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Cohen-Macaulay modules over the plane curve singularity of type T_{36}

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Abstract. For a wide class of Cohen–Macaulay modules over the local ring of the plane curve singularity of type T_{36} we describe explicitly the corresponding matrix factorizations. The calculations are based on the technique of matrix problems, in particular, representations of bunches of chains.

1. Introduction

Let $\Bbbk$ be an algebraically closed field, $\boldsymbol{S} = \Bbbk[[x, y]]$. Recall that the complete local ring of the plane curve singularity of type T_{36} is $\boldsymbol{R} = \boldsymbol{S}/(F)$, where $F = x(x - y^2)(x - \lambda y^2)$ and $\lambda \in \Bbbk \setminus \{0, 1\}$. In this paper we present explicit description of a wide class of maximal Cohen–Macaulay modules over the ring $\boldsymbol{R}$ called *modules of the first level*. Note that T_{36} is one of the critical singularities of tame Cohen-Macaulay representation type [7]. Till now, only for the singularities of type T_{44} matrix factorizations have been described [8, 9].

2. Matrix problem and the first reduction

So, let $\boldsymbol{R} = \Bbbk[[x, y]]/(F)$, where $F = x(x-y^2)(x-\lambda y^2)$ ($\lambda \in \Bbbk\setminus\{0, 1\}$). We consider $\boldsymbol{R}$ as the subring of the direct product $\tilde{\boldsymbol{R}} = \boldsymbol{R}_1 \times \boldsymbol{R}_2 \times \boldsymbol{R}_3$, where all $\boldsymbol{R}_i = \Bbbk[[t]]$, generated by the elements $x = (0, t^2, \lambda t^2)$ and $y = (t, t, t)$. We denote by $\boldsymbol{R}_{12}$ the projection of $\boldsymbol{R}$ onto $\boldsymbol{R}_1 \times \boldsymbol{R}_2$. It is

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generated by $(t,t,0)$ and $(0,t^2,0)$ and $\boldsymbol{R}_{12} \simeq \Bbbk[[x,y]]/x(x-y^2)$. It is a singularity of type A_2, so all indecomposable $\boldsymbol{R}_{12}$-modules are $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_{12}$ and $\boldsymbol{R}'_{12} = \boldsymbol{R}_{12}[t_1]$, where $t_1 = (t,0,0)$. We also set $t_2 = (0,t,0)$.

Denote $\tilde{M} = \tilde{\boldsymbol{R}}M = M_1 \oplus M_2 \oplus M_3$, where M_i is an $\boldsymbol{R}_i$-module. There is an exact sequence:

$$0 \to M_{12} \to M \to M_3 \to 0,$$

where $M_{12} = M \cap (M_1 \oplus M_2)$ is an $\boldsymbol{R}_{12}$-module, hence $M_{12} = m_1\boldsymbol{R}_1 \oplus m_2\boldsymbol{R}_2 \oplus m_{12}\boldsymbol{R}_{12} \oplus m'_{12}\boldsymbol{R}'_{12}$. So M gives an element $\xi \in \operatorname{Ext}^1_{\boldsymbol{R}}(M_3, M_{12})$. There is an exact sequence

$$0 \to (x-\lambda y^2)\boldsymbol{R} \to \boldsymbol{R} \to \boldsymbol{R}_3 \to 0,$$

whence

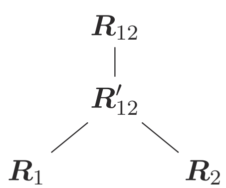
$$\operatorname{Ext}^1_{\boldsymbol{R}}(M_3, M_{12}) = M_{12}/(x-\lambda y^2)M_{12}$$

In the table below we present bases of the modules $\operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_3, N)$, where $N \in \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_{12}, \boldsymbol{R}'_{12}$. In this table each element actually denotes its residue class modulo $x-\lambda y^2$, 1_s $(s \in \{1,2,12\})$ is the identity element of $\boldsymbol{R}_s$, $t_{12} = t_1 + t_2$ and $\mu = (1-\lambda)/\lambda$.

$\boldsymbol{R}_1$	$1_1,\ t1_1$
$\boldsymbol{R}_2$	$1_2,\ t_2$
$\boldsymbol{R}'_{12}$	$1_{12},\ t_1,\ t_2,\ t_1^2 = \mu t_2^2$
$\boldsymbol{R}_{12}$	$1_{12},\ t_{12},\ t_1^2 = \mu t_2^2,\ t_1^3 = \mu t_2^3$

Homomorphisms $\boldsymbol{R}_{12} \to \boldsymbol{R}_i$ induce the maps $\operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_3, \boldsymbol{R}_{12}) \to \operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_3, \boldsymbol{R}_i)$ which map $1_{12} \mapsto 1_i$ and $t_{12} \mapsto t_i$. The embedding $\boldsymbol{R}'_{12} \to \boldsymbol{R}_{12}$ which maps $1_{12} \mapsto t_{12}$ induces the map $\operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_3, \boldsymbol{R}'_{12}) \to \operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_3, \boldsymbol{R}_{12})$ that coincide with the multiplication by t_{12}. The embeddings $\boldsymbol{R}_i \to \boldsymbol{R}'_{12}$ such that $1_i \mapsto t_i$ induce the maps $\operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_3, \boldsymbol{R}_i) \to \operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_3, \boldsymbol{R}'_{12})$ that coincide with the multiplication by t_i.

In particular, if we only consider free terms, we obtain representations of the partially ordered set of width 2:



(see [12]), hence the matrix A_0 of free terms can be reduced to the form:

$$A_0 = \left(\begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & I_1 & 0 & 0 \\ 0 & I_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

where I_s means $1_s I$ for the identity matrices I of the appropriate size (maybe different for different matrices). In what follows we always suppose that A_0 is of this form.

3. Modules of the first level

Let now $A = A_0 + tA_1$ where A_1 is also divided into blocks in the same way as A_0. Using automorphisms of M_3 we can make zero the 1st, 4th, 7th and 9th rows of A_1, as well as one of the 2rd or 5th rows.

Using automorphisms of M_{12} we can also make zero all columns in A_1, except the 1st one and the parts of the 2nd, 3rd and 4th columns in the 10th row, where we can only delete all terms containing t^2 or t^3. In particular, the terms 1_{12} from the last vertical stripe become direct summands of the whole matrix A. So in what follows we can omit this column. We always suppose that A_1 has this form.

Let A_1^0 be the free term of the matrix A_1. The non-zero part of its 10th row can be considered as representation of the partially ordered set:

$$\begin{array}{ccc} & 2 & \\ / & & \backslash \\ 3 & & 4 \\ \backslash & & / \\ & 1 & \end{array}$$

Hence it can be reduced to the form

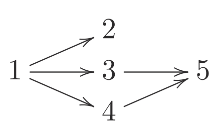
$$\left(\begin{array}{cc|cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

where I is again an identity matrix of the appropriate size (maybe different for different matrices). Then the whole matrix A modulo t^2 can be reduced to the form:

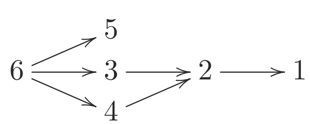
	1	2			3		4	5	
	0	0	0	0	0	0	I_1	0	0
	0	0	0	0	0	0	0	I_1	0
1	$A_{11}t^*$	0	I_1	0	0	0	0	0	0
2	$A_{21}t^*$	0	0	I_1	0	0	0	0	0
3	$A_{31}t_1$	0	0	0	0	0	0	0	0
	0	0	0	0	I_2	0	0	0	0
	0	0	0	0	0	I_2	0	0	0
	0	0	I_2	0	0	0	0	0	0
	0	0	0	I_2	0	0	0	0	0
4	$A_{41}t_2$	0	0	0	0	0	0	0	0
5	$A_{51}t^*$	$A_{52}t^*$	0	0	0	0	0	0	I_{12}
6	A_{61}	$A_{62}t^*$	0	0	$A_{63}t_1$	0	$A_{64}t_2$	$A_{65}t^*$	0
	0	$t_{12}I$	0	0	0	0	0	0	0
	0	0	$t_{12}I$	0	0	0	0	0	0
	0	0	0	0	0	$t_{12}I$	0	$t_{12}I$	0
	0	0	0	0	0	0	0	0	0

Here the symbol t^* means that in this block $t_1 = -t_2$, and A_{61} is a matrix pencil $X_1t_1 + X_2t_2$. The horizontal lines show the division of A into the stripes such that the 1st stripe corresponds to R_1, the 2nd to R_2, the 3rd to R'_{12} and the 4th to R_{12}. Moreover, as in the matrix A we have $t_1^2 = \mu t_2^2$ with $\mu \neq -1$, one can delete all terms with t_i^2 and t_i^3 everywhere except the last block of the first column.

The endomorphisms of M_3 and M_{12} which do not destroy the shape of the matrices A_0 and A_1^0 induce the transfromations of columns that can be described by the scheme



and the transformations of rows that can be described by the scheme



For the matrix A_{61} it means that we can add the rows of X_1 to those of A_{i1} for $i \in \{1,2,3\}$ and the rows of X_2 to the rows of A_{i1} for $i \in \{1,2,4\}$. In the same way, the columns of X_1 can be added to those of A_{6j} for $j \in \{3,5\}$, while the columns of X_2 can be added to those of A_{6j} for $j \in \{4,5\}$.

The indecomposable matrix pencils (representations of the Kronecker quiver) are described in [11,13]. In [13] the morphisms between indecomposable representations are also described. It implies that the matrix A_{61} is a direct sum of the following matrices:

$$A(n) = \begin{pmatrix} t_1 & t_2 & 0 & \dots & 0 \\ 0 & t_1 & t_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_2 \\ 0 & 0 & 0 & \dots & t_1 \end{pmatrix}, \qquad B(n) = \begin{pmatrix} t_2 & t_1 & 0 & \dots & 0 \\ 0 & t_2 & t_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_1 \\ 0 & 0 & 0 & \dots & t_2 \end{pmatrix},$$

$$C(n) = \begin{pmatrix} t_1 & t_2 & 0 & \dots & 0 & 0 \\ 0 & t_1 & t_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_1 & t_2 \end{pmatrix}, \qquad D(n) = C(n)^\top$$

It is easy to see that:

- If $A_{61} = A(n)$ than we can make zero all matrices above A_{61} except the 1st column of A_{41}, and all matrices to the right of A_{61} except the last row of A_{64}.
- If $A_{61} = B(n)$ than we can make zero all matrices above A_{61} except the 1st column of A_{31}, and all matrices to the right of A_{61} except the last row of A_{63}.
- If $A_{61} = C(n)$ than we can make zero all matrices to the right of A_{61}, and all matrices above A_{61} except the last column of A_{31}, the 1st column of A_{41} and one (any chosen) of the columns of the matrix A_{51}.
- If $A_{61} = D(n)$ than we can make zero all matrices above A_{61}, and all matrices to the right of A_{61} except the last row of A_{63}, the 1st row of A_{64} and one (any chosen) of the rows of the matrix A_{62}.

Hence in the non-zero part of A_{51} (and, respectively, A_{62}) we can left one non-zero element above each block of $C(n)$ (and, respectively, $D(n)$). Therefore, except the summands $A(n), B(n), C(n), D(n)$ in the blocks A_{ij}, $(i = 5,6,\ j = 1,2)$, we will also have the summands of the form $C'(n)$, with one additional element in A_{51}-part as compared to $C(n)$, and $D'(n)$, with one additional element in A_{62}-part as compared to $D(n)$). So we can suppose that $C'(n)$ looks like $B(n)^\top$, but with the 1st row from A_{51}, and $D'(n)$ looks like $A(n)^\top$, but with the last column from A_{62}.

One can see that now we can make zero all elements of the matrix A_{52} except those which are in the zero rows of A_{51} and zero columns of A_{62}. The remaining part of A_{51} can be reduced to the form $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. It gives direct summands of the whole matrix A of the form $\begin{pmatrix} t_1 \\ \hline t_{12} \end{pmatrix}$ (certainly, t_1 can be replace here by t_2). Therefore, in what follows we can suppose that $A_{52} = 0$. Analogously, we can suppose that the matrices A_{11}, A_{12} and A_{65} are also zero. Otherwise we obtain direct summands of A. For instance, if $A_{65} \neq 0$, all non-zero elements are in the rows which do not belong to the non-zero parts of A_{66} and A_{65}. So they give direct summands of the form

$$\begin{pmatrix} 0 & 1_1 \\ \hline 1_2 & 0 \\ \hline 0 & t_2 \\ \hline t_{12} & t_{12} \end{pmatrix}.$$

The description of homomorphisms between the representations of the Kronecker quuiver [13] show that we can add the non-zero columns over $A(n)$ (respectively, $B(n)$) to those over $A(m)$ (respectively, $B(m)$) for $m > n$, and the same for the non-zero rows to the right of $A(n)$ or $B(n)$. We can also add the non-zero columns over $C(n)$ (respectively, non-zero rows to the right of $D(n)$) to those of $C(m)$ (respectively, of $D(m)$), where $n < m$, as well as to those of $A(k)$ and $B(k)$ for any k. It means that the possible transformations of these columns and rows can be considered as *representations of a bunch of chains* in the sense of [1] or [2, Appendix B] (we use the formulation of the second paper). Namely, we have the next pairs of chains:

- $\mathcal{E}_1 = \{a_i, d_i, d'_i \,|\, i \in \mathbb{N}\}$, $\mathcal{F}_1 = \{c_3\}$
- $\mathcal{E}_2 = \{b_i, \tilde{d}_i, \tilde{d}'_i \,|\, i \in \mathbb{N}\}$, $\mathcal{F}_2 = \{c_4\}$
- $\mathcal{E}_3 = \{r_3\}$, $\mathcal{F}_3 = \{\tilde{a}_i, c_i, c'_i \,|\, i \in \mathbb{N}\}$
- $\mathcal{E}_4 = \{r_4\}$, $\mathcal{F}_4 = \{\tilde{b}_i, \tilde{c}_i, \tilde{c}'_i \,|\, i \in \mathbb{N}\}$

with the relation $\sim$:

$$a_i \sim \tilde{a}_i, \quad b_i \sim \tilde{b}_i, \quad c_i \sim \tilde{c}_i, \quad d_i \sim \tilde{d}_i, \quad c'_i \sim \tilde{c}'_i, \quad d'_i \sim \tilde{d}'_i \qquad (i \in \mathbb{N}).$$

Here r_3, r_4 corresponds to A_{31}, A_{41} respectively and c_3, c_4 corresponds to A_{63}, A_{64} respectively.

Now we use the description of the indecomposable representations of this bunch of chains from [1, 2]. In our case they correspond to the following words in the alphabet $\{a_i, \tilde{a}_i, b_i, \tilde{b}_i, c_i, \tilde{c}_i, d_i, \tilde{d}_i, c'_i, \tilde{c}'_i, d'_i, \tilde{d}'_i, c_3, r_4, c_4, r_3, -, \sim\}$:

- 4 type of words with a_i, $i \in \mathbb{N}$: $\mathbf{w}_a(i) = r_3 - \tilde{a}_i \sim a_i - c_3$ and 3 shorter words: $r_3 - \tilde{a}_i \sim a_i$, $\tilde{a}_i \sim a_i - c_3$, $\tilde{a}_i \sim a_i$;
- 4 type of words with b_i, $i \in \mathbb{N}$: $\mathbf{w}_b(i) = r_4 - \tilde{b}_i \sim b_i - c_4$ and 3 shorter words: $r_4 - \tilde{b}_i \sim b_i$, $\tilde{b}_i \sim b_i - c_4$, $\tilde{b}_i \sim b_i$;
- 4 type of words with c_i, $i \in \mathbb{N}$: $\mathbf{w}_c(i) = r_4 - \tilde{c}_i \sim c_i - r_3$ and 3 shorter words: $r_4 - \tilde{c}_i \sim c_i$, $\tilde{c}_i \sim c_i - r_3$, $\tilde{c}_i \sim c_i$;
- 4 type of words with d_i, $i \in \mathbb{N}$: $\mathbf{w}_d(i) = c_4 - \tilde{d}_i \sim d_i - c_3$ and 3 shorter words: $c_4 - \tilde{d}_i \sim d_i$, $\tilde{d}_i \sim d_i - c_3$, $\tilde{d}_i \sim d_i$;
- 4 type of words with c'_i, $i \in \mathbb{N}$: $\mathbf{w}'_c(i) = r_4 - \tilde{c}'_i \sim c'_i - r_3$ and 3 shorter words: $r_4 - \tilde{c}'_i \sim c'_i$, $\tilde{c}'_i \sim c'_i - r_3$, $\tilde{c}'_i \sim c'_i$;
- 4 type of words with d'_i, $i \in \mathbb{N}$: $\mathbf{w}'_d(i) = c_4 - \tilde{d}'_i \sim d'_i - c_3$ and 3 shorter words: $c_4 - \tilde{d}'_i \sim d'_i$, $\tilde{d}'_i \sim d'_i - c_3$, $\tilde{d}'_i \sim d'_i$;

Following the construction of indecomposable representations from [1], we construct the matrices corresponding to these words:

$$
P_a(n) = \left(\begin{array}{c|c} 0 & 1 \\ \hline t_2\mathbf{e}_1 & 0 \\ \hline A(n) & t_2\mathbf{e}_n^\top \end{array}\right), \qquad
P_b(n) = \left(\begin{array}{c|c} t_1\mathbf{e}_1 & 0 \\ \hline 0 & 1 \\ \hline B(n) & t_1\mathbf{e}_n^\top \end{array}\right),
$$

$$
P_c(n) = \left(\begin{array}{c} t_1\mathbf{e}_1 \\ \hline t_2\mathbf{e}_1 \\ \hline C(n) \end{array}\right), \qquad
P_d(n) = \left(\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline D(n) & t_2\mathbf{e}_{n+1}^\top & t_1\mathbf{e}_{n+1}^\top \end{array}\right),
$$

$$
P'_c(n) = \left(\begin{array}{c|c} t_1\mathbf{e}_1 & 0 \\ \hline t_2\mathbf{e}_1 & 0 \\ \hline C'(n) & t_{12}\mathbf{e}_1^\top \end{array}\right), \quad
P'_d(n) = \left(\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline D'(n) & t_2\mathbf{e}_{n+1}^\top & t_1\mathbf{e}_{n+1}^\top \\ \hline t_{12}\mathbf{e}_{n+1} & 0 & 0 \end{array}\right).
$$

Here $t_r = y1_r$ and $\mathbf{e}_n = (0, 0, \dots, 0, 1)$, $\mathbf{e}_1 = (1, 0, \dots, 0)$ and $^\top$ means the transposition.

4. Generators and relations. Example

Now we calculate matrix factorizations of the polynomial $F = x(x - y^2)(x - \lambda y^2)$ corresponding to the indecomposable Cohen-Macaulay modules over $\boldsymbol{R}$. In other words, we find minimal sets of generators for these modules and minimal sets of relations for these generators.

In order to make smaller the arising matrices, we denote $z = x - y^2$ an $z' = x - \lambda y^2$. Thus $F = xzz'$.

We do detailed calculations for the word $\mathbf{w}_a(2) = r_3 - \tilde{a}_2 \sim a_2 - c_3$. Since all calculations are similar, for other words we just write the resulting matrices.

$$P_a(2) = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ \hline ye_2 & 0 & 0 \\ \hline ye_1 & ye_2 & 0 \\ 0 & ye_1 & ye_2 \end{array}\right)$$

Here the first two stripes belongs to R_1 and R_2 respectively and the last stripe belongs to R'_{12}. So we have generators:

$$\begin{gathered} v_1, v_2, v_3 \in R_3, \\ u^1 \in R_1, \qquad u^2 \in R_2, \\ u_1^{12}, \bar{u}_1^{12}, u_2^{12}, \bar{u}_2^{12} \in R'_{12}. \end{gathered} \tag{$*$}$$

Note that $ye_1 u_i^{12} = \bar{u}_i^{12}$ and $ye_2 u_i^{12} = yu_i^{12} - \bar{u}_i^{12}$ for $u_i^{12} \in R'_{12}$, $i = 1, 2$. Then we have the following relations for these generators:

$$\begin{gathered} z'v_1 = ye_2 u^2 + ye_1 u_1^{12}, \qquad z'v_2 = ye_2 u_1^{12} + ye_1 u_2^{12}, \\ z'v_3 = u^1 + ye_2 u_2^{12} \end{gathered}$$

It implies that

$$\begin{gathered} \bar{u}_1^{12} = z'v_1 - yu^2, \qquad \bar{u}_2^{12} = z'v_1 - yu^2 + z'v_2 - yu_1 12, \\ u^1 = z'v_1 - yu^2 + z'v_2 - yu_1 12 + z'v_3 - yu_2^{12} \end{gathered}$$

Now we can exclude generators $\bar{u}_1^{12}, \bar{u}_2^{12}, u^1$. It is important to note that $\bar{u}_i^{12}$, $i = 1, 2$ are annihilated by x. Since $u^1 \in R_1$ is also annihilated by x, $u^2 \in R_2$ is annihilated by z and $u_1^{12}, u_2^{12} \in R'_{12}$ are annihilated by xz, we have the following relations for $v_1, v_2, v_3, u^2, u_1^{12}, u_2^{12}$ from $(*)$:

$$\begin{gathered} zu^2 = 0, \qquad xzu_1^{12} = 0, \qquad xzu_2^{12} = 0, \\ xz'v_1 - xyu^2 = 0, \qquad xz'v_2 - xyu_1^{12} = 0, \qquad xz'v_3 - xyu_2^{12} = 0. \end{gathered}$$

It gives the following matrix factorization with columns corresponding to $u^2, u_1^{12}, u_2^{12}, v_1, v_2, v_3$, in this order:

$$Q_a(2) = \begin{pmatrix} z & 0 & 0 & 0 & 0 & 0 \\ 0 & xz & 0 & 0 & 0 & 0 \\ 0 & 0 & xz & 0 & 0 & 0 \\ -xy & 0 & 0 & xz' & 0 & 0 \\ 0 & -xy & 0 & 0 & xz' & 0 \\ 0 & 0 & -xy & 0 & 0 & xz' \end{pmatrix}$$

For the other three words with a_i, namely $r_3 - \tilde{a}_i \sim a_i$, $\tilde{a}_i \sim a_i - c_3$, $\tilde{a}_i \sim a_i$ we obtain the matrix factorizations by excluding some generators and the appropriate srows and columns:

- Excluding u^2 from the list of generators $(*)$ and deleting the first row and 1st column from the matrix $Q_a(2)$ we get the matrix factorization for $\tilde{a}_i \sim a_i - c_3$.
- Excluding v_3 from the list of generators $(*)$ and deleting the last row and the last column from the matrix $Q_a(2)$ we get the matrix factorization for $r_3 - \tilde{a}_i \sim a_i$.
- Excluding both u^2, v_3 from the list of generators $(*)$ and deleting the first and the last rows and the first and the last columns from the matrix $Q_a(2)$ we get the matrix factorization for $\tilde{a}_i \sim a_i$.

Now one can easily see how the matrix factorization $Q_a(i)$ for the word $\mathbf{w}_a(i) = r_3 - \tilde{a}_i \sim a_i - c_3$ looks like for $i > 2$.

5. Generators and relations. Other words

For other modules of the first level the corresponding matrix factorizations are calculated in a similar way. We only present the results for $n = 2$, since otherwise we obtain too cumbersome matrices.

For the word $\mathbf{w}_b(2) = r_4 - \tilde{b}_2 \sim b_2 - c_4$ we have the matrix of correspondences with columns corresponding to $u^1, u_1^{12}, u_2^{12}, v_1, v_2, v_3$:

$$Q_b(2) = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 \\ 0 & xz & 0 & 0 & 0 & 0 \\ 0 & 0 & xz & 0 & 0 & 0 \\ -xy & -xy & 0 & xz' & 0 & 0 \\ 0 & 0 & xy & 0 & xz' & 0 \\ -zy & -zy & zy & zz' & zz' & zz' \end{pmatrix}$$

For the word $\mathbf{w}_c(2) = r_4 - \tilde{c}_2 \sim c_2 - r_3$ we have the matrix of correspondences with columns corresponding to $u^2, u_1^{12}, u_2^{12}, v_3, v_2, v_1$:

$$Q_c(2) = \begin{pmatrix} z & 0 & 0 & 0 & 0 & 0 \\ 0 & xz & 0 & 0 & 0 & 0 \\ 0 & 0 & xz & 0 & 0 & 0 \\ 0 & 0 & -xy & xz' & 0 & 0 \\ 0 & -xy & 0 & 0 & xz' & 0 \\ -xy & 0 & 0 & 0 & 0 & xz' \end{pmatrix}$$

For the word $\mathbf{w}_d(2) = c_4 - \tilde{d}_2 \sim d_2 - c_3$ we have the matrix of correspondences with columns corresponding to $u^2, u_1^{12}, u_2^{12}, u_3^{12}, v_4, v_3, v_2, v_1$:

$$Q_d(2) = \begin{pmatrix} z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & xz & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & xz & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & xz & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & xz' & 0 & 0 & 0 \\ 0 & 0 & 0 & -xy & 0 & xz' & 0 & 0 \\ 0 & 0 & 0 & -xy & 0 & 0 & xz' & 0 \\ 0 & 0 & -xy & 0 & 0 & 0 & 0 & xz' \end{pmatrix}$$

For the word $\mathbf{w}'_c(2) = r_4 - \tilde{c}'_2 \sim c'_2 - r_3$ we have the matrix of correspondences with columns corresponding to $u^1, u^2, u_1^{12}, u_2^{12}, v_4, v_3, v_2, v_1$:

$$Q'_c(2) = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & xz & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & xz & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & xzz' & 0 & 0 & 0 \\ 0 & 0 & 0 & -xy & 0 & xz' & 0 & 0 \\ 0 & 0 & -xy & 0 & 0 & 0 & xz' & 0 \\ -xy & -xy & 0 & 0 & -xyz' & 0 & 0 & xz' \end{pmatrix}$$

For the word $\mathbf{w}'_d(2) = c_4 - \tilde{d}'_2 \sim d'_2 - c_3$ we have the matrix of correspondences with columns corresponding to $u^2, u_1^{12}, u_2^{12}, u_3^{12}, v_5, v_4, v_3, v_2, v_1$, namely $Q'_d(2)$ equals:

$$Q'_d(2) \begin{pmatrix} z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & xz & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & xz & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & xz & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & xz' & 0 & 0 & 0 & \\ -x & 0 & 0 & 0 & xzz' & -xzz' & 0 & 0 & 0 \\ 0 & 0 & 0 & -xy & 0 & 0 & xz' & 0 & 0 \\ 0 & 0 & 0 & -xy & 0 & 0 & 0 & xz' & 0 \\ 0 & 0 & -xy & 0 & 0 & 0 & 0 & 0 & xz' \end{pmatrix}$$

For the truncated words (without the first or the last letter) we apply the procedure analogous to that described at the end of the preceding section.

In this way we obtain all matrix factorizations of the polynom F corresponding to the modules of the first level.

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