isoclasses of S. The direct sum in $\operatorname{rep} \mathcal{A}$ is induced by the direct sum in $\operatorname{add} \mathsf{K}$. It turns $\operatorname{rep} \mathcal{A}$ into a fully additive category, and the Krull-Schmidt theorem holds in $\operatorname{rep} \mathcal{A}$ (see [5, 8]). The following result is clear.

Lemma 2. *Let* $\mathsf{p} = (\mathsf{p}_0, \mathsf{p}_1) : \mathcal{A} \longrightarrow \mathcal{A}_{\mathrm{red}} = (\mathsf{K}/\operatorname{Ann}_{\mathsf{K}} \mathsf{V}, \mathsf{V})$ *be a* natural *bimodule problem morphism, where* $\mathsf{p}_0 : \mathsf{K} \longrightarrow \mathsf{K}_{\mathrm{red}}$ *is the canonical projection and* $\mathsf{p}_1 = 1_{\mathsf{V}}$. *Then the functor* $\operatorname{rep} \mathsf{p} : \operatorname{rep} \mathcal{A} \longrightarrow \operatorname{rep} \mathcal{A}_{\mathrm{red}}$ *between the representation categories induced by* p *is an epimorphism on the morphisms, preserves isomorphisms and* $\operatorname{rep} \mathsf{p}(\operatorname{ind} \mathcal{A}) = \operatorname{ind} \mathcal{A}_{\mathrm{red}}$.

The category $\operatorname{rep} \mathcal{A}$ is fully additive, and it is called of *finite representation type* provided $\operatorname{rep} \mathcal{A}$ has finitely many isoclasses of indecomposable objects, and of *infinite representation type* in the opposite case.

A representation $M \in \operatorname{rep} \mathcal{A}$ is called *sincere* provided $(\dim M)_A \neq 0$ for any $A \in \operatorname{Ob} \mathsf{K}$. In this case $\operatorname{Ob} \mathsf{K}$ is obviously finite. A bimodule problem

$\mathcal{A}$ is called *sincere* if there exists a sincere indecomposable representation $M \in \operatorname{rep} \mathcal{A}$.

A representation $M \in \operatorname{ind} \mathcal{A}$ is called *schurian* provided it has only scalar endomorphisms. A bimodule problem $\mathcal{A}$ is called *schurian* if every $M \in \operatorname{ind} \mathcal{A}$ is schurian ([10, 14]).

Lemma 3 ([11, 14]). *Let $\mathcal{A}$ be a finite dimensional schurian bimodule problem. Then $\mathcal{A}$ is representation finite, its Tits form $q_{\mathcal{A}}$ is unit integral and WP, the map $\dim_{\mathcal{A}} : \operatorname{ind} \mathcal{A}/_{\simeq} \to \Re^{+}_{q_{\mathcal{A}}}$ is a bijection, where $\operatorname{ind} \mathcal{A}/_{\simeq}$ denote the set of all isoclasses of indecomposable representations.*

Some results on representations of partially ordered sets. Let $\mathcal{A} = (\mathsf{K},\mathsf{V})$ be a faithful admitted bimodule problem with $\operatorname{Ob} \mathsf{K}^{-} = \{O\}$. If $\dim_{\Bbbk} \mathsf{V}(O, A) = 1$ for every $A \in \operatorname{Ob} \mathsf{K}^{+}$, then we say that $\mathcal{A}$ describes *representations of a poset* $\mathrm{Q} = \mathrm{Q}(\mathcal{A})$ defined in the following way ([12], 4.1). The elements of Q are Σ_0^{+}, and for $A, B \in \mathrm{Q}$, $A > B$ if and only if $\mathsf{K}(A, B) \neq 0$ (in this case $\dim_{\Bbbk} \mathsf{K}(A, B) = 1$), and the composition of two non-zero morphisms is again non-zero. The category $\operatorname{rep} \mathcal{A}$ is isomorphic to the representation category of the poset Q.

Lemma 4 ([12]). *Let $\mathcal{A}$ describes the representation of the poset* Q.

1) *If $q_{\mathcal{A}}$ is WP, then $\mathcal{A}$ is schurian.*

2) *$\mathcal{A}$ is of finite type if and only if $q_{\mathcal{A}}$ is WP.*

3) *$\mathcal{A}$ is of finite type if and only if Q does not contain the subposets $\hat{1} \sqcup \hat{1} \sqcup \hat{1} \sqcup \hat{1}$, $\hat{2} \sqcup \hat{2} \sqcup \hat{2}$, $\hat{1} \sqcup \hat{3} \sqcup \hat{3}$, $\hat{1} \sqcup \hat{2} \sqcup \hat{5}$, $\hat{4} \sqcup \hat{2} \nwarrow \hat{2}$. The quadratic forms of these posets are critical (critical posets are indicated in [13]).*

4) *If Q contains a chain of the length 3, $x \in \dim_{\mathcal{A}}$ is sincere positive and $x_O \leqslant 2$, then every $M \in \operatorname{rep} \mathcal{A}$ with $\dim_{\mathcal{A}} M = x$ is decomposable.*

2. Schurity Problem

Sincere schurian bimodule problem. The following lemma states that each sincere schurian bimodule problem is faithful.

Lemma 5. *If bimodule problem $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ is non-faithful, then every sincere $M \in \operatorname{ind} \mathcal{A}$ is non-schurian.*

Proof. Let $\operatorname{Ann}_{\mathsf{K}} \mathsf{V}(A, B) \neq 0$, $A, B \in \operatorname{Ob} \mathsf{K}$. Then the subspace generated by $\operatorname{Ann}_{\mathsf{K}} \mathsf{V}(A, B)$ in $\operatorname{add} \mathsf{K}(M_{\mathsf{K}}, M_{\mathsf{K}})$ consists of nilpotent endomorphisms of M. Since M is sincere, M is non-schurian. □

Minimal non-schurian bimodule problem. Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be a sincere non-schurian bimodule problem such that $q_{\mathcal{A}} \in \mathrm{WP}$, and for every proper subset $S \subset \mathrm{Ob}\,\mathsf{K}$, the restriction $\mathcal{A}_S$ is schurian. Then $\mathcal{A}$ is called a *minimal non-schurian bimodule problem.*

Let $\mathcal{B}, \mathcal{C}$ be bimodule problems, defined by their basic bigraphs $(\Sigma_{\mathcal{B}})_0 = \{X\}$, $(\Sigma_{\mathcal{B}})_1 = \varnothing$, and $(\Sigma_{\mathcal{C}})_0 = \{X, Y\}$, $(\Sigma_{\mathcal{C}})_1 = \{a : X \to Y\}$. For a bimodule problem $\mathcal{A}$ having basis with at most 2 vertices, $\mathcal{A}$ is sincere schurian if and only if either $\mathcal{A} = \mathcal{B}$ or $\mathcal{A} = \mathcal{C}$.

The following lemma is similar to the result given in [4] for box problems (see comments on Theorem 1).

Lemma 6. *Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be a minimal non-schurian admitted bimodule problem with a basis Σ and $|\Sigma_0| \geqslant 3$. If $\mathcal{A}_{\mathrm{red}}$ is a sincere schurian, then there exist two uniquely defined vertices $A, B \in \Sigma_0^+$ such that:*

1) *for any sincere $M \in \mathrm{ind}\,\mathcal{A}$, the vector $\dim_{\mathcal{A}} M$ is a basic root of Tits quadratic form $q_{\mathcal{A}_{\mathrm{red}}}$ with the singular vertices A, B;*

2) *if $\mathrm{Ann}_{\mathsf{K}}\,\mathsf{V}(A_1, B_1) \neq 0$, then the sets $\{A_1, B_1\}$, $\{A, B\}$ coincide.*

Proof. Since $\mathcal{A}_{\mathrm{red}}$ is sincere, the set $\mathrm{Ob}\,\mathsf{K}$ is finite. By Lemma 3, the map $\dim_{\mathcal{A}} : \mathrm{ind}\,\mathcal{A}_{\mathrm{red}}/_{\simeq} \longrightarrow \Re^+_{q_{\mathcal{A}_{\mathrm{red}}}}$ is a bijection. Let us consider a pair $A, B \in \Sigma_0^+$ such that $\mathrm{Ann}_{\mathsf{K}}\,\mathsf{V}(A, B) \neq 0$. Since $|\Sigma_0| \neq 1$ and $\mathcal{A}$ is minimal non-schurian, $A \neq B$. If there exists sincere non-basic $y \in \Re^+_{q_{\mathcal{A}_{\mathrm{red}}}}$, then by Lemma 1 there exists non-sincere $z \in \Re^+_{q_{\mathcal{A}_{\mathrm{red}}}}$ such that $A, B \in \mathrm{supp}\,z$ and subproblem $\mathcal{A}_{\mathrm{supp}\,z}$ is not schurian by Lemma 5.

If x is a basic root of $q_{\mathcal{A}_{\mathrm{red}}}$ with singular vertices A_1, B_1, then $x - e_{A_1}$ and $x - e_{B_1}$ are the roots, non-sincere in A_1 and B_1 correspondingly. If $\{A, B\} \neq \{A_1, B_1\}$, then by Lemma 5 at least one of the subproblems $\mathcal{A}_{\Sigma_0 \setminus \{A\}}$ or $\mathcal{A}_{\Sigma_0 \setminus \{B\}}$ is non-shurian. □

A minimal non-schurian bimodule problem $\mathcal{A}$ satisfying the conditions of Lemma 6 is called *standard minimal non-schurian bimodule problem with singular vertices A, B.*

Small minimal non-schurian bimodule problems. The following lemma is proved by direct verification.

Lemma 7. *Let $\mathcal{A}$ be an admitted minimal non-schurian bimodule problem with a basis $\Sigma = \Sigma_{\mathcal{A}}$. Then:*

1) *$|\Sigma_0| \neq 2$;*

2) *if $|\Sigma| = 1, 3$, then $\mathcal{A}$ is non-working contour;*

3) *if* $|\Sigma_0| = 4$, *then* $\mathcal{A}_{\rm red} = \mathscr{D}_4$ *is given by its bigraph* Ω *where* $\Omega_0 = \{A_1, A_2, E_1, E_2\}$, $\Omega_1^0 = \{x_i : E_1 \to A_i, y_i : E_2 \to A_i, i = 1, 2\}$, $\Omega_1^1 = \{\varphi : A_1 \to A_2\}$ *and* $\mathrm{P}_\varphi = \{(x_1, x_2), (y_1, y_2)\}$. *The dimension of non-schurian representation with singular vertices* A_1 *and* A_2 *is* $e_{A_1} + e_{A_2} + e_{E_1} + e_{E_2}$. *In this case we say that the minimal non-schurian* $\mathcal{A}$ *is of the type* $\mathscr{D}_4$.

If $\mathcal{A}$ is a minimal non-schurian bimodule problem, Σ is its bigraph, and $\Sigma_0^- = \varnothing$, then $|\Sigma_0^+| = 1$, $\Sigma_1^0 = \varnothing$, $\Sigma_1^1 \neq \varnothing$, hence Σ is a union of dotted loops. Unless otherwise indicated, we exclude this case. Moreover, we will assume that Tits quadratic form $q \in \mathrm{WP}$, and so Σ does not contain parallel solid arrows.

Lemma 8. *Let* $\mathcal{A}$ *be an admitted finite dimensional minimal non-schurian bimodule problem with Tits form* $q_{\mathcal{A}_{\rm red}} \in \mathrm{WP}$. *Then* $\mathcal{A} \in \mathsf{C}$, *and therefore,* $\mathcal{A}_{\rm red}$ *possesses a quasi multiplicative basis, and the bigraph of* $\mathcal{A}_{\rm red}$ *does not contain a solid cycle* O_4.

Proof. Let the bigraph of $\mathcal{A}_{\rm red}$ contain one of the listed in [3, Lemma 2] bigraphs or a solid cycle O_4 as a subbigraph. Since $q_{\mathcal{A}_{\rm red}} \in \mathrm{WP}$, there exists two vertices $A, B \in \Sigma_0^+$ such that $\mathrm{Ann}_{\mathsf{K}} \mathsf{V}(A, B) \neq 0$. Hence $\mathcal{A}_{\rm red}$ contains a non-shurian proper subproblem (see Lemma 7). By [3, Theorem 1] $\mathcal{A}_{\rm red}$ possesses a quasi multiplicative basis. $\square$

Reduction of direct summand of bimodule. We will use a partial case of so called *reduction functor* presented in [5, 7]. We assume that:

1. $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ is a bimodule problem.

2. $\mathsf{W}, \bar{\mathsf{V}} \subset \mathsf{V}$ are subbimodules of V such that $\mathsf{V} = \bar{\mathsf{V}} \oplus \mathsf{W}$, in particular, for every $M = (M_{\mathsf{K}}, M_{\mathsf{V}}) \in \operatorname{rep} \mathcal{A}$, there exists a canonical decomposition $M_{\mathsf{V}} = M_{\bar{\mathsf{V}}} \oplus M_{\mathsf{W}}$, $M_{\bar{\mathsf{V}}} \in \operatorname{add} \bar{\mathsf{V}}$, $M_{\mathsf{W}} \in \operatorname{add} \mathsf{W}$. Let $i_{\mathsf{W}} : \operatorname{add} \mathsf{W} \to \operatorname{add} \mathsf{V}$ be the morphism induced by the canonical inclusion $\mathsf{W} \hookrightarrow \mathsf{V}$.

3. Let $\bar{\mathcal{A}} = (\mathsf{K}, \bar{\mathsf{V}})$ be the induced bimodule problem, let $L \subset \operatorname{rep} \bar{\mathcal{A}}$ be a set of representations of $\bar{\mathcal{A}}$, let $\mathsf{L} = \operatorname{rep} \mathcal{A}|_L$, let $i_{\mathsf{L}} : \mathsf{L} \hookrightarrow \operatorname{rep} \bar{\mathcal{A}}$ be the inclusion, let j_{L} be the composition $j_{\mathsf{L}} : \mathsf{L} \xrightarrow{i_{\mathsf{L}}} \operatorname{rep} \bar{\mathcal{A}} \xrightarrow{\mathcal{F}_{\bar{\mathcal{A}}}} \operatorname{add} \mathsf{K}$ of i_{L} with the identical on morphisms forgetful functor $\mathcal{F} (= \mathcal{F}_{\mathcal{A}}) : \operatorname{rep} \mathcal{A} \longrightarrow \operatorname{add} \mathsf{K}$ such that $\mathcal{F}(M) = M_{\mathsf{K}}$, and let $j_{\mathsf{W}} : \mathsf{W}_{\mathsf{L}} \longrightarrow \operatorname{add} \mathsf{V}$ be the corresponding L-bimodule morphism. Without loss of generality we assume that the representations of L are pairwise non-isomorphic.

We construct a bimodule problems $\mathcal{B} = (\mathsf{L}, \mathsf{W}_{\mathsf{L}})$ and define the functor $F_{\mathsf{L}} : \operatorname{rep} \mathcal{B} \longrightarrow \operatorname{rep} \mathcal{A}$ as follows:

- if $M = (M_{\mathsf{L}}, M_{\mathsf{W}_{\mathsf{L}}}) \in \operatorname{Ob} \operatorname{rep} \mathcal{B}$ for $M_{\mathsf{W}_{\mathsf{L}}} \in \operatorname{add} \mathsf{W}_{\mathsf{L}}(M_{\mathsf{L}}, M_{\mathsf{L}})$, $M_{\mathsf{L}} \in \operatorname{add} \mathsf{L}$, then $F_{\mathsf{L}}(M)$ is a couple $(j_{\mathsf{L}}(M_{\mathsf{L}}), (M_{\mathsf{L}})_{\bar{\mathsf{V}}} \oplus j_{\mathsf{W}}(M_{\mathsf{W}_{\mathsf{L}}}))$;

- if $f : M \longrightarrow N$, then $F_{\mathsf{L}}(f) = j_{\mathsf{L}}(f)$.

The functor F_{L} is called the *reduction functor* for $\mathcal{A}$ with respect to subbimodule $\bar{\mathsf{V}} \subset \mathsf{V}$.

Lemma 9. *F_{L} is an equivalence on its image $F_{\mathsf{L}}(\operatorname{rep}\mathcal{B})$.*

Proof. If $M = (M_{\mathsf{L}}, M_{\mathsf{W}_{\mathsf{L}}})$, $N = (N_{\mathsf{L}}, N_{\mathsf{W}_{\mathsf{L}}}) \in \operatorname{rep}\mathcal{B}$, then

$$\operatorname{rep}\mathcal{B}(M,N) = \{f \in \operatorname{add}\mathsf{L}(M_{\mathsf{L}}, N_{\mathsf{L}}) \mid M_{\mathsf{W}_{\mathsf{L}}} f - f N_{\mathsf{W}_{\mathsf{L}}} = 0\}. \qquad (1)$$

To include $\operatorname{rep}\mathcal{B}$ in $\operatorname{rep}\mathcal{A}$ as a subcategory, we identify $f \in \operatorname{rep}\mathcal{B}(M,N)$ and $\operatorname{add} i_{\mathsf{L}}(f) : M_{\mathsf{K}} \to N_{\mathsf{K}}$. Then f is a morphism in $\operatorname{add} i_{\mathsf{L}}(\operatorname{rep}\mathcal{B})$ if and only if $M_{\bar{\mathsf{V}}} f - f N_{\bar{\mathsf{V}}} = 0$ (for $\operatorname{add}\bar{\mathsf{V}}$ this means that $f \in \operatorname{rep}\bar{\mathcal{A}}(M,N)$), and (1) holds. But these conditions mean that $f : M_{\mathsf{K}} \to N_{\mathsf{K}}$ defines a morphism from $M = (M_{\mathsf{K}}, M_{\bar{\mathsf{V}}} \oplus M_{\mathsf{W}})$ to $N = (N_{\mathsf{K}}, N_{\bar{\mathsf{V}}} \oplus N_{\mathsf{W}})$. The composition in $\operatorname{rep}\mathcal{A}$ and in $\operatorname{add} i_{\mathsf{L}}(\operatorname{rep}\mathcal{B})$ coincide with the one in $\operatorname{add}\mathsf{K}$. $\square$

Remark that reduction functor F_{L} induces a homomorphism $\dim_{F_{\mathsf{L}}} : \dim_{\mathcal{B}} \to \dim_{\mathcal{A}}$ which keeps the order on dimensions.

Reduction for an admitted bimodule problem. Let $P \subset \operatorname{Ob}\mathsf{K}^-$, and $\mathsf{W} = V(P) \subset \mathsf{V}$ be a subbimodule such that $\mathsf{W}(X,Y) = \mathsf{V}(X,Y)$ if $X \notin P$ and $\mathsf{W}(X,Y) = 0$ otherwise. Since $\mathcal{A}$ is admitted, then $\mathsf{V} = \mathsf{W} \oplus \bar{\mathsf{V}}$ with $\bar{\mathsf{V}} = \mathsf{V}(\operatorname{Ob}\mathsf{K}^- \backslash P)$ be a K-subbimodule in V. Let $\bar{\mathcal{A}} = (\mathsf{K}, \bar{\mathsf{V}})$ be the induced bimodule problem. There are defined the morphisms of bimodule problems $p_P = (p_0, p_1) : \mathcal{A} \longrightarrow \bar{\mathcal{A}} = (\mathsf{K}, \bar{\mathsf{V}})$ where $p_0 = 1_{\mathsf{K}}$ and $p_1 : \mathsf{V} \longrightarrow \bar{\mathsf{V}}$ is a projection, and $i_P = (i_0, i_1) : \bar{\mathcal{A}} \longrightarrow \mathcal{A}$ where $i_0 = 1_{\mathsf{K}}$ and $i_1 : \bar{\mathsf{V}} \longrightarrow \mathsf{V}$ is an inclusion. For $M \in \operatorname{rep}\bar{\mathcal{A}}$, let $M|_P = \operatorname{rep} p_P(M)$.

For $M \in \operatorname{rep}\mathcal{A}$, let $\operatorname{Ob}\mathsf{L} = \{M_1, \ldots, M_t\}$ where $\mathsf{L} \subset \operatorname{rep}\bar{\mathcal{A}}$ is the full subcategory defined above. Let $M|_P = M_1^{k_1} \oplus \ldots \oplus M_t^{k_t}$ be the decomposition in indecomposables in $\operatorname{rep}\bar{\mathcal{A}}$. Then the reduction morphism $\operatorname{rep}\mathcal{B} \xrightarrow{F_{\mathsf{L}}} \operatorname{rep}\mathcal{A} \xrightarrow{\mathcal{F}_{\mathcal{A}}} \operatorname{add}\mathsf{K}$ is called *M–reduction* of $\mathcal{A}$ (with respect to P) and write $R_{P,M}$ instead of F_{L} and $\mathcal{A}_{P,M} = (\mathsf{K}_{P,M}, \mathsf{V}_{P,M})$ instead of $\mathcal{B}$.

Lemma 10. 1) *The bimodule problem $\mathcal{A}_{P,M}$ is admitted.*

2) *The set $\operatorname{Ob}\mathsf{K}_{P,M}$ is a disjoint union $(\operatorname{Ob}\mathsf{K}\backslash\bar{P}) \cup \operatorname{Ob}\mathsf{L}$ where $\bar{P} \supset P$ is the set of all vertices in Σ_0 incident to the arrows starting in P.*

3) *There exists $M_P \in \operatorname{rep}\mathcal{A}_{P,M}$ such that $M \simeq \operatorname{rep} R_{P,M}(M_P)$.*

4) *Restrictions of $\dim_{\mathcal{A}_{P,M}} M_P$ and $\dim_{\mathcal{A}} M$ on $\operatorname{Ob}\mathsf{K}\backslash\bar{P}$ coincide.*

5) *If M is sincere representation, then M_P is sincere.*

6) *Let $a \in \Sigma_1^0(E, A)$, $A \in \bar{P}$, $E \notin P$. Then the bigraph $\Sigma_{\mathcal{A}_{P,M}}$ of $\mathcal{A}_{P,M}$ contains $p = \dim_{\Bbbk} M_i(A)$ solid arrows $a_k : E \to M_i$ such that $R_{P,M}(a_1) + \cdots + R_{P,M}(a_p) = M(a)$.*

7) *Let* $\mathrm{E}_\varphi = \{s(a), (a,b) \in \mathrm{P}_\varphi\}$. *If* $\mathrm{E}_\varphi \cap P = \varnothing$ *for* $\varphi \in \Sigma_1^1(A,B)$, $A, B \in \bar{P}$, *then there are* pq *(*$p = \dim_{\Bbbk} M_i(A)$, $q = \dim_{\Bbbk} M_j(B)$*) dotted arrows* $\varphi_{kl} : M_i \to M_j$ *in* $\Sigma_{\mathcal{A}_{P,M}}$ *such that* $R_{P,M}(\varphi_{kl}) : M_{i\mathsf{K}}(A) \to M_{j\mathsf{K}}(B)$ *having only one non-zero component* $[R_{P,M}(\varphi_{kl})](\varphi)$, $1 \leqslant k \leqslant p$, $1 \leqslant l \leqslant q$.

Proof. Statements 3)-6) are the corollaries of the reduction construction. The property of a bimodule problem to be admitted depends only on its bigraph which implies 1). To prove 7), remark that $\mathsf{K}_{P,M}(M_i, M_j) = \operatorname{rep} \bar{\mathcal{A}}(M_i, M_j)$. The conditions on φ imply $\varphi \in \operatorname{Ann}_{\mathsf{K}} \bar{\mathsf{V}}$. Without loss of generality suppose that M_i and M_j are presented as matrices with coefficients in V. Then the matrix $[\varphi_{ij}]$ with only non-zero entry $[\varphi_{kl}]_{(A,k),(B,l)} = \varphi$ defines a morphism $\varphi_{kl} : M_i \longrightarrow M_j$. □

We call the arrows a_k and φ_{kl} from Lemma 10 the *components* of a and φ by the reduction $R_{P,M}$ correspondingly.

Minimal pairs and reductions. Further we will carry out the proof for pairs $(\mathcal{A}, M)$ where $M \in \operatorname{ind} \mathcal{A}$ is a sincere. The pair $(\mathcal{A}, M)$ is called *minimal* if every $N \in \operatorname{ind} \mathcal{A}$ with $\dim N < \dim M$ is schurian.

Lemma 11. *Let* $\mathcal{A} = (\mathsf{K}, \mathsf{V}) \in \mathsf{C}$ *be a faithful connected finite dimensional bimodule problem with basis* Σ *and Tits form* $q \in \mathrm{WP}$. *If* $M \in \operatorname{ind} \mathcal{A}$, $x = \dim_{\mathcal{A}} M$, $(\mathcal{A}, M)$ *is a minimal pair and* $P \subset \operatorname{Ob} \mathsf{K}^-$, *then:*

1) $(\mathcal{A}_{P,M}, M_P)$ *is a minimal pair.*
2) *If* $A, B \in \bar{P}$ *and* $\varphi \in \Sigma_1^1(A,B)$ *is such that* $E_\varphi \cap P = \varnothing$, *then:*
   (i) *There does not exist* $M_i \in \operatorname{ind} \mathcal{A}$ *such that* $M_i(A) \neq 0$, $M_i(B) \neq 0$. *If* $M_i(A) \neq 0$ *(*$M_i(B) \neq 0$*), then* $M_i(A) \simeq A$ *(*$M_i(B) \simeq B$*).*
   (ii) *If* $M_i(A) \neq 0$, $M_j(B) \neq 0$, *then* $\mathsf{V}_{P,M}(E, M_i) = \Bbbk a$, $\mathsf{V}_{P,M}(E, M_j) = \Bbbk b$, *and* $a \| b\varphi$ *in* $\mathsf{V}_{P,M}$, *where* $\varphi = \varphi_{ij}$.
   (iii) *Let subcategory* $\mathsf{I}_\varphi \subset \operatorname{add} \mathsf{K}$ *consists of all representations* F *such that* $F(\varphi) = 0$. *Then* $R_{P,M0}(\operatorname{Ann}_{\mathsf{K}_{P,M}}(\mathsf{V}_{P,M})) \subset \mathsf{I}_\varphi$.

Proof. By [3, Theorem 1], we can assume that the basis Σ is quasi multiplicative. Statement 1) is obvious. Suppose there is $i = 1, \dots, t$ such that M_i is sincere in both A and B. The condition on φ gives that $\varphi \in \operatorname{Ann}_{\mathsf{K}} \bar{\mathsf{V}}$, and M_i is non-schurian by Lemma 5. Hence $N = \operatorname{rep} i_P(M_i) \in \operatorname{ind} \mathcal{A}$ is non-schurian and $\dim_{\mathcal{A}} N < x$. If for some i, $M_i(A) \simeq A^p$, $p \geqslant 2$, then by Lemma 10, 6), $|\Sigma^0_{\mathcal{A}_{P,M}1}(E, M_i)| = p \geqslant 2$. Since $q_{\mathcal{A}_{P,M}} \in \mathrm{WP}$, $|\Sigma^0_{\mathcal{A}_{P,M}1}(M_i, M_i)| \geqslant 2$ and M_i is non-schurian, that proves (i). By definition of reduction $\mathsf{V}_{P,M}(E, M_i) = \operatorname{add} \mathsf{V}(E_{\mathsf{K}}, M_{i\mathsf{K}}) = \Bbbk a$, and $\mathsf{V}_{P,M}(E, M_j) = \Bbbk b$. Statement $a \| b\varphi$ follows from the definition of $\mathsf{V}_{P,M}$.

To prove (iii) suppose there exists $F \in \mathrm{Ann}_{\mathsf{K}_{P,M}}(\mathsf{V}_{P,M})$ such that $f = R_{P,M0}(F) \notin \mathsf{I}_{\varphi}$, or equivalently, $f(\varphi) \neq 0$. Obviously, $F \in \mathsf{K}_{P,M}(M_i,M_j)$ for M_i, M_j defined in (ii). Since for every $E \in \mathrm{E}_{\varphi}$, F annihilated $\mathsf{V}_{P,M}(E, M_i)$, then $R_{P,M1}(\mathsf{V}_{P,M}(E, M_i))R_{P,M0}(F) = 0$. Using the reduction of direct summand of bimodule we obtain that the equality is equivalent to $\operatorname{add}\mathsf{V}(E_{\mathsf{K}}, M_{i\mathsf{K}})f = 0$. For $v \in \operatorname{add}\mathsf{V}(E_{\mathsf{K}}, M_{i\mathsf{K}})$ with the unique non-zero component $v(a)$, we have $vf(b) \neq 0$, that proves (iii). □

Special bimodule problems. We use in this section the following class of admitted bimodule problems. We call a faithful bimodule problem $\mathcal{S} = (\mathsf{L}, \mathsf{W})$ with a quasi multiplicative basis Φ *special* provided $\Phi_0 = \{E_1, E_2; A_1, \ldots, A_m\}$, $\Phi_1^0 = \{x_i : E_1 \longrightarrow A_i, y_i : E_2 \to A_i, i = 1, \ldots, m\}$ and the following equivalent conditions hold:

- if $1 \neq i < j \leqslant m$, then both pairs x_i, x_j and y_i, y_j are comparable;
- if $1 \leqslant i < j \leqslant m$, then $\mathcal{S}|_{E_k,A_i,A_j}$, $k = 1, 2$ is not a non-working contour, and $\mathcal{S}|_{E_1,E_2,A_i,A_j}$ is not the cycle O_4.

Note that the bimodule problem $\mathscr{D}_4$ belongs to the defined class.

Lemma 12. *Let $\mathcal{S}$ be a special bimodule problem with a basis Φ. Then:*

1) *$\mathcal{S}$ is schurian;*

2) *the set $\mathrm{D}_{\mathcal{S}} \subset \dim_{\mathcal{S}}$ of the dimensions of representations from* $\operatorname{ind}\mathcal{S}$ *is the subset of the set $\mathscr{D}_{\Phi} = \{e_{E_1}, e_{E_2}; e_{A_i}, e_{E_1} + e_{A_i}, e_{E_2} + e_{A_i}, e_{E_1} + e_{E_2} + e_{A_i}, i = 1, \ldots, m; e_{E_1} + e_{E_2} + e_{A_i} + e_{A_j}, 1 \leqslant i \neq j \leqslant m\}$;*

3) *there exists $M \in \operatorname{ind}\mathcal{A}$, $\dim_{\mathcal{S}} M = e_{E_1} + e_{E_2} + e_{A_i} + e_{A_j}$ if and only if A_i and A_j are connected in Φ by an unique joint arrow.*

4) *For a sincere $M \in \operatorname{rep}\mathcal{S}$ and $F \in \operatorname{rep}\mathcal{S}(M, M)$ such that $F \neq 0$, $F(1_{E_1}) = 0$, $F(1_{E_2}) = 0$, and once $F(\varphi) \neq 0$ for $\varphi \in \Phi_1^1$, then φ is joint. Then M contains a trivial direct summand U_{A_i} for some $i = 1, \ldots, m$ which corresponds to a simple root e_{A_i}.*

Proof. The statements 1), 2), 3) can be verified immediately. Let us prove 4). Let $M = \overset{t}{\underset{k=1}{\oplus}} M_k^{x_k}$ be a decomposition in indecomposables in $\operatorname{rep}\mathcal{S}$. For every non-trivial M_i, $i = 1, \ldots, t$, any $N \in \operatorname{add}\mathsf{L}$ and $G \in \operatorname{add}\mathsf{L}(M_{i\mathsf{L}}, N)$ such that $G(\varphi) \neq 0$ for joint $\varphi \in \Sigma_1^1$, there holds $M_{i\mathsf{L}}G \neq 0$. For $F \in \operatorname{rep}\mathcal{S}(M, M)$, $M_{\mathsf{W}}F - FM_{\mathsf{W}} = 0$. The condition $F(1_{E_i}) = 0$ for $i = 1, 2$ implies that $M_{\mathsf{W}}F = 0$. □

The main result. Now we are ready to prove that every minimal non-schurian finite dimensional admitted bimodule problem is standard.

Theorem 1. *Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be a minimal non-schurian finite dimensional connected admitted bimodule problem with basic bigraph Σ having at least 3 vertices and weakly positive Tits quadratic form $q_{\mathcal{A}}$. Then $q_{\mathcal{A}_{\rm red}} \in \mathrm{WP}$ and $\mathcal{A}$ is standard minimal non-schurian bimodule problem.*

In fact, Theorem 1 is a special case of the main Theorem of [4] on DGC problems formulated in the other terms. Note that Theorem from [4] is also given in [7] using the language of box problems which are equivalent to DGC ones. For our class of bimodule problems, we give an alternative proof using, in addition to the basic ideals of [4], the specifics of the bimodule language.

Proof. According to Lemma 7, bigraph Σ does not contain dotted loops and, since $q = q_{\mathcal{A}}$ is weakly positive, parallel solid arrows. Let

- $\mathcal{B} = \mathcal{A}_{\rm red} = (\mathsf{K}_{\rm red}, \mathsf{V})$, $\mathsf{K}_{\rm red} = \mathsf{K}/\mathrm{Ann}_{\mathsf{K}}\,\mathsf{V}$, be the faithful part of $\mathcal{A}$;
- Ω be the bigraph of $\mathcal{B}$, $\Omega_0 = \Sigma_0$, $\Omega_1^0 = \Sigma_1^0$, $\Omega_1^1 \subset \Sigma_1^1$;
- $p(x) = q_{\mathcal{B}}(x)$, $p(x) = q(x) - \sum_{(A,B)\in\Sigma_0\times\Sigma_0} |\mathrm{Ann}_{\mathsf{K}}(\mathsf{V})(A,B)|_{\Bbbk} x_A x_B$.

By Lemma 2, $\mathrm{Ob\,rep}\,\mathcal{A} = \mathrm{Ob\,rep}\,\mathcal{B}$. By assumption, for any proper subset $S \subset \Sigma_0$, the problem $\mathcal{A}_S$ is schurian.

The proof of the Theorem is carried out in several steps by induction on the lexicographically ordered pairs $(|\Sigma_0^-|, |\Sigma_0^+|)$.

Step 1. $|\Sigma_0^-| = 1$.

Let $\Sigma_0^- = \{O\}$. Then $\mathcal{B}$ is a bimodule problem corresponding to a poset $\mathrm{Q} = \mathrm{Q}(\mathcal{B})$. To show that the Tits form p of $\mathcal{B}$ is WP, we formulate the following preciser statement under Theorem conditions.

Lemma 13. *If $p \notin \mathrm{WP}$, $M \in \mathrm{ind}\,\mathcal{B}$ is sincere, then there exists $y \in \Re_p^+$ such that $0 < y < x = \dim M$, and:*

1) $p|_{\mathrm{supp}\,y} \in \mathrm{WP}$ *and there exists $N \in \mathrm{ind}\,\mathcal{B}$ of dimension y;*

2) $\mathrm{Ann}_{\mathsf{K}}\,\mathsf{V}|_{\mathrm{supp}\,y} \neq 0$*, in particular, the indecomposable N is non-schurian in the category* $\mathrm{rep}\,\mathcal{A}$.

Proof. Since $q \in \mathrm{WP}$, $p \notin \mathrm{WP}$, there are $A, B \in \Sigma_0^+$ with $\mathrm{Ann}_{\mathsf{K}}\mathsf{V}(A,B) \neq 0$.

If $x_O = 1$, then $\mathrm{Q} = \sqcup_{i=1}^k \hat{1}$ for $k \geqslant 4$ since $\mathcal{B}$ is sincere and $p \notin \mathrm{WP}$. Then we set $y = e_O + e_A + e_B$.

Thereafter we assume $x_O \geqslant 2$. By Lemma 4, Q contains a proper subset $\mathrm{C} \subset \mathrm{Q}$ such that restriction $p|_{\mathrm{C}}$ is critical. To construct the vector y, we need to consider the following three cases.

(1) If $\mathrm{C} = \hat{1} \sqcup \hat{1} \sqcup \hat{1} \sqcup \hat{1}$ then we set $y = e_O + e_A + e_B$.

(2) Let $\mathrm{C} = \hat{2} \sqcup \hat{2} \sqcup \hat{2}$, $\mathrm{C} = \hat{1} \sqcup \hat{3} \sqcup \hat{3}$, or $\mathrm{C} = \hat{1} \sqcup \hat{2} \sqcup \hat{5}$. Then every pair of vertices of C, in particular (A, B), belongs to a subposet $\mathrm{C}' \subset \mathrm{C}$, $\mathrm{C}' = (A_1, A_2, A_3 < A_4\} \simeq \hat{1} \sqcup \hat{1} \sqcup \hat{2}$. We set $y = 2e_O + \sum_{i=1}^4 e_{A_i}$.

(3) Assume that $\mathrm{C} = \hat{4} \sqcup \hat{2} \nwarrow \hat{2}$. By Lemma 4, 4), $x_O \geqslant 3$. Every pair of vertices (in particular A, B) belongs to $\mathrm{C}' \subset \mathrm{C}$ of the form $\{A_1 < A_2; B_2 > B_1 < C_2 > C_1\} \simeq \hat{2} \sqcup \hat{2} \nwarrow \hat{2}$. Then $y = 3e_O + e_{A_1} + e_{A_2} + e_{B_1} + e_{B_2} + e_{C_1} + e_{C_2}$.

For the constructed y, the restriction $p|_{\operatorname{supp} y}$ is WP since it is a proper subform of the critical form. Then, by Lemma 3, there exists $N \in \operatorname{ind} \mathcal{B}$ of dimension y which is non-schurian in $\operatorname{rep} \mathcal{A}$ by Lemma 5. □

In the case $|\Sigma_0^-| = 1$, the assumption $p \notin$ WP contradicts to the minimal non-schurity of $\mathcal{A}$, hence $p \in$ WP. Then $\mathcal{B}$ is schurian by Lemma 4, and we can apply Lemma 6 to $\mathcal{A}$. So the induction base is proved.

Step 2. General case $|\Sigma_0^-| \geqslant 2$.

To prove the Theorem, it is enough to verify that $\mathcal{B}$ is a sincere schurian bimodule problem, and apply Lemma 6.

Substep 2.1. $p \in$ WP.

Proof. Assume $p \notin$ WP, then there exists $S \subset \Omega_0$ such that $p|_S$ is a critical form and μ is corresponding critical vector. Since $q \in$ WP, there exist $A, B \in S$ such that $\operatorname{Ann}_{\mathsf{K}} \mathsf{V}(A, B) \neq 0$. Then there is $y \in \Re_p^+$ such that $A, B \in \operatorname{supp} y$, but $\operatorname{supp} y \neq S$. By assumption, $\mathcal{A}_{\operatorname{supp} y}$ and $\mathcal{B}_{\operatorname{supp} y}$ are schurian and, by Lemma 3, there exists $Y \in \operatorname{ind} \mathcal{B}$ such that $\dim_{\mathcal{A}} Y = y$. By Lemma 2, $Y \in \operatorname{ind} \mathcal{A}$ is non-schurian in $\operatorname{rep} \mathcal{A}$ which implies contradiction by Lemma 5. □

By Lemma 8, $\mathcal{B} \in \mathsf{C}$ and possesses a quasi multiplicative basis.

Remark 1. Let $(\mathcal{A}, M)$ be a minimal pair with a non-schurian $M \in \operatorname{ind} \mathcal{A}$, let $E \in \Sigma_0^-$, $P = \Sigma_0^- \backslash \{E\}$, $R_{P,M} : \mathcal{A}_{P,M} \longrightarrow \mathcal{A}$ be the reduction constructed in Lemma 11, and let $\mathcal{A}_{P,M} = (\mathsf{K}_{P,M}, \mathsf{V}_{P,M})$ for $M_P \in \operatorname{rep} \mathcal{A}_{P,M}$ with $\operatorname{rep} R_{P,M}(M_P) \simeq M$. Then $\mathcal{A}_{P,M}$ is the standard non-schurian, and $\mathcal{S} = (\mathcal{A}_{P,M})_{\mathrm{red}}$ is a poset bimodule problem.

Proof. By Lemma 11, $(\mathcal{A}_{P,M}, M_P)$ is a minimal pair. The bigraph of $\mathcal{A}_{P,M}$ has a unique minus-vertex, so the fact follows from the step 1. □

To show that $\mathcal{B}$ is schurian, we prove that every nilpotent endomorphism of any $M \in \operatorname{ind} \mathcal{B}$ is equal to zero. Then, by Fitting's lemma ([15]), every endomorphism of any $M \in \operatorname{ind} \mathcal{B}$ is invertible, and thus scalar.

Let $F \in \operatorname{rep} \mathcal{B}(M, M)$ be a nilpotent endomorphism of M, i. e. $F^k = 0$ for some integer $k \geqslant 1$. Obviously, to show that $F = 0$ we need to prove that $F(\varphi) = 0$ for any $\varphi \in \Sigma_1^1$, and $F(1_A) = 0$ for each $A \in \Sigma_0$.

Substep 2.2. $F(1_E) = 0$ for every $E \in \Sigma_0^-$.

Proof. Since $|\Sigma_0^-| \geqslant 2$, there exists $E_1 \in \Sigma_0^- \backslash \{E\}$. Then, according to remark 1, we apply M-reduction $R_{M,P} : \mathcal{A}_{M,P} \longrightarrow \mathcal{A}$ for $P = \{E_1\}$. Then $M_P(E) = M(E)$ for a representation $M_P \in \operatorname{rep} \mathcal{A}_{M,P}$ such that $\operatorname{rep} R_{M,P}(M_P) = M$, and if $F_P : M_P \longrightarrow M_P$ is such that $\operatorname{rep} R_{M,P}(F_P) = F$, then $F_P(1_E) = F(1_E)$. Applying the induction assumption and Lemma 11 to the minimal non-schurian problem $\mathcal{A}_{M,P}$ and to the sincere representation M_P, we obtain $F_P(1_E) = 0$. □

Substep 2.3. If $\varphi \in \Sigma_1^1(A, B)$ is a single arrow, then $F(\varphi) = 0$.

Proof. We apply $R_{P,M}$ for $P = \operatorname{Ob} \mathsf{K}^- \backslash \mathrm{E}_\varphi$ where $\mathrm{E}_\varphi = \{E\}$. By step 1, $\mathcal{A}_{M,P}$ is standard minimal non-schurian bimodule problem, and by Lemma 11, $F_P \in \operatorname{add} \operatorname{Ann}_{\mathsf{K}_{P,M}}(\mathsf{V}_{P,M})$ and $F(\varphi) = 0$. □

Substep 2.4. If $\varphi : A \to B$ is a joint arrow, then $F(\varphi) = 0$.

Proof. Let $\mathrm{E}_\varphi = \{E_1, E_2\}$, $P = \operatorname{Ob} \mathsf{K}^- \backslash \mathrm{E}_\varphi$. We apply $R_{P,M}$. If $F_P \in \operatorname{add} \operatorname{Ann}_{\mathsf{K}_{P,M}}(\mathsf{V}_{P,M})$, then, as in the substep 2.3, all is proved. In opposite case, there exist non-zero nilpotent $F_P \in \operatorname{ind} \mathcal{S}(M_P, M_P)$ for $\mathcal{S} = \mathcal{A}_{P,M} = (\mathsf{K}_{\mathrm{red}}, \mathsf{W})$. We choose the basis $\Psi = \Sigma_{\mathcal{S}}$, and extend it to the basis $\Sigma_{\mathcal{A}_{P,M}}$.

By substep 2.3, $F_P(\psi) = 0$ for any single $\psi \in \Psi_1^1$, and so $F(\psi) = 0$.

For any joint $\psi \in \Psi_1^1$, $F_P(\psi) = 0$. Indeed, let $A_1, \ldots, A_m \in \Psi_0$ be all vertices incident to the joint arrows from Ψ_1^1. Consider a special bimodule problem $\mathcal{S}_T$ with $T = \{E_1, E_2, A_1 \ldots, A_m\}$. Then $F \in \operatorname{add} \mathsf{K}_{\mathrm{red}\mathcal{S}_T}$, and F defines an endomorphism of M_P. By Lemma 12, 4), we obtain that $M_P \in \operatorname{rep} \mathcal{S}_T$ has a trivial direct summand which is the direct summand of $M \in \operatorname{rep} \mathcal{S}$ obviously. So $F_P \in \operatorname{add} \operatorname{Ann}_{\mathsf{K}_{P,M}}(\mathsf{V}_{P,M})$. □

Substep 2.5. $F = 0$.

Proof. It is enough to prove that $F(1_A) = 0$ for all $A \in \Sigma_0^+$. Suppose it fails for some $A \in \Sigma_0^+$. We denote by $a_j : E_j \longrightarrow A, j = 1, \ldots, i$, all the solid arrows incident to A. We consider the $(\overset{i}{\underset{j=1}{\oplus}} M(E_j), M(A))$-cut of M and the matrix $[M]^A = \big([M](a_1) \ldots [M](a_i)\big)$. The rank of this matrix equals $\dim M_A$, otherwise M contains a trivial direct summand U_A. Since $F(\varphi) = 0$ for all $\varphi \in \Sigma_1^0$, by the substeps 2.3 and 2.4, the equation

$[M][F]-[F][M]=0$ implies $[F](\bigoplus_{j=1}^{i} 1_{E_j})[M]^A-[M]^A[F](1_A)=0$. Since $F(1_{E_j})=0$ for any j, by the substep 2.2, $[M]^A[F](1_A)=0$. □

Corollary 1. *Let $\mathcal{A}$ be a finite dimensional connected admitted bimodule problem with at least 3 objects and Tits form $q_{\mathcal{A}} \in \mathrm{WP}$. If $q_{\mathcal{A}_{\mathrm{red}}}$ is not WP, then $\mathcal{A}$ can not be a minimal non-schurian bimodule problem.*

Conclusion

The paper is devoted to the study of schurity property for a class of matrix problems called one-sided bimodule problems, and is the second step after [3] in research of the representation finiteness problem for a mentioned class. By means of the constructed quasi multiplicative basis, we prove that the minimal non-schurian admitted bimodule problem having more than two vertices is standard minimal non-schurian.

References

[1] Babych, V. M. and Golovashchuk, N. S. An Application of Covering Techniques. *Scientific Bulletin of Uzhgorod Univ. Series Mathematics and Informatics* **8** (2003) 4–14.

[2] Babych, V. M., Golovashchuk, N. S. and Ovsienko, S. A. Quasi multiplicative bases for bimodule problems from some class. *Scientific Bulletin of Uzhgorod Univ. Series Mathematics and Informatics* **22** (2011) 12–17.

[3] Babych, V. M., Golovashchuk, N. S. and Ovsienko, S. A. Generalized multiplicative bases for one-sided bimodule problems. *Algebra and Discrete Mathematics* **12**, Num. 2 (2011) 1–24.

[4] Bondarenko, V. M., Golovashuk, N. S., Ovsienko, S. A. and Roiter, A. V. Schurian matrix problems and quadratic forms. *Inst. of Math., Academy of Sci. USSR*, Preprint 78.25 (1978) 19–38. [in Russian]

[5] Crawley-Boewey, W. W. Matrix problems and Drozd's theorem. *Banach Center publ.* **26**, part 1 (1990) 199–222.

[6] Drozd Yu. A. Coxeter transformations and representations of partially ordered sets. *Funkts. Anal. Prilozhen.* **8** (1974) 34–42. [in Russian]

[7] Drozd, Yu. A. Reduction algorithm and representations of boxes and algebras. *Comptes Rendue Math. Acad. Sci. Canada* **23** (2001) 97–125.

[8] Drozd, Yu. A. Tame and wild matrix problems. *In: Representations and quadratic forms*, Kiev (1979) 39–74. [English translation in AMS Translations v. 128]

[9] Drozd, Yu. A., Ovsienko, S. A. and Furchin, B. Yu. Categorical constructions in representations theory. *In: Algebraic Structures and Their Applications*, Kiev State Univ., Kiev (1988) 17–43.

[10] Gabriel, P. Auslander-Reiten sequences and representation-finite algebras. *LNM, Springer* **831** (1980) 1–71.

[11] Gabriel, P. Indecomposable representations II. *Symposia Math. Ist. Naz. Alta Mat.* **11** (1973) 81–104.

[12] Gabriel, P. and Roiter, A. *Representations of finite-dimensional algebras.* Algebra VIII, Encyclopaedia of Math. Sci. 73, Springer, New York, 1992.

[13] Kleiner, M. M. Partially ordered sets of finite type. *Studies in the theory of representations, Zap. Nauchn. Sem. LOMI* **28**, Nauka, Leningrad. Otdel., Leningrad (1972) 32–41. [in Russian]

[14] Kleiner, M. M. and Roiter, A. V. Representations of differential graded categories. *LNM, Springer* **488** (1975) 316–340.

[15] Krause, H. Krull-Schmidt categories and projective covers. *Expo. Math.* **33**, 4 (2015) 535–549.

[16] Ovsienko, S. A. Bimodule and matrix problems. *Progress in Mathematics* **173** (1999) 323–357.

[17] Peña, J. A. de la. On the dimension of the module-varieties of tame and wild algebras. *Commun. Algebra* **19** (1991) 1795–1807.

[18] Ringel, C. M. *Tame Algebras and Integral Quadratic Forms.* LNM, 1099, Springer, 1984.

Contact information

V. Babych,
N. Golovashchuk

Department of Mechanics and Mathematics,
Taras Shevchenko National University of Kyiv, 64,
Volodymyrs'ka St., Kyiv, Ukraine
E-Mail(s): vyacheslav.babych@univ.kiev.ua,
golova@univ.kiev.ua
Web-page(s): sites.google.com/view/cgtds/

Received by the editors: 06.10.2019
and in final form 19.11.2019.