© Algebra and Discrete Mathematics
Volume **32** (2021). Number 2, pp. 280–298
DOI:10.12958/adm1439

RESEARCH ARTICLE

Some commutativity criteria for 3-prime near-rings

A. Raji

Communicated by G. Pilz

Abstract. In the present paper, we introduce the notion of $*$-generalized derivation in near-ring N and investigate some properties involving that of $*$-generalized derivation of a $*$-prime near-ring N which forces N to be a commutative ring. Some properties of generalized semiderivations have also been given in the context of 3-prime near-rings. Consequently, some well known results have been generalized. Furthermore, we will give examples to demonstrate that the restrictions imposed on the hypothesis of various results are not superfluous.

1. Introduction

Throughout this paper N will be a near-ring with multiplicative center $Z(N)$. A near-ring N is said to be 3-prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$. It is called 3-semiprime if $xNx = \{0\}$ implies $x = 0$. Recalling that N is called 2-torsion free if $(N, +)$ has no elements of order 2; and N is called zero-symmetric if $0 \cdot x = x \cdot 0 = 0$ for all $x \in N$. For any $x, y \in N$; as usual $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie and Jordan products respectively. An additive mapping $* : N \to N$ is called an involution if $(xy)^* = y^* x^*$ and $(x^*)^* = x$ for all $x, y \in N$. A near-ring N equipped with an involution $*$ is said to be $*$-prime if $xNy = xNy^* = \{0\}$ implies $x = 0$ or $y = 0$. Obviously a left

2020 MSC: 16N60, 16W25, 16Y30.

Key words and phrases: 3-prime near-rings, 3-semiprime near-rings, involution, $*$-derivation, semiderivation, commutativity.

(resp. right) near-ring with involution is in fact distributive, and hence also zero-symmetric. Indeed, let N be a left near-ring with involution $*$, and let $x, y, z, \in N$. Then

$$\begin{aligned} ((x+y)z)^* &= z^*(x^*+y^*) \\ &= z^*x^* + z^*y^*. \end{aligned}$$

Applying $*$ to both sides, we obtain

$$\begin{aligned} (x+y)z &= (z^*x^* + z^*y^*)^* \\ &= x^{**}z^{**} + y^{**}z^{**} \\ &= xz + yz. \end{aligned}$$

Clearly, a near-ring with an involution is not necessarily a ring. Indeed, in Example 1, since the addition law in N is not commutative, N cannot be a ring.

The commutativity of 3-prime near-rings with derivation was initiated by Bell and Mason in [3]. Over the last two decades, a lot of work has been done on this subject. Recently, in [2], Ashraf and Siddeeque defined the following notations: An additive mapping $d : N \to N$ is called a $*$-derivation if there exists an involution $* : N \to N$ such that $d(xy) = d(x)y^* + xd(y)$, for all $x, y \in N$. An additive mapping $F : N \to N$ is called a left $*$-multiplier if $F(xy) = F(x)y^*$ hold for all $x, y \in N$. Motivated by these concepts, we introduce the concepts of $*$-generalized derivation in near-rings as follows: An additive mapping $F : N \to N$ is called a $*$-generalized derivation if there exist a $*$-derivation d of N such that

$$F(xy) = F(x)y^* + xd(y) \text{ for all } x, y \in N.$$

One may observe that the concept of $*$-generalized derivation includes the concept of $*$-derivation and left $*$-multiplier. Hence it should be interesting to extend some results concerning these notions to $*$-generalized derivation. An additive mapping $\delta : N \to N$ is said to be a derivation if $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in N$, or equivalently, as noted in [10], that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in N$. An additive mapping $f : N \to N$ is said to be a generalized derivation on N if there exists a derivation $\delta : N \to N$ such that $f(xy) = f(x)y + x\delta(y)$ for all $x, y \in N$. An additive mapping $d : N \to N$ is called semiderivation if there is a function $g : N \to N$ such that $d(xy) = xd(y) + d(x)g(y) = g(x)d(y) + d(x)y$ and $d(g(x)) = g(d(x))$ for all $x, y \in N$, or equivalently, as noted in [5], that $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$

for all $x, y \in N$. Obviously, any derivation is a semiderivation, but the converse is not true in general (see [5]). An additive mapping $F : N \to N$ is called generalized semiderivation if there is a semiderivation $d : N \to N$ associated with a function $g : N \to N$ such that

$$F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y) \text{ for all } x, y \in N.$$

Of course any semiderivation is a generalized semiderivation. Moreover, if g is the identity map of N, then all generalized semiderivations associated with g are merely generalized derivations of N. Recently, many researchers have studied commutativity of prime and semiprime rings admitting suitably constrained additive mappings, as: automorphisms, derivations and involutions acting on appropriate subsets of the rings (see, for example, Bell and Mason [3]; Bresar and Vukman [6]; Fong [7]; Oukhtite and Salhi [8]). Being motivated by their invaluable research, it is natural then to ask if we can apply the involution so as to study the structure of a near-ring. In this paper, firstly we would like to study the structure of a $*$-prime left near-ring and 3-semiprime left near-rings as having an involution. More precisely, we shall prove that a $*$-prime near-ring must be a commutative ring. Secondly, some results concerning composition of generalized semiderivation on a 3-prime right near-ring will be exposed. As for terminologies used here without mention, we refer to Pilz [9].

2. Main results

In [6] M. Bresar and J.Vukman introduced the notion of $*$-derivation (resp. reverse $*$-derivation) in rings and proved that if a prime $*$-ring admits a nonzero $*$-derivation (resp. reverse $*$-derivation), then the ring is commutative. This result was extended to 3-semiprime ring by S. Ali [1], who proved that if a 3-semiprime $*$-ring R admits a $*$-derivation (resp. reverse $*$-derivation) d, then d maps R into $Z(R)$ where $Z(R)$ is the center of R. Further, Ashraf and Seddeeque [2] showed that a 3-prime $*$-near-ring N with a nonzero $*$-derivation d, must be commutative. Motivated by these results mentioned above, we succeeded in establishing the following results:

Theorem 1. *Let N be a $*$-prime near ring. If N admits a nonzero $*$-generalized derivation F, then N is a commutative ring.*

Proof. For all $x, y, z \in N$, we have

$$\begin{aligned} F(xyz) &= F(xy)z^* + xyd(z) \\ &= (F(x)y^* + xd(y))z^* + xyd(z) \\ &= F(x)y^*z^* + xd(y)z^* + xyd(z). \end{aligned}$$

On the other hand,

$$\begin{aligned} F(xyz) &= F(x)(yz)^* + xd(yz) \\ &= F(x)z^*y^* + x(d(y)z^* + yd(z)) \\ &= F(x)z^*y^* + xd(y)z^* + xyd(z). \end{aligned}$$

Combining the both expressions for $F(xyz)$, we get

$$F(x)y^*z^* = F(x)z^*y^* \text{ for all } x, y, z \in N. \tag{1}$$

Replacing y and z by y^* and z^*, respectively, in (1), we obtain

$$F(x)yz = F(x)zy \text{ for all } x, y, z \in N. \tag{2}$$

Taking yt instead of y in (2), where $t \in N$, and using it again we arrive at

$$F(x)yzt = F(x)ytz \text{ for all } x, y, z, t \in N. \tag{3}$$

So that,

$$F(x)y[z, t] = 0 \text{ for all } x, y, z, t \in N. \tag{4}$$

Replacing z and t by t^* and z^*, respectively, in (4), we obtain

$$F(x)y[t^*, z^*] = 0 \text{ for all } x, y, z, t \in N. \tag{5}$$

Since $[t^*, z^*] = [z, t]^*$, equation (5) can be written as

$$F(x)y[z, t]^* = 0 \text{ for all } x, y, z, t \in N. \tag{6}$$

From (4) and (6), we find that

$$F(x)N[z, t] = F(x)N[z, t]^* = \{0\} \text{ for all } x, z, t \in N. \tag{7}$$

In view of the $*$-primeness of N, (7) shows that

$$F(x) = 0 \text{ or } [z, t] = 0 \text{ for all } x, z, t \in N.$$

Since $F \neq 0$, we conclude that $[z, t] = 0$ for $z, t \in N$. And therefore, the multiplicative law of N is commutative. Now, let $x, y, z \in N$. Then we have $(x+x)(y+z) = (y+z)(x+x)$. It follows that $(x+x)y + (x+x)z = (y+z)x + (y+z)x$ for all $x, y, z \in N$. The previous relation can be rewritten as $y(x+x) + z(x+x) = x(y+z) + x(y+z)$. After simplifying, we have $yx + zx = xz + xy$ which implies that

$$x((y+z) - (z+y)) = 0 \text{ for all } x, y, z \in N. \tag{8}$$

Replacing y and z by y^* and z^*, respectively, in (8), we arrive at

$$x((y+z) - (z+y))^* = 0 \text{ for all } x, y, z \in N. \tag{9}$$

Now, comparing the identities (8) and (9), we find that

$$x((y+z) - (z+y)) = x((y+z) - (z+y))^* = 0 \text{ for all } x, y, z \in N.$$

Next putting xt, where $t \in N$, in place of x, we obtain

$$xt((y+z) - (z+y)) = xt((y+z) - (z+y))^* = 0 \text{ for all } x, y, z, t \in N.$$

It follows that

$$xN((y+z) - (z+y)) = xN((y+z) - (z+y))^* = \{0\} \text{ for all } x, y, z \in N. \tag{10}$$

In the light of the $*$-primeness of N and $N \neq \{0\}$, (10) shows that $(y+z) - (z+y) = 0$ for all $y, z \in N$, i.e, $(N, +)$ is abelian. Consequently, N is a commutative ring. This completes the proof of our theorem. $\square$

The following example proves that the hypothesis of $*$-primeness in the previous Theorem is not superfluous.

Example 1. Let $N = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y \in S \right\}$, where S is a zero-symmetric left near-ring such that the addition in S is not commutative. Of course N with matrix addition and matrix multiplication is a left near-ring. Define mappings $*, d, F : N \to N$ such that

$$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, \; d\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } F\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that N is not $*$-prime near-ring and F is a nonzero $*$-generalized derivation associated with a $*$-derivation d. But, since the addition in N is not commutative, N cannot be a commutative ring.

Theorem 2. *Let N be a 3-semiprime near-ring. If N admits a nonzero $*$-generalized derivation F, then N contains a nonzero commutative ring under the operations of N.*

Proof. Given that there exists a $*$-generalized derivation F of N. Arguing as in the proof of Theorem 1, we have relation (4)

$$F(x)y[z,t] = 0 \text{ for all } x, y, z, t \in N. \tag{11}$$

Taking $z = F(x)$ and $y = ty$ in (11) and applying $*$, we find that

$$[F(x), t]^* y^* t^* (F(x))^* = 0 \text{ for all } x, y, t \in N. \tag{12}$$

On the other hand, left multiplying (11) by t and replacing z by $F(x)$, we obtain

$$tF(x)y[F(x), t] = 0 \text{ for all } x, y, t \in N.$$

Now, applying $*$ for this relation, we get

$$[F(x), t]^* y^* (F(x))^* t^* = 0 \text{ for all } x, y, t \in N. \tag{13}$$

Combining the identities (12) and (13), we conclude that

$$[F(x), t]^* y^* [t^*, (F(x))^*] = 0 \text{ for all } x, y, t \in N,$$

which means that

$$[F(x), t]^* y^* [F(x), t]^* = 0 \text{ for all } x, y, t \in N. \tag{14}$$

Once again applying $*$ to (14), we obtain

$$[F(x), t]y[F(x), t] = 0 \text{ for all } x, y, t \in N.$$

Accordingly,

$$[F(x), t]N[F(x), t] = \{0\} \text{ for all } x, t \in N.$$

In view of the semiprimeness of N, the last relation assures that $[F(x), t] = 0$ for all $x, t \in N$; in other words $F(N) \subseteq Z(N)$. As $F \neq 0$, then there is an element $x_0 \in N$ such that $F(x_0) \neq 0$. Now let H the subset of N be defined by:

$$H = \{F(x_0)r \mid r \in N\}.$$

Our goal is to show that H is a commutative ring included in N. Firstly, we have $H \neq \{0\}$. Indeed, suppose that $F(x_0)r = 0$ for all $r \in N$. Right multiplying by $F(x_0)$ we get $F(x_0)rF(x_0) = 0$ for all $r \in N$. Using the semiprimeness of N, we find that $F(x_0) = 0$; a contradiction. Secondly, showing that $(H, +, .)$ is a semiprime left near-ring. As N is a left near-ring and $F(x_0) \in Z(N)$, then with a simple calculation, it is easy to see that H is a nonzero left near-ring. Next, we show that H is a semiprime near-ring. Indeed, let $a \in N$ such that

$$F(x_0)aHF(x_0)a = \{0\}.$$

Then

$$F(x_0)aF(x_0)rF(x_0)a = 0 \text{ for all } r \in N. \tag{15}$$

Right multiplying (15) by $F(x_0)$ and using the fact that $F(x_0) \in Z(N)$, we obtain

$$F(x_0)aF(x_0)rF(x_0)aF(x_0) = \Big(\big(F(x_0)\big)^2 a\Big) r \Big(\big(F(x_0)\big)^2 a\Big) = 0$$

for all $r \in N$. It follows that

$$\Big(\big(F(x_0)\big)^2 a\Big) N \Big(\big(F(x_0)\big)^2 a\Big) = \{0\}.$$

In view of the semiprimeness of N, the preceding equation shows that $\big(F(x_0)\big)^2 a = 0$. Once again left multiplying this equation by ar, where $r \in N$, we get

$$F(x_0)arF(x_0)a = 0 \text{ for all } r \in N. \tag{16}$$

Since N is a semiprime near-ring, (16) implies that $F(x_0)a = 0$. And therefore, H is a 3-semiprime left near-ring. Now, returning to equation (2) of the previous theorem, this equation remains valid in this theorem. So we have $F(x)[y, z] = 0$ for all $x, y, z \in N$. Taking $x = x_0$ and left multiplying by r, we obtain $F(x_0)r[y, z] = 0$ for all $r, y, z \in N$. Setting $t = F(x_0)r$, where $r \in N$, then the last relation yields $t[y, z] = 0$ for all $t \in H, y, z \in N$. In particular, for $y, z \in H$, we have $t[y, z] = 0$ for all $t, y, z \in H$ which implies that

$$[y, z]t[y, z] = 0 \text{ for all } t, y, z \in H.$$

Accordingly,

$$[y, z]H[y, z] = \{0\} \text{ for all } y, z \in H.$$

Using the fact that H is a 3-semiprime near-ring, the last equation shows that $[y, z] = 0$ for all $y, z \in H$. Hence, the multiplicative law of H is commutative; thereby, for all $x, y, z \in H$ we have $(x+x)(y+z) = (y+z)(x+$

x). Arguing as in the proof of Theorem 1, we arrive at $x((y+z)-(z+y)) = 0$ for all $x, y, z \in H$. Replacing x by $((y+z)-(z+y))x$, we get

$$((y+z)-(z+y))x((y+z)-(z+y)) = 0 \text{ for all } x, y, z \in H.$$

Consequently, $((y+z)-(z+y))H((y+z)-(z+y)) = \{0\}$ for all $y, z \in H$ and semiprimeness of H yields that $(y+z)-(z+y) = 0$ for all $y, z \in H$. So, we conclude that H is a commutative ring and hence N contains a nonzero commutative ring. Then, the Theorem is proved. □

Theorem 3. *Let N be a 3-semiprime near-ring and F be a $*$-generalized derivation of N associated with a $*$-derivation d which commutes with $*$. Then F must be a left $*$-multiplier if $F^2 = 0$.*

Proof. By the hypothesis, we have $F^2(xy) = F(x)d(y^*) + F(x)(d(y))^* + xd^2(y) = 0$ for all $x, y \in N$. Replacing x by $F(x)$ in the preceding equation, we obtain

$$F(x)d^2(y) = 0 \text{ for all } x, y \in N. \tag{17}$$

By defining F, we have $F(xyz^*) = F(xy)z + xyd(z^*)$ and $F(xyz^*) = F(x)zy^* + xd(y)z + xyd(z^*)$ for all $x, y, z \in N$. Comparing the above expressions of $F(xyz^*)$, we find that $F(xy)z = F(x)zy^* + xd(y)z$ for all $x, y, z \in N$. Now, taking xt instead of x in (17) and applying the last result, we arrive at

$$xd(t)d^2(y) = 0 \text{ for all } x, t, y \in N.$$

Writing $d(y)$ in the place of t and using ([2], Theorem 3.9), we get $d^2(y)xd^2(y) = 0$ for all $x, y \in N$. By the semiprimeness of N, we conclude that $d^2 = 0$, furthermore $d = 0$ by ([3], Lemma 2.5). And therefore, $F(xy) = F(x)y^*$ for all $x, y \in N$ which means that F is a left $*$-multiplier. □

3. Conditions involving generalized semiderivations

In this section, our near-rings are right near-rings. In order to prove our main theorems, we shall need the following lemmas.

Lemma 1. *Let N be a 3-prime near-ring.*
(i) ([4], Lemma 1.2 (iii)) *If $z \in Z(N)\setminus\{0\}$ and $xz \in Z(N)$, then $x \in Z(N)$.*
(ii) ([4], Lemma 1.5) *If $N \subseteq Z(N)$, then N is a commutative ring.*

Lemma 2. *Let N be an arbitrary right near-ring admitting a semiderivation d associated with a function g. Then N is a zero-symmetric near-ring. Moreover, if N is 3-prime and $d \neq 0$, then $g(0) = 0$.*

Proof. Since N is a right near-ring, we have $0 \cdot x = 0$ for all $x \in N$. Now, let d be a semiderivation of N associated with a function g and let x, y be two arbitrary elements of N, by defining the property of d, we have

$$\begin{aligned} d(x \cdot 0) &= xd(0) + d(x)g(0) = x \cdot 0 + d(x)g(0) \\ &= g(x)d(0) + d(x) \cdot 0 = g(x) \cdot 0 + d(x) \cdot 0 \end{aligned}$$

and

$$\begin{aligned} d((x \cdot 0) \cdot y) &= x \cdot 0 \cdot d(y) + d(x \cdot 0)g(y) = x \cdot 0 + d(x \cdot 0)g(y) \\ &= x \cdot 0 + (x \cdot 0 + d(x)g(0))g(y) \\ &= x \cdot 0 + (g(x) \cdot 0 + d(x) \cdot 0)g(y). \end{aligned}$$

So, $d((x \cdot 0) \cdot y) = x \cdot 0 + x \cdot 0 \cdot g(y) + d(x)g(0)g(y) = x \cdot 0 + g(x) \cdot 0 \cdot g(y) + d(x) \cdot 0 \cdot g(y)$ which implies that $x \cdot 0 + d(x)g(0)g(y) = g(x) \cdot 0 + d(x) \cdot 0$. As $g(x) \cdot 0 + d(x) \cdot 0 = x \cdot 0 + d(x)g(0)$ by defining $d(x \cdot 0)$, and after simplifying, the preceding result shows that

$$d(x)g(0)g(y) = d(x)g(0) \text{ for all } x, y \in N.$$

On the other hand, we have

$$d(x \cdot (0 \cdot y)) = xd(0 \cdot y) + d(x)g(0 \cdot y) = x \cdot 0 + d(x)g(0) \text{ for all } x, y \in N.$$

From the preceding expressions of $d(x \cdot 0 \cdot y)$, we find that $x \cdot 0 + d(x)g(0) = x \cdot 0 + x \cdot 0 + d(x)g(0)g(y)$ for all $x, y \in N$. After simplifying, the latter equation assures that $x \cdot 0 = 0$ for all $x \in N$. Consequently, N is a zero-symmetric near-ring. Assume now that N is 3-prime and $d \neq 0$, by applying the last result, we get $d(x \cdot 0) = 0 = x \cdot 0 + d(x)g(0) = d(x)g(0)$ for all $x \in N$. Replacing x by xt gives $d(xt)g(0) = 0 = \big(g(x)d(t) + d(x)t\big)g(0)$ for all $x, t \in N$, which implies that $d(x)tg(0) = 0$ for all $x, t \in N$. Accordingly, $d(x)Ng(0) = \{0\}$ for all $x \in N$. Since N is 3-prime and $d \neq 0$, we conclude that $g(0) = 0$. $\square$

In right near-ring N, left distributive property does not hold in general. However, the following Lemma has its own significance.

Lemma 3. *Let N be a near-ring admitting a semiderivation d associated with a function g such that $g(xy) = g(x)g(y)$ for all $x, y \in N$. Then N satisfies the following partial distributive law:*

$$x(yd(z) + d(y)g(z)) = xyd(z) + xd(y)g(z) \ \textit{ for all } x, y, z \in N.$$

Proof. We have $d((xy)z) = xyd(z) + d(xy)g(z) = xyd(z) + xd(y)g(z) + d(x)g(y)g(z)$ and $d(x(yz)) = xd(yz) + d(x)g(yz) = x(yd(z) + d(y)g(z)) + d(x)g(yz)$. Comparing these two equations gives the desired result. □

Lemma 4. *Let N be a near-ring. If N admits a semiderivation d associated with an onto map g, then $d(z) \in Z(N)$ for all $z \in Z(N)$.*

Proof. Calculate $d(xz)$ and $d(zx)$ where $x \in N, z \in Z(N)$ and compare. □

Theorem 4. *Let N be a 2-torsion free 3-prime near-ring admitting a generalized semiderivation F associated with a nonzero semiderivation d which is associated with an onto map g such that $g(xy) = g(x)g(y)$ for all $x, y \in N$. If $F([x, y]) = 0$ for all $x, y \in N$, then N is a commutative ring.*

Proof. By the hypothesis given, we have $F(xy) = F(yx)$ for all $x, y \in N$, hence

$$F(x)y + g(x)d(y) = d(y)g(x) + yF(x) \ \text{ for all } x, y \in N.$$

Replacing x by $[r, s]$ in the previous equation, we get

$$g([r, s])d(y) = d(y)g([r, s]) \text{ for all } y, r, s \in N. \tag{18}$$

Substituting $yd(t)$ for y in (18) and invoking Lemma 3, we obtain

$$g([r, s])yd^2(t) + g([r, s])d(y)g(d(t)) = yd^2(t)g([r, s]) + d(y)g(d(t))g([r, s])$$

for all $y, r, s, t \in N$. Taking into account that $g(d(t)) = d(g(t))$ for all $t \in N$ and using (18), we find that

$$g([r, s])yd^2(t) = yd^2(t)g([r, s]) \text{ for all } y, r, s, t \in N. \tag{19}$$

Replacing y by yn in (19) and using (19) again, we get

$$\begin{aligned} g([r, s])ynd^2(t) &= ynd^2(t)g([r, s]) \\ &= y\big(nd^2(t)g([r, s])\big) \\ &= yg([r, s])nd^2(t) \text{ for all } y, n, r, s, t \in N, \end{aligned}$$

which leads to

$$[g([r,s]),y]Nd^2(t) = \{0\} \text{ for all } y,r,s,t \in N. \tag{20}$$

By the 3-primeness of N, equation (20) assures that

$$d^2(t) = 0 \text{ or } g([r,s]) \in Z(N) \text{ for all } r,s,t \in N. \tag{21}$$

Suppose that $d^2(t) = 0$ for all $t \in N$. In particular, we have

$$\begin{aligned} 0 &= d^2(tu) \\ &= d(td(u) + d(t)g(u)) \\ &= 2d(t)d(g(u)) \text{ for all } t,u \in N. \end{aligned}$$

In view of the 2-torsion freeness of N, the above relation implies that $d(t)d(g(u)) = 0$ for all $t,u \in N$. Now, replacing t by xt, where $x \in N$, in the preceding equation, we find that $d(x)td(g(u)) = 0$ for all $x,t,u \in N$. It follows that $d(x)Nd(g(u)) = \{0\}$ for all $x,u \in N$ which, as N is 3-prime and $d \neq 0$, implies that $d(g(u)) = 0 = g(d(u))$ for all $u \in N$. Now, by the defining property of d, we have

$$\begin{aligned} d(vd(u)) &= vd^2(u) + d(v)g(d(u)) \\ &= g(v)d^2(u) + d(v)d(u) \text{ for all } u,v \in N. \end{aligned}$$

Comparing these two expressions for $d(vd(u))$ gives $d(v)d(u) = 0$ for all $u,v \in N$. Taking $v = vt$ and using the same arguments as used above, we conclude that $d = 0$. But this contradicts our assumption that $d \neq 0$; hence, equation (21) forces $g([r,s]) \in Z(N)$ for all $r,s \in N$. Replacing s by sr in the previous result and noting that $[r,sr] = [r,s]r$, we obtain

$$g([r,sr]) = g([r,s]r) = g([r,s])g(r) \in Z(N) \text{ for all } r,s \in N.$$

According to Lemma 1(i), we conclude that

$$g([r,s]) = 0 \text{ or } g(r) \in Z(N) \text{ for all } r,s \in N. \tag{22}$$

Suppose there exist two elements $r,s \in N$ such that $g([r,s]) = 0$. In this case, using Lemma 2 and g is onto, we find that $[r,s] = 0$ which means that $rs = sr$. And therefore $g(rs) = g(sr)$, hence $g(r)g(s) = g(s)g(r)$. So, equation (22) reduces to

$$g(r)g(s) = g(s)g(r) \text{ or } g(r) \in Z(N) \text{ for all } r,s \in N. \tag{23}$$

But it is clear that $g(r) \in Z(N)$ implies $g(r)g(s) = g(s)g(r)$ for all $s \in N$. Consequently, (23) yields $g(r)g(s) = g(s)g(r)$ for all $r, s \in N$. Since g is onto, the latter equation shows that $rs = sr$ for all $r, s \in N$. Which means that $N \subseteq Z(N)$ and our result follows by Lemma 1(ii). $\square$

The following result proves that the conclusion of the previous Theorem is not valid if we replace the product $[x, y]$ by $x \circ y$. Indeed,

Theorem 5. *Let N be a 2-torsion free 3-prime near-ring. There is no nonzero generalized semiderivation F associated with a semiderivation d which is associated with an onto map g such that $g(xy) = g(x)g(y)$ for all $x, y \in N$ satisfies $F(x \circ y) = 0$ for all $x, y \in N$.*

Proof. Suppose that

$$F(x \circ y) = 0 \text{ for all } x, y \in N. \tag{24}$$

It follows that

$$F(xy) + F(yx) = 0 \text{ for all } x, y \in N.$$

By the defining property of F, we get

$$F(x)y + g(x)d(y) + d(y)g(x) + yF(x) = 0 \text{ for all } x, y \in N. \tag{25}$$

Substituting $u \circ v$ for x in (25) and invoking (24), we obtain

$$g(u \circ v)d(y) + d(y)g(u \circ v) = 0 \text{ for all } y, u, v \in N. \tag{26}$$

Taking yt instead of y in (26) and using Lemma 3, we get

$$g(u \circ v)yd(t) + g(u \circ v)d(y)g(t) + d(y)g(t)g(u \circ v) + yd(t)g(u \circ v) = 0$$

for all $u, v, y, t \in N$. Replacing t by $u \circ v$ and using (26), we arrive at

$$g(u \circ v)yd(u \circ v) = -yd(u \circ v)g(u \circ v) \text{ for all } u, v, y \in N. \tag{27}$$

Substituting yt for y in (27), we find that

$$\begin{aligned} g(u \circ v)ytd(u \circ v) &= -ytd(u \circ v)g(u \circ v) \\ &= (-y)(td(u \circ v)g(u \circ v)) \\ &= (-y)(-g(u \circ v)td(u \circ v)) \\ &= (-y)(-g(u \circ v))td(u \circ v) \text{ for all } u, v, y, t \in N. \end{aligned}$$

Hence,

$$\Big(g(u\circ v)y-(-y)(-g(u\circ v))\Big)td(u\circ v)=0 \text{ for all } u,v,y,t\in N. \quad (28)$$

Therefore, (28) can be rewritten as

$$\Big(-(-g(u\circ v))y+y(-g(u\circ v))\Big)Nd(u\circ v)=\{0\} \text{ for all } u,v,y\in N. \quad (29)$$

By 3-primeness of N, equation (29) implies that

$$-g(u\circ v)\in Z(N) \text{ or } d(u\circ v)=0 \text{ for all } u,v\in N. \quad (30)$$

So,

$$-g(u\circ v)\in Z(N) \text{ or } g(d(u\circ v))=0 \text{ for all } u,v\in N. \quad (31)$$

Suppose there are two elements u_0 and v_0 of N such that $-g(u_0\circ v_0)\in Z(N)$. If $d(Z(N))=\{0\}$, then $0=d(-g(u_0\circ v_0))=-d(g(u_0\circ v_0))=d(g(u_0\circ v_0))$ which implies that $g(d(u_0\circ v_0))=0$. On the other hand, if $d(Z(N))\neq\{0\}$, in this case, returning to (26) and replacing y by an element $z_0\in Z(N)$ such that $d(z_0)\neq 0$, also taking $u=u_0$ and $v=v_0$, according to Lemma 4, we obtain $2g(u_0\circ v_0)d(z_0)=0$. By 2-torsion freeness the latter relation shows that

$$g(u_0\circ v_0)d(z_0)=0 \quad (32)$$

Right multiplying (32) by n, where $n\in N$, we obtain

$$g(u_0\circ v_0)nd(z_0)=0 \text{ for all } n\in N.$$

Which implies that

$$g(u_0\circ v_0)Nd(z_0)=\{0\}.$$

In view of the 3-primeness of N and $d(z_0)\neq 0$, we conclude that $g(u_0\circ v_0)=0$, hence $d(g(u_0\circ v_0))=0$ which yields $g(d(u_0\circ v_0))=0$. Thus, in the both cases, i.e $d(Z(N))=\{0\}$ or $d(Z(N))\neq\{0\}$, we find that $g(d(u_0\circ v_0))=0$. Consequently, (31) reduces to

$$g(d(u\circ v))=0 \text{ for all } u,v\in N.$$

According to Lemma 2 and g is onto, the last relation yields

$$d(u\circ v)=0 \text{ for all } u,v\in N. \quad (33)$$

Taking vu instead of v in (33) and using (33), we get

$$(u \circ v)d(u) = 0 \text{ for all } u, v \in N$$

that is,

$$uvd(u) = -vud(u) \text{ for all } u, v \in N.$$

Next putting vt, where $t \in N$, in place of v we arrive at

$$uvtd(u) = (-v)(-u)td(u) \text{ for all } u, v, t \in N.$$

It follows that

$$(uv + v(-u))td(u) = 0 \text{ for all } u, v, t \in N.$$

Taking $-u$ instead of u, hence the last relation can be rewritten as

$$(-uv + vu)Nd(-u) = \{0\} \text{ for all } u, v \in N.$$

Which, in the light of the primeness of N, yields

$$u \in Z(N) \text{ or } d(u) = 0 \text{ for all } u \in N. \tag{34}$$

Let $u_0 \in Z(N)$, from (33) and 2-torsion freeness, we have $d(u_0 \circ v) = 0 = d(u_0 v)$ for all $v \in N$. By defining d, we get $g(u_0)d(v) + d(u_0)v = 0$ for all $v \in N$. Replacing v by vu_0 in the last result, we find that $d(u_0)vu_0 = 0$ for all $v \in N$. Once again using the 3-primeness of N, we conclude that $d(u_0) = 0$ or $u_0 = 0$, given that $d(u_0) = 0$. And therefore, (34) proves that $d(u) = 0$ for all $u \in N$.

Our goal in what follows is to show that $d = 0$ implies $F = 0$. Let $x, y \in N$, by defining F and our hypothesis $d = 0$, equation (24) yields $F(x \circ y) = 0 = F(xy) + F(yx) = xF(y) + yF(x)$ for all $x, y \in N$. Replacing x by $x \circ t$, we get $(x \circ t)F(y) = 0$ for all $x, y, t \in N$, which means that

$$xtF(y) = -txF(y) \text{ for all } x, y, t \in N.$$

Taking tm instead of t in the last equation, where $m \in N$, and using the same equation we arrive at

$$xtmF(y) = (-t)(-x)mF(y) \text{ for all } x, y, t, m \in N.$$

It follows that

$$(xt + t(-x))NF(y) = \{0\} \text{ for all } x, y, t \in N.$$

Putting $-x$ instead of x, we get

$$(-xt + tx)NF(y) = \{0\} \text{ for all } x, y, t \in N.$$

In the light of the 3-primeness of N, we conclude that $N \subseteq Z(N)$ or $F = 0$. But if $N \subseteq Z(N)$, N is a commutative ring by Lemma 1(ii). In this case, returning to (24) and using the fact that N is 2-torsion free, we obtain $F(xy) = 0 = xF(y)$ for all $x, y \in N$. Replacing x by xt, we get $xtF(y) = 0$ for all $x, y, t \in N$. Once again N is 3-prime, the last expression yields $F = 0$, which is contrary to our hypothesis. This completes the proof of our Theorem. □

Theorem 6. *Let N be a 2-torsion free 3-prime near-ring admitting a generalized semiderivation F associated with a nonzero semiderivation d which is associated with an automorphism map g such that $F([x, y]) = [x, y]$ for all $x, y \in N$, then N is a commutative ring.*

Proof. Assume that

$$F([x, y]) = [x, y] \text{ for all } x, y \in N. \tag{35}$$

Replacing y by yx in (35), we get

$$F([x, y])x + g([x, y])d(x) = [x, y]x \text{ for all } x, y \in N. \tag{36}$$

In view of (35) and g is an homomorphism, equation (36) can be rewritten as

$$g(x)g(y)d(x) = g(y)g(x)d(x) \text{ for all } x, y \in N.$$

Once again, since g is onto, the last equation yields

$$g(x)yd(x) = yg(x)d(x) \text{ for all } x, y \in N. \tag{37}$$

Taking yt instead of y in (37) and using (37) again, we find that

$$\begin{aligned} g(x)ytd(x) &= ytg(x)d(x) \\ &= y(tg(x)d(x)) \\ &= yg(x)td(x) \text{ for all } x, y, t \in N \end{aligned}$$

implying that

$$(g(x)y - yg(x))td(x) = 0 \text{ for all } x, y, t \in N$$

that is,

$$[g(x), y]Nd(x) = \{0\} \text{ for all } x, y \in N. \tag{38}$$

By the use of 3-primeness of N, (38) yields

$$g(x) \in Z(N) \text{ or } d(x) = 0 \text{ for all } x \in N \tag{39}$$

in the latter case, we see that if $d(x) = 0$, then $g(d(x)) = d(g(x)) = 0$. So, (39) proves that either

$$g(x) \in Z(N) \text{ or } d(g(x)) = 0 \text{ for all } x \in N.$$

Since g is onto, the above relation shows that $x \in Z(N)$ or $d(x) = 0$ for all $x \in N$. And according to Lemma 4 we conclude that $d(x) \in Z(N)$ for all $x \in N$. Replacing x by xy, gives

$$d(xy) = xd(y) + d(x)g(y) \in Z(N) \text{ for all } x, y \in N. \tag{40}$$

Using Lemma 3, (40) implies that

$$x^2d(x) + xd(x)g(y) = x^2d(x) + d(x)g(y)x \text{ for all } x, y \in N,$$

this is reduced to

$$[g(y), x]Nd(x) = \{0\} \text{ for all } x, y \in N. \tag{41}$$

Since N is 3-prime and g is onto, (41) shows that

$$x \in Z(N) \text{ or } d(x) = 0 \text{ for all } x \in N. \tag{42}$$

If there is an element x_0 of N such that $d(x_0) = 0$, according to the equation (40), we obtain

$$x_0d(y) \in Z(N) \text{ for all } y \in N. \tag{43}$$

Since $d \neq 0$, then Lemma 1(i) assures that $x_0 \in Z(N)$, and therefore (42) reduces to $x \in Z(N)$ for all $x \in N$. The Lemma 1(ii) demonstrates that N is a commutative ring. □

Theorem 7. *Let N be a 2-torsion free 3-prime near-ring. There is no generalized semiderivation F associated with a nonzero semiderivation d which is associated with an automorphism map g such that $F(x \circ y) = x \circ y$ for all $x, y \in N$.*

Proof. Suppose that there is F which indicates the following

$$F(x \circ y) = x \circ y \text{ for all } x, y \in N. \tag{44}$$

Substituting yx for y in (44), because of $x \circ yx = (x \circ y)x$, we obtain

$$F((x \circ y)x) = (x \circ y)x \text{ for all } x, y \in N.$$

By defining F, we get

$$F(x \circ y)x + g(x \circ y)d(x) = (x \circ y)x \text{ for all } x, y \in N. \tag{45}$$

From (44) and (45), we find that

$$g(x \circ y)d(x) = 0 \text{ for all } x, y \in N.$$

As g is an homomorphism, we have

$$g(x)g(y)d(x) = -g(y)g(x)d(x) \text{ for all } x, y \in N.$$

Since g is onto, the last equation shows that

$$g(x)yd(x) = -yg(x)d(x) \text{ for all } x, y \in N. \tag{46}$$

Replacing y by yt in (46) and using (46) again, we have

$$\begin{aligned} g(x)ytd(x) &= -ytg(x)d(x) \\ &= (-y)(tg(x)d(x)) \\ &= (-y)(-g(x)td(x)) \\ &= (-y)(-g(x))td(x) \text{ for all } x, y, t \in N. \end{aligned}$$

But since g is an additive map, we have $-g(x) = g(-x)$ and $g(x)ytd(x) = (-y)g(-x)td(x)$ for all $x, y, t \in N$, which can be rewritten as

$$(-g(-x)y + yg(-x))Nd(x) = \{0\} \text{ for all } x, y \in N.$$

Taking $-x$ instead of x in the latter relation and using the 3-primeness of N, we arrive at

$$g(x) \in Z(N) \text{ or } d(x) = 0 \text{ for all } x \in N.$$

To complete the proof, we only need to consider the same arguments as used after (39) in the proof of Theorem 6, we arrive at a conclusion N is a commutative ring. Now, returning to the assumptions of theorem, we obtain

$$F(xy) = F(x)y + g(x)d(y) = xy \quad \text{for all } x, y \in N.$$

Putting xz instead of x, we arrive at

$$g(x)g(z)d(y) = 0 \text{ for all } x, y, z \in N. \tag{47}$$

Taking into account that g is onto, (47) yields

$$xNd(y) = \{0\} \text{ for all } x, y \in N.$$

In the light of the 3-primeness of N, the laste equation forces that $d = 0$, which is a contradiction. □

The following example shows that the condition of 3-primeness in the hypothesis of Theorems 4, 5, 6 and 7 is crucial.

Example 2. Let S be a noncommutative 2-torsion free zero-symmetric right near-ring. Let us define $N, F, d, g : N \to N$ by:

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y \in S \right\},\ F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } g = d.$$

It can be checked that N with matrix addition and matrix multiplication is not a 3-prime right near-ring and F is a nonzero generalized semiderivation of N associated with a nonzero semiderivation d which is associated with the automorphism map g such that the following situations hold:
(i) $F([A, B]) = 0$, (ii) $F(A \circ B) = 0$, (iii) $F([A, B]) = [A, B]$, (iv) $F(A \circ B) = A \circ B$ for all $A, B \in N$. However, N is not a commutative ring.

References

[1] S. Ali, *On generalized $*$-derivations in $*$-rings*, Palestine J. Math. **1** (2012), 32-37.

[2] M. Ashraf and M. A. Siddeeque, *On $*$-derivations in near-rings with involution*, J. Adv. Res. Pure Math. **6** (2) (2014), 1-12.

[3] H. E. Bell and G. Mason, *On derivations in near-rings*, (G. Betsch editor), North-Holland/ American Elsevier, Amsterdam **137** (1987), 31-35.

[4] H. E. Bell, *On derivations in near-rings, II,* Kluwer Academic Publishers Netherlands (1997), 191-197.

[5] A. Boua and L. Oukhtite, *Semiderivations satisfying certain algebraic identities on prime near-rings*, Asian Eur. J. Math. **6** (3) (2013), 1350043 (8 pages).

[6] M. Bresar, J. Vukman, *On some additive mappings in rings with involution*, Aequat. Math. **38** (1989), 178-185.

[7] Y. Fong, *Derivation in near-ring theory*, Contemp. Mathematics **264** (2000), 91-94.

[8] L. Oukhtite and S. Salhi, *On commutativity of σ-prime rings*, Glasnik Mathematicki **41** (1) (2006), 57-64.

[9] G. Pilz, *Near-Rings. 2nd Edition*, North Holland /American Elsevier, Amsterdam, 1983.

[10] X. K. Wang, *Derivations in prime near-rings*, Proc. Amer. Math. Soc. **121** (1994), 361-366.

CONTACT INFORMATION

Abderrahmane Raji LMACS Laboratory
Faculty of Sciences and Technology
Sultan Moulay Slimane University
P.O.Box 523, 23000, Beni Mellal, Morocco
E-Mail(s): rajiabd2@gmail.com

Received by the editors: 18.08.2019
and in final form 26.04.2021.