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About the spectra of a real nonnegative matrix and its signings

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Abstract. For a complex matrix M, we denote by $\mathrm{Sp}(M)$ the spectrum of M and by $|M|$ its absolute value, that is the matrix obtained from M by replacing each entry of M by its absolute value. Let A be a nonnegative real matrix, we call a *signing* of A every real matrix B such that $|B| = A$. In this paper, we characterize the set of all signings of A such that $\mathrm{Sp}(B) = \alpha\,\mathrm{Sp}(A)$ where α is a complex unit number. Our motivation comes from some recent results about the relationship between the spectrum of a graph and the skew spectra of its orientations.

1. Introduction

Throughout this paper, all matrices are complex, unless otherwise noted. The identity matrix of order n is denoted by I_n and the transpose of a matrix A by A^T. Let Σ be a subgroup of $\mathbb{C}^*$, the group of nonzero complex numbers under multiplication. Two square matrices A and B are Σ-*diagonally similar* if $B = \Lambda^{-1} A \Lambda$ for some diagonal matrix Λ with diagonal entries in Σ. A square matrix A is *reducible* if there exists a permutation matrix P, so that A can be reduced to the form $PAP^T = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ where X and Z are square matrices. A square matrix which is not reducible is

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said to be *irreducible*. A real matrix A is nonnegative, (we write $A \geqslant 0$), if all its entries are nonnegative.

Let A be an $n \times n$ real or complex matrix. The multiset $\{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues of A is called the *spectrum* of A and is denoted by $\mathrm{Sp}(A)$. We usually assume that $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$. The *spectral radius* of A is $\rho(A) := |\lambda_1|$.

The relationship between the spectrum of a graph and the skew spectra of its orientations is studied in many papers (see for example [1,2,4–6,8,10]). Our work is closely related to the result of Shader and So [8]. To state this result, we need to introduce some definitions and notations.

Let G be a finite simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G is the symmetric matrix $A(G) = (a_{ij})_{1 \leqslant i,j \leqslant n}$ where $a_{ij} = a_{ji} = 1$ if $\{v_i, v_j\}$ is an edge of G and $a_{ij} = a_{ji} = 0$ otherwise. Since the matrix $A(G)$ is symmetric, its eigenvalues are real. The *adjacency spectrum* $\mathrm{Sp}\,(G)$ of G is defined as the spectrum of $A(G)$. Let G^σ be an orientation of G, which assigns to each edge a direction so that the resultant graph G^σ becomes an oriented graph. The *skew-adjacency* matrix of G^σ is the real skew-symmetric matrix $S(G^\sigma) = (a^\sigma_{ij})_{1 \leqslant i,j \leqslant n}$ where $a^\sigma_{ij} = -a^\sigma_{ji} = 1$ if (v_i, v_j) is an arc of G^σ and $a^\sigma_{ij} = 0$ otherwise. The *skew-spectrum* $\mathrm{Sp}\,(G^\sigma)$ of G^σ is defined as the spectrum of $S(G^\sigma)$. Note that $\mathrm{Sp}(G^\sigma)$ consists of only purely imaginary eigenvalues because $S(G^\sigma)$ is a real skew-symmetric matrix.

Let G be a bipartite graph with bipartition $[I, J]$, the orientation G^ε that assigns to each edge of G a direction from I to J is called the *canonical orientation*. Shader and So [8] showed that $\mathrm{Sp}\,(G^\varepsilon) = i\,\mathrm{Sp}\,(G)$. Moreover, they proved that a graph G is bipartite if and only if $\mathrm{Sp}\,(G^\sigma) = i\,\mathrm{Sp}\,(G)$ for some orientation G^σ of G.

Consider now two orientations G^σ and G^τ of G. We say that G^σ and G^τ are *switching-equivalent* if there exists a subset W of $V(G)$ such that G^σ is obtained from G^τ by reversing the direction of all arcs between W and $V(G)\backslash W$. Clearly, the skew-adjacency matrices of switching-equivalent orientations are $\{-1, 1\}$-diagonally similar. Hence, they have the same spectrum. When G is bipartite, Anuradha et al. [1] proved that $\mathrm{Sp}\,(G^\sigma) = i\,\mathrm{Sp}\,(G)$ if and only if G^σ is switching-equivalent to the canonical orientation.

These results can be stated in term of matrices as follows.

Proposition 1. *Let A be a $\{0, 1\}$-symmetric matrix. Then the following statements are equivalent :*

i) *There exists a real skew-symmetric matrix* B *such that* $|B| = A$ *and* $\mathrm{Sp}(B) = i\,\mathrm{Sp}(A)$*;*

ii) *There exists a permutation matrix* P *such that*

$$PAP^T = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}$$

where the zero diagonal blocks are square.

Proposition 2. *Let* $A = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}$ *be a* $\{0,1\}$*-symmetric matrix and let* B *be a skew-symmetric matrix such that* $|B| = A$*. Then, the following statements are equivalent :*

i) $\mathrm{Sp}(B) = i\,\mathrm{Sp}(A)$*;*

ii) B *is* $\{-1,1\}$*-diagonally similar to* $\widetilde{A} = \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix}$.

For a $\{0,1\}$-symmetric matrix A, the propositions above characterize the set of all skew-symmetric signings B of A, such that $\mathrm{Sp}(B) = i\,\mathrm{Sp}(A)$. In this paper, we consider the more general problem.

Problem 1. Let A be a nonnegative real matrix and let α be a complex unit number. Characterize the set of all signings B of A such that $\mathrm{Sp}(B) = \alpha\,\mathrm{Sp}(A)$.

We solve this problem when A is an irreducible matrix. To state our main result, we need some terminology. A *digraph* D is a pair consisting of a finite set $V(D)$ of *vertices* and a subset $E(D)$ of ordered pairs of vertices called *arcs*. Let v, v' be two vertices of D, a *path* P from v to v' is a finite sequence $v_0 = v, \ldots, v_k = v'$ such that $(v_0, v_1), \ldots, (v_{k-1}, v_k)$ are arcs of D. The *length* of P is the number k of its arcs. If $v_0 = v_k$, we say that P is a *closed path.* A digraph is said to be *strongly connected* if for any two vertices v and v', there is a path joining v to v'. It is easy to see that a strongly connected digraph contains a closed path. The *period* of a strongly connected digraph is the greatest common divisor of the lengths of its closed paths.

With each $n \times n$ matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$, we associate a digraph D_A on the vertex set $[n] = \{1, \ldots, n\}$ and with arc set $\{(i,j) : a_{ij} \neq 0\}$. It is easy to show that A is irreducible if and only if D_A is strongly connected. The *period* of an irreducible matrix is the period of its associate digraph. For example, if A is the adjacency matrix of a connected graph G, then its period is either 1 or 2. Moreover, the period of A is 2 if and only if G is bipartite.

Let A be an irreducible nonnegative real matrix of period p. For each complex unit number α, we denote by $\mathcal{M}(\alpha, A)$ the set of all signings B of A such that $\mathrm{Sp}(B) = \alpha\,\mathrm{Sp}(A)$.

In Corollary 2, we prove that if $\mathcal{M}(\alpha, A)$ is nonempty, then $\alpha = e^{\frac{i\pi k}{p}}$ for some $k \in \{0, \dots, 2p-1\}$. Moreover, we prove in Proposition 6 that $e^{\frac{i2\pi}{p}} sp(A) = sp(A)$. This implies that,

$$\mathcal{M}(1, A) = \mathcal{M}(e^{\frac{2i\pi}{p}}, A) = \cdots = \mathcal{M}(e^{\frac{2(p-1)i\pi}{p}}, A)$$
$$\mathcal{M}(e^{\frac{i\pi}{p}}, A) = \mathcal{M}(e^{\frac{3i\pi}{p}}, A) = \cdots = \mathcal{M}(e^{\frac{(2p-1)i\pi}{p}}, A)$$

Therefore, it suffices to characterize $\mathcal{M}(1, A)$ and $\mathcal{M}(e^{\frac{i\pi}{p}}, A)$.

In the proof of Corollary 1, we give an explicit construction of a matrix $B_0 \in \mathcal{M}(e^{\frac{i\pi}{p}}, A)$ which is used in our main theorem below.

Theorem 1. *Under the notation above, the following statements hold*

i) $\mathcal{M}(1, A)$ *is the set of matrices* $\{-1, 1\}$*-diagonally similar to* A*;*

ii) $\mathcal{M}(e^{\frac{i\pi}{p}}, A)$ *is the set of matrices* $\{-1, 1\}$*-diagonally similar to* B_0*.*

2. Some properties of $\mathcal{M}(\alpha, A)$

Throughout, A is an $n \times n$ irreducible nonnegative matrix, p its period and α a unit complex number. We will use the following theorem due to Helmut Wielandt [9].

Theorem 2. *Let* B *be a complex* $n \times n$ *matrix such that* $|B| \leqslant A$*. Then* $\rho(B) \leqslant \rho(A)$*. Moreover, if equality holds (i.e.,* $\rho(A)e^{i\theta} \in \mathrm{Sp}(B)$ *for some real number* θ*) then* $B = e^{i\theta} L A L^{-1}$*, where* L *is a complex diagonal matrix such that* $|L| = I_n$*.*

We will use Theorem 2 to prove the following.

Proposition 3. *Let* B *be a signing of* A *such that* $\rho(B) = \rho(A)$*. If* λ *is an eigenvalue of* B *such that* $|\lambda| = \rho(A)$*, then* $\lambda = \rho(A)e^{\frac{i\pi k}{p}}$ *for some* $k \in \{0, \dots, 2p-1\}$*.*

Proof. Let $A := (a_{ij})_{1 \leqslant i,j \leqslant n}$, $B := (b_{ij})_{1 \leqslant i,j \leqslant n}$ and $\lambda = \rho(A)e^{i\theta}$. By Theorem 2, we have $B = e^{i\theta} L A L^{-1}$ where L is a complex diagonal matrix such that $|L| = I_n$. It follows that $b_{ij} = e^{i\theta} l_i a_{ij} l_j^{-1}$ for $i, j \in \{1, \dots, n\}$, where $l_1, \ldots, l_n$ are the diagonal entries of L. Consider now a closed path $C = (i_1, i_2, \ldots, i_r, i_1)$ of D_A. By the previous equality, we have

$$\frac{b_{i_1 i_2} \ldots b_{i_{r-1} i_r} b_{i_r i_1}}{a_{i_1 i_2} \ldots a_{i_{r-1} i_r} a_{i_r i_1}} = (e^{i\theta} l_{i_1} l_{i_2}^{-1}) \ldots (e^{i\theta} l_{i_{r-1}} l_{i_r}^{-1})(e^{i\theta} l_{i_r} l_{i_1}^{-1}) = (e^{i\theta})^r$$

Then $(e^{i\theta})^r \in \{1, -1\}$ because $|B| = A$.

Since p is the greatest common divisor of the lengths of the closed paths in D_A, we have $(e^{i\theta})^p \in \{1, -1\}$ and then $\lambda = \rho(A) e^{\frac{i\pi k}{p}}$ for some $k \in \{0, \ldots, 2p-1\}$. □

Remark 1. Let λ be an eigenvalue of A such that $|\lambda| = \rho(A)$. By applying Proposition 3 to $B = A$, we have $\lambda = \rho(A) e^{\frac{i\pi k}{p}}$ for some $k \in \{0, \ldots, 2p-1\}$.

The following result gives a necessary condition under which $\mathcal{M}(\alpha, A)$ is nonempty.

Corollary 1. *If $\mathcal{M}(\alpha, A)$ is nonempty then $\alpha = e^{\frac{i\pi k}{p}}$ for some $k \in \{0, \ldots, 2p-1\}$, or equivalently $\alpha^p = \pm 1$.*

Proof. Let λ be an eigenvalue of A such that $|\lambda| = \rho(A)$. By Remark 1, we have $\lambda = \rho(A) e^{\frac{i\pi k}{p}}$ for some $k \in \{0, \ldots, 2p-1\}$. Let $B \in \mathcal{M}(\alpha, A)$. Then $\alpha\rho(A) e^{\frac{i\pi k}{p}} \in \mathrm{Sp}(B)$ because $\mathrm{Sp}(B) = \alpha \, \mathrm{Sp}(A)$. It follows from Proposition 3 that $\alpha\rho(A) e^{\frac{i\pi k}{p}} = \rho(A) e^{\frac{i\pi h}{p}}$ for some $h \in \{0, \ldots, 2p-1\}$ and hence $\alpha = e^{\frac{i\pi(h-k)}{p}}$. □

It is easy to see that if $B \in \mathcal{M}(\alpha, A)$, then $\Lambda^{-1} B \Lambda \in \mathcal{M}(\alpha, A)$ for every $\{-1, 1\}$-diagonal matrix Λ. Conversely,

Proposition 4. *The matrices in the set $\mathcal{M}(\alpha, A)$ are all $\{-1, 1\}$diagonally similar.*

Proof. Let $B_1, B_2 \in \mathcal{M}(\alpha, A)$. Then $\mathrm{Sp}(B_1) = \mathrm{Sp}(B_2) = \alpha \, \mathrm{Sp}(A)$. It follows that B_1 and B_2 have a common eigenvalue of the form $\rho(A) e^{i\theta}$ for some real number θ. By Theorem 2, we have $B_1 = e^{i\theta} L_1 A L_1^{-1}$ and $B_2 = e^{i\theta} L_2 A L_2^{-1}$ where L_1, L_2 are complex diagonal matrices such that $|L_1| = |L_2| = I_n$. It follows that $B_1 = (L_2 L_1^{-1})^{-1} B_2 L_2 L_1^{-1}$. To conclude, it suffices to apply Lemma 1 below. □

Lemma 1. *Let B, B' be two signings of A. If there exists a complex diagonal matrix Γ such that $B' = \Gamma B \Gamma^{-1}$ and $|\Gamma| = I_n$ then B and B' are $\{-1, 1\}$-diagonally similar.*

Proof. Let $A := (a_{ij})_{1\leqslant i,j\leqslant n}$, $B := (b_{ij})_{1\leqslant i,j\leqslant n}$ and $B' := (b'_{ij})_{1\leqslant i,j\leqslant n}$. We denote by $\gamma_1, \ldots, \gamma_n$ the diagonal entries of Γ. Let $\Delta := \gamma_1^{-1}\Gamma$. Clearly, we have $\Delta B \Delta^{-1} = \Gamma B \Gamma^{-1} = B'$. Hence, to prove our lemma, it suffices to check that Δ is a $\{-1,1\}$-diagonal matrix. For this, let $j \in \{2, \ldots, n\}$. As A is irreducible, the digraph D_A is strongly connected and then there is a path $j = i_1, \ldots, i_r = 1$ of D_A from j to 1. By definition of D_A, we have $a_{i_1 i_2} \neq 0, \ldots, a_{i_{r-1} i_r} \neq 0$. It follows that $b_{i_1 i_2} \neq 0, \ldots, b_{i_{r-1} i_r} \neq 0$ and $b'_{i_1 i_2} \neq 0, \ldots, b'_{i_{r-1} i_r} \neq 0$ because $|B| = |B'| = A$. Moreover, from the equality $B' = \Gamma B \Gamma^{-1}$ we have $b'_{i_1 i_2} = \gamma_{i_1} b_{i_1 i_2} \gamma_{i_2}^{-1}$, $b'_{i_2 i_3} = \gamma_{i_2} b_{i_2 i_3} \gamma_{i_3}^{-1}, \ldots,$ $b'_{i_{r-1} i_r} = \gamma_{i_{r-1}} b_{i_{r-1} i_r} \gamma_{i_r}^{-1}$. Then $b'_{i_1 i_2} \ldots b'_{i_{r-1} i_r} = \gamma_{i_1} \gamma_{i_r}^{-1} b_{i_1 i_2} \ldots b_{i_{r-1} i_r}$. But by hypothesis, B, B' are real matrices and $|B| = |B'|$, then $b'_{i_1 i_2} \ldots b'_{i_{r-1} i_r} = \pm b_{i_1 i_2} \ldots b_{i_{r-1} i_r}$ and hence $\gamma_j \gamma_1^{-1} = \gamma_{i_1} \gamma_{i_r}^{-1} \in \{-1, 1\}$, which completes the proof of the lemma. □

3. Proof of the main theorem

Assertion *i.* (resp. assertion *ii.* for $p = 1$) follows from Proposition 4 and the fact that $A \in \mathcal{M}(1, A)$ (resp. $-A \in \mathcal{M}(-1, A)$). To prove assertion *ii.* for $p > 1$, we will use the cyclic form of irreducible matrices with period p. To define k-cyclic matrices, let n be a positive integer and let $\{r_1, \ldots, r_k\}$ be a partition of n, that is $r_1, \ldots, r_k$ are positive integers and $r_1 + \cdots + r_k = n$. For $i = 1, \ldots, k-1$, let A_i be a $r_i \times r_{i+1}$ matrix and let A_k be a $r_k \times r_1$ matrix. The matrix

$$\begin{pmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & A_{k-1} \\ A_k & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is denoted by $Cyc(A_1, A_2, \ldots, A_k)$. Each matrix of this form is called *k-cyclic.*

The characterization of irreducible matrices with period $p > 1$ is given by the following result due to Frobenius.

Proposition 5. *Let A be an irreducible nonnegative real matrix with period $p > 1$, then there exists a permutation matrix P such that PAP^T is p-cyclic.*

Proposition 6. *Let $A = Cyc(A_1, A_2, \ldots, A_p)$ be a nonnegative p-cyclic matrix where A_i is a $r_i \times r_{i+1}$ matrix for $i = 1, \ldots, p-1$ and A_p is a $r_p \times r_1$ matrix. Let $\widetilde{A}$ be the matrix obtained from A by replacing the block A_p by $-A_p$. Let $k \in \{0, \ldots, 2p-1\}$, then*

i) *if k is even, $e^{\frac{i\pi k}{p}} A$ is diagonally similar to A, in particular $\mathrm{Sp}(A) = e^{\frac{i\pi k}{p}} \mathrm{Sp}(A)$;*

ii) *if k is odd, $e^{\frac{i\pi k}{p}} A$ is diagonally similar to $\widetilde{A}$, in particular $\mathrm{Sp}(\widetilde{A}) = e^{\frac{i\pi k}{p}} \mathrm{Sp}(A)$.*

Proof. Let

$$L := \begin{pmatrix} I_{r_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{\frac{i\pi k}{p}} I_{r_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & e^{\frac{i\pi k(p-1)}{p}} I_{r_p} \end{pmatrix}.$$

It easy to check that if k is even, $e^{\frac{i\pi k}{p}} LAL^{-1} = A$ and if k is odd, $e^{\frac{i\pi k}{p}} LAL^{-1} = \widetilde{A}$. □

The next corollary is a consequence of the above proposition and Proposition 5.

Corollary 2. *Let A be an irreducible nonnegative matrix with period p. Then $\mathcal{M}(e^{\frac{i\pi}{p}}, A)$ is nonempty.*

Proof. As $(-A) \in \mathcal{M}(-1, A)$, we can assume that $p > 1$. By Proposition 5, there exists a permutation matrix P such that PAP^T is $p-$cyclic. Let $A' := PAP^T := Cyc(A_1^{'}, A_2^{'}, \ldots, A_p^{'})$ and let $\widetilde{A'}$ be the matrix obtained from A' by replacing the block $A_p^{'}$ by $-A_p^{'}$. It follows from Proposition 6 that $\mathrm{Sp}(\widetilde{A'}) = e^{\frac{i\pi}{p}} \mathrm{Sp}(A')$, and hence $\mathrm{Sp}(P^T\widetilde{A'}P) = e^{\frac{i\pi}{p}} \mathrm{Sp}(P^TA'P) = e^{\frac{i\pi}{p}} \mathrm{Sp}(A)$. Let $B_0 := P^T\widetilde{A'}P$. Since $\left|\widetilde{A'}\right| = A'$, we have $|B_0| = P^TA'P = A$ and then $B_0 \in \mathcal{M}(e^{\frac{i\pi k}{p}}, A)$. □

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