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RESEARCH ARTICLE

Interassociativity and three-element doppelsemigroups

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ABSTRACT. In the paper we characterize all interassociates of some non-inverse semigroups and describe up to isomorphism all three-element (strong) doppelsemigroups and their automorphism groups. We prove that there exist 75 pairwise non-isomorphic three-element doppelsemigroups among which 41 doppelsemigroups are commutative. Non-commutative doppelsemigroups are divided into 17 pairs of dual doppelsemigroups. Also up to isomorphism there are 65 strong doppelsemigroups of order 3, and all non-strong doppelsemigroups are not commutative.

Introduction

Given a semigroup $(S,\dashv)$, consider a semigroup $(S,\vdash)$ defined on the same set. We say that $(S,\vdash)$ is an *interassociate* of $(S,\dashv)$ provided $(x\dashv y)\vdash z=x\dashv(y\vdash z)$ and $(x\vdash y)\dashv z=x\vdash(y\dashv z)$ for all $x,y,z\in S$. In 1971, Zupnik [20] coined the term interassociativity in a general groupoid setting. However, he required only one of the two defining equations to hold. The present concept of interassociativity for semigroups originated in 1986 in Drouzy [4], where it is noted that every group is isomorphic to each of its interassociates. In 1983, Gould and Richardson [8] introduced *strong interassociativity*, defined by the above equations along with $x\dashv(y\vdash z)=x\vdash(y\dashv z)$. J. B. Hickey in 1983 [9] and 1986 [10] dealt with the special case of interassociativity in which the operation $\vdash$ is defined by specifying

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$a \in S$ and stipulating that $x \vdash y = x \dashv a \dashv y$ for all $x, y \in S$. Clearly $(S, \vdash)$, which Hickey calls a *variant* of $(S, \dashv)$, is a semigroup that is an interassociate of $(S, \dashv)$. It is easy to show that if $(S, \dashv)$ is a monoid, every interassociate $(S, \vdash)$ must satisfy the condition $x \vdash y = x \dashv a \dashv y$ for some fixed element $a \in S$ and for all $x, y \in S$, that is $(S, \vdash)$ is a variant of $(S, \dashv)$. Methods of constructing interassociates were developed, for semigroups in general and for specific classes of semigroups, in 1997 by Boyd, Gould and Nelson [1]. The description of all interassociates of finite monogenic semigroups was presented by Gould, Linton and Nelson in 2004, see [7].

A *doppelsemigroup* is an algebraic structure $(D, \dashv, \vdash)$ consisting of a non-empty set D equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(x \dashv y) \vdash z = x \dashv (y \vdash z), \tag{D_1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z). \tag{D_2}$$

Thus, we can see that in any doppelsemigroup $(D, \dashv, \vdash)$, $(D, \vdash)$ is an interassociate of $(D, \dashv)$, and conversely, if a semigroup $(D, \vdash)$ is an interassociate of a semigroup $(D, \dashv)$ then $(D, \dashv, \vdash)$ is a doppelsemigroup. A doppelsemigroup $(D, \dashv, \vdash)$ is called *commutative* [13] if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative. A doppelsemigroup $(D, \dashv, \vdash)$ is said to be *strong* [15] if it satisfies the axiom $x \dashv (y \vdash z) = x \vdash (y \dashv z)$.

Many classes of doppelsemigroups were studied by A. Zhuchok and Y. Zhuchok. The free product of doppelsemigroups, the free (strong) doppelsemigroup, the free commutative (strong) doppelsemigroup, the free n-nilpotent (strong) doppelsemigroup and the free rectangular doppelsemigroup were constructed in [13, 15, 19]. Relatively free doppelsemigroups were studied in [17]. The free n-dinilpotent (strong) doppelsemigroup was constructed in [12, 15]. In [14] A. Zhuchok described the free left n-dinilpotent doppelsemigroup. Representations of ordered doppelsemigroups by binary relations were studied by Y. Zhuchok and J. Koppitz [18].

Until now, the task of describing all pairwise non-isomorphic (strong) doppelsemigroups of order 3 has not been solved. The goal of the present work is to characterize all interassociates of some non-inverse semigroups, and use these characterizations in describing up to isomorphism all three-element (strong) doppelsemigroups and their automorphism groups.

1. Preliminaries

A semigroup S is called an *inflation* of its subsemigroup T (see [3], Section 3.2) provided that there is an surjective map $r : S \to T$ such that

$r^2 = r$ and $r(a)r(b) = ab$ for all $a, b \in S$. In the described situation S is often referred to as an *inflation of* T *with an associated map* r (or just *with a map* r*)*. It is immediate that if S is an inflation of T then T is a retract of S (that is the image under a *retraction* r in the sense that $r(a) = a$ for all $a \in T$) and $S^2 \subset T$.

A semigroup S is called *monogenic* if it is generated by some element $a \in S$ in the sense that $S = \{a^n\}_{n\in\mathbb{N}}$. If a monogenic semigroup is infinite then it is isomorphic to the additive semigroup $\mathbb{N}$ of positive integer numbers. A finite monogenic semigroup $S = \langle a\rangle$ also has simple structure, see [11]. There are positive integer numbers r and m called the *index* and the *period* of S such that

- $S = \{a, a^2, \dots, a^{r+m-1}\}$ and $r + m - 1 = |S|$;
- $a^{r+m} = a^r$;
- $C_m := \{a^r, a^{r+1}, \dots, a^{r+m-1}\}$ is a cyclic and maximal subgroup of S with the neutral element $e = a^n \in C_m$ and generator a^{n+1}, where $n \in (m \cdot \mathbb{N}) \cap \{r, \dots, r + m - 1\}$.

We denote by $\mathrm{M}_{r,m}$ a finite monogenic semigroup of index r and period m.

Recall that an *isomorphism* between $(S, *)$ and $(S', \circ)$ is a bijective function $\psi : S \to S'$ such that $\psi(x * y) = \psi(x) \circ \psi(y)$ for all $x, y \in S$. If there exists an isomorphism between $(S, *)$ and $(S', \circ)$ then $(S, *)$ and $(S', \circ)$ are said to be *isomorphic*, denoted $(S, *) \cong (S', \circ)$. An isomorphism between $(S, *)$ and $(S, *)$ is called an *automorphism* of a semigroup $(S, *)$. By $\mathrm{Aut}(S, *)$ we denote the automorphism group of a semigroup $(S, *)$.

An element e of a semigroup $(S, *)$ is called an *idempotent* if $e * e = e$. The semigroup is a *band*, if all its elements are idempotents. Commutative bands are called *semilattices*. By L_n we denote the *linear semilattice* $\{0, 1, \dots, n - 1\}$ of order n, endowed with the operation of minimum.

If $(S, *)$ is a semigroup then the semigroup $(S, *^d)$ with operation $x *^d y = y * x$ is called *dual* to $(S, *)$.

A non-empty subset I of a semigroup $(S, *)$ is called an *ideal* if $I * S \cup S * I \subset I$. An element z of a semigroup S is called a *zero* (resp. a *left zero*, a *right zero*) in S if $a * z = z * a = z$ (resp. $z * a = z$, $a * z = z$) for any $a \in S$. If $(D, \dashv, \vdash)$ is a doppelsemigroup and $z \in D$ is a zero (resp. a left zero, a right zero) of a semigroup $(D, \dashv)$ then (D_1) and (D_2) imply that z is a zero (resp. a left zero, a right zero) of a semigroup $(D, \vdash)$, and vice versa. Thus, any interassociate of a semigroup with zero is a semigroup with zero as well.

A semigroup $(S, *)$ is called a *null semigroup* if there exists an element $z \in S$ such that $x * y = z$ for any $x, y \in S$. In this case z is a zero of S.

All null semigroups on the same set are isomorphic. By O_X we denote a null semigroup on a set X. If X is finite of cardinality $|X| = n$ then instead of O_X we use O_n. It is easy to see that a null semigroup is a strong interassociate of each semigroup with the same zero.

Let X be a set, $z \in X$ and $A \subset X \setminus \{z\}$. Define the binary operation $*$ on X in the following way:

$$x * y = \begin{cases} x & \text{if } y = x \in A \\ z & \text{otherwise.} \end{cases}$$

It is easy to check that a set X endowed with the operation $*$ is a semigroup with zero z, and we denote this semigroup by O_X^A. If $A = X \setminus \{z\}$ then O_X^A is a semilattice. In the case $A = \varnothing$, O_X^A coincides with a null semigroup with zero z. The semigroups O_X^A and O_Y^B are isomorphic if and only if $|X| = |Y|$ and $|A| = |B|$. If X is a finite set of cardinality $|X| = n$ and $|A| = m$ then we use O_n^m instead of O_X^A.

Let $(S, *)$ be a semigroup and $e \notin S$. The binary operation $*$ defined on S can be extended to $S \cup \{e\}$ putting $e * s = s * e = s$ for all $s \in S \cup \{e\}$. The notation $(S, *)^{+1}$ denotes a monoid $(S \cup \{e\}, *)$ obtained from $(S, *)$ by adjoining the extra identity e (regardless of whether $(S, *)$ is or is not a monoid).

Let $(S, *)$ be a semigroup and $0 \notin S$. The binary operation $*$ defined on S can be extended to $S \cup \{0\}$ putting $0 * s = s * 0 = 0$ for all $s \in S \cup \{0\}$. The notation $(S, *)^{+0}$ denotes a semigroup $(S \cup \{0\}, *)$ obtained from $(S, *)$ by adjoining the extra zero 0 (regardless of whether $(S, *)$ has or has not the zero).

Let $(M, *)$ be a monoid with identity e, and $\tilde{1} \notin M$. The binary operation $*$ defined on M can be extended to $M \cup \{\tilde{1}\}$ putting $\tilde{1} * \tilde{1} = e$ and $\tilde{1} * m = m * \tilde{1} = m$ for all $m \in M$. The notation $(M, *)^{\tilde{1}}$ denotes the semigroup obtained from $(M, *)$ by adjoining an extra element $\tilde{1}$. Note that $(M, *)^{\tilde{1}}$ is not a monoid and $(M, *)^{\tilde{1}}$ is an inflation of a monoid $(M, *)$.

Let $(D, \dashv, \vdash)$ be a doppelsemigroup and $0 \notin D$. The binary operations defined on D can be extended to $D \cup \{0\}$ putting $0 \dashv d = d \dashv 0 = 0 = 0 \vdash d = d \vdash 0$ for all $d \in D \cup \{0\}$. The notation $(D, \dashv, \vdash)^{+0}$ denotes a doppelsemigroup $(D \cup \{0\}, \dashv, \vdash)$ obtained from $(D, \dashv, \vdash)$ by adjoining the extra zero 0. If $(D, \dashv, \vdash)$ is a strong doppelsemigroup then $(D, \dashv, \vdash)^{+0}$ is a strong doppelsemigroup as well. It is easy to see that $\mathrm{Aut}((D, \dashv, \vdash)^{+0}) \cong \mathrm{Aut}(D, \dashv, \vdash)$.

A semigroup $(S, *)$ is said to be a *left (right) zero semigroup* if $a * b = a$ $(a * b = b)$ for any $a, b \in S$. By LO_X and RO_X we denote a left zero semigroup and a right zero semigroup on a set X, respectively. It is easy to see that the semigroups LO_X and RO_X are dual. If X is finite of cardinality $|X| = n$ then instead of LO_X and RO_X we use LO_n and RO_n, respectively.

Let X be a set, $A \subset X$ and $0 \notin X$. Define the binary operation $*$ on $X^0 = X \cup \{0\}$ in the following way:

$$x * y = \begin{cases} x & \text{if } y \in A \\ 0 & \text{if } y \in X^0 \setminus A. \end{cases}$$

It is easy to check that a set X^0 endowed with the operation $*$ is a semigroup with zero 0, and we denote this semigroup by $LO^{\sim 0}_{A \leftarrow X}$. If $A = X$ then $LO^{\sim 0}_{A \leftarrow X}$ coincides with LO^{+0}_X. In the case $A = \varnothing$, $LO^{\sim 0}_{A \leftarrow X}$ coincides with a null semigroup O_{X^0} with zero 0. The semigroups $LO^{\sim 0}_{A \leftarrow X}$ and $LO^{\sim 0}_{B \leftarrow Y}$ are isomorphic if and only if $|X| = |Y|$ and $|A| = |B|$. If X is a finite set of cardinality $|X| = n$ and $|A| = m$ then we use $LO^{\sim 0}_{m \leftarrow n}$ instead of $LO^{\sim 0}_{A \leftarrow X}$.

Let X be a set, $A \subset X$ and $0 \notin X$. Define the binary operation $*$ on $X^0 = X \cup \{0\}$ in the following way:

$$x * y = \begin{cases} y & \text{if } x \in A \\ 0 & \text{if } x \in X^0 \setminus A. \end{cases}$$

It is easy to check that a set X^0 endowed with the operation $*$ is a semigroup with zero 0, and we denote this semigroup by $RO^{\sim 0}_{A \leftarrow X}$. If $A = X$ then $RO^{\sim 0}_{A \leftarrow X}$ coincides with RO^{+0}_X. In the case $A = \varnothing$, $RO^{\sim 0}_{A \leftarrow X}$ coincides with a null semigroup on X^0 with zero 0. Semigroups $RO^{\sim 0}_{A \leftarrow X}$ and $RO^{\sim 0}_{B \leftarrow Y}$ are isomorphic if and only if $|X| = |Y|$ and $|A| = |B|$. If X is a finite set of cardinality $|X| = n$ and $|A| = m$ then we use $RO^{\sim 0}_{m \leftarrow n}$ instead of $RO^{\sim 0}_{A \leftarrow X}$.

It is easy to see that the semigroups $LO^{\sim 0}_{A \leftarrow X}$ and $RO^{\sim 0}_{A \leftarrow X}$ are dual.

Let a and c be different elements of a set X. Define the associative binary operation $\dashv^a_c$ on X in the following way:

$$x \dashv^a_c y = \begin{cases} x & \text{if } x \neq c \\ a & \text{if } x = c \text{ and } y \neq c \\ c & \text{if } x = y = c. \end{cases}$$

It follows that $(X, \dashv^a_c)$ is a non-commutative band in which all elements $z \neq c$ are left zeros.

It is not difficult to check that for any different $b, d \in X$, the semigroups $(X, \dashv^a_c)$ and $(X, \dashv^b_d)$ are isomorphic. We denote by LOB_X a model semigroup of the class of semigroups isomorphic to $(X, \dashv^a_c)$. If X is a finite set of cardinality $|X| = n$ then we use LOB_n instead of LOB_X.

The semigroup ROB_X is defined dually.

Let a and c be different elements of a set X. Define the associative binary operation $\vdash^a_c$ on X in the following way:

$$x \vdash^a_c y = \begin{cases} x & \text{if } x \neq c \\ a & \text{if } x = c. \end{cases}$$

It follows that $(X, \vdash^a_c)$ is a non-commutative non-regular semigroup in which all elements $z \neq c$ are left zeros.

It is not difficult to check that for any $b \neq c$, the semigroups $(X, \vdash^a_c)$ and $(X, \vdash^b_c)$ are isomorphic. We denote by $LO_{X\setminus\{c\}\leftarrow X}$ a model semigroup of the class of semigroups isomorphic to $(X, \vdash^a_c)$. If X is a finite set of cardinality $|X| = n$ then we use $LO_{(n-1)\leftarrow n}$ instead of $LO_{X\setminus\{c\}\leftarrow X}$.

Dually we define the semigroups $RO_{X\setminus\{c\}\leftarrow X}$ and $RO_{(n-1)\leftarrow n}$.

A transformation $l : S \to S$ of a semigroup $(S, *)$ is called a *left translation* if $l(x * y) = l(x) * y$ for all $x, y \in S$. By Corollary 2.2. from [1] for any semigroup $(S, *)$ and for any its left translation l, the semigroup $(S, *_l)$, where $x *_l y = x * l(y)$, is an interassociate of $(S, *)$. Thus, $(S, *, *_l)$ is a doppelsemigroup for any left translation $l : S \to S$.

The following lemma was proved in [1].

Lemma 1.1. *Let $(S, *)$ be an inflation of an inverse Clifford semigroup $(A, *)$ and let $r : S \to A$ denote the associated retraction. If $(S, \circ)$ is a semigroup that is an interassociate of $(S, *)$ then A is an ideal of $(S, \circ)$ and $(A, \circ) = (A, *_l)$ for some left translation l of $(A, *)$. Moreover, r is a homomorphism of $(S, \circ)$ onto $(A, \circ)$.*

2. Isomorphisms of doppelsemigroups

A bijective map $\psi : D_1 \to D_2$ is called an *isomorphism of doppelsemigroups* $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ if

$$\psi(a \dashv_1 b) = \psi(a) \dashv_2 \psi(b) \quad \text{and} \quad \psi(a \vdash_1 b) = \psi(a) \vdash_2 \psi(b)$$

for all $a, b \in D_1$.

If there exists an isomorphism between the doppelsemigroups $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ then $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are said to be *isomorphic*, denoted $(D_1, \dashv_1, \vdash_1) \cong (D_2, \dashv_2, \vdash_2)$. An isomorphism $\psi : D \to D$ is called an *automorphism* of a doppelsemigroup $(D, \dashv, \vdash)$. By $\mathrm{Aut}(D, \dashv, \vdash)$ we denote the automorphism group of a doppelsemigroup $(D, \dashv, \vdash)$.

Proposition 2.1. *Let* $(D_1, \dashv_1, \vdash_1)$ *and* $(D_2, \dashv_2, \vdash_2)$ *be doppelsemigroups such that* $(D_1, \dashv_1)$ *and* $(D_2, \dashv_2)$ *are null semigroups. If the semigroups* $(D_1, \vdash_1)$ *and* $(D_2, \vdash_2)$ *are isomorphic then the doppelsemigroups* $(D_1, \dashv_1, \vdash_1)$ *and* $(D_2, \dashv_2, \vdash_2)$ *are isomorphic as well.*

Proof. Let z_1 and z_2 be zeros of null semigroups $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$, respectively. Then z_1 and z_2 are zeros of the semigroups $(D_1, \vdash_1)$ and $(D_2, \vdash_2)$, respectively. Let $\psi : D_1 \to D_2$ is an isomorphism of the semigroups $(D_1, \vdash_1)$ and $(D_2, \vdash_2)$. Since zeros are preserved by isomorphisms of semigroups, $\psi(z_1) = z_2$. Taking into account that $|D_1| = |D_2|$ and any map between two null semigroups of the same order that preserves zeros is a isomorphism of these semigroups, we conclude that $\psi : D_1 \to D_2$ is an isomorphism of the doppelsemigroups $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$. □

Proposition 2.2. *Let* $(D_1, \dashv_1, \vdash_1)$ *and* $(D_2, \dashv_2, \vdash_2)$ *be doppelsemigroups, and* $(D_1, \vdash) \cong (D_1, \vdash_1)$ *implies* $\vdash = \vdash_1$ *for any interassociate* $(D_1, \vdash)$ *of* $(D_1, \dashv_1)$. *If* $(D_2, \dashv_2) \cong (D_1, \dashv_1)$ *and* $(D_2, \vdash_2) \cong (D_1, \vdash_1)$ *then* $(D_2, \dashv_2, \vdash_2) \cong (D_1, \dashv_1, \vdash_1)$.

Proof. Let $\psi : D_2 \to D_1$ be an isomorphism of semigroups $(D_2, \dashv_2)$ and $(D_1, \dashv_1)$. For any $a, b \in D_1$ define the operation $\vdash_\psi$ on D_1 in the following way:

$$a \vdash_\psi b = \psi(\psi^{-1}(a) \vdash_2 \psi^{-1}(b)).$$

It follows that $\psi : D_2 \to D_1$ is an isomorphism from $(D_2, \dashv_2, \vdash_2)$ to $(D_1, \dashv_1, \vdash_\psi)$, and thus $(D_1, \dashv_1, \vdash_\psi)$ is a doppelsemigroup as an isomorphic image of the doppelsemigroup $(D_2, \dashv_2, \vdash_2)$. Taking into account that $(D_1, \vdash_\psi)$ is an interassociate of $(D_1, \dashv_1)$ and $(D_1, \vdash_\psi) \cong (D_2, \vdash_2) \cong (D_1, \vdash_1)$, we conclude that $\vdash_\psi = \vdash_1$. Therefore, $(D_2, \dashv_2, \vdash_2) \cong (D_1, \dashv_1, \vdash_1)$. □

Proposition 2.3. *If* $(D, \dashv, \vdash)$ *is a doppelsemigroup such that* $(D, \dashv)$ *is a null semigroup then* $\mathrm{Aut}(D, \dashv, \vdash) = \mathrm{Aut}(D, \vdash)$.

Proof. Let z be a zero of a null semigroup $(D, \dashv)$. Then z is a zero of $(D, \vdash)$. If $\psi : D \to D$ is an automorphism of $(D, \vdash)$ then $\psi(z) = z$. Using the similar arguments as in the proof of Proposition 2.1, we conclude that $\psi : D \to D$ is an automorphism of $(D, \dashv)$. It follows that $\psi \in \mathrm{Aut}(D, \dashv, \vdash)$. Therefore, $\mathrm{Aut}(D, \dashv, \vdash) = \mathrm{Aut}(D, \vdash)$. □

Using the fact that all bijections of a left (right) zero semigroup are its automorphisms and the similar arguments as in the proof of Proposition 2.3, one can prove the following proposition.

Proposition 2.4. *If* $(D,\dashv,\vdash)$ *is a doppelsemigroup such that the semigroup* $(D,\dashv)$ *is isomorphic to* LO_X^{+0} *or* RO_X^{+0} *then* $\mathrm{Aut}(D,\dashv,\vdash)=\mathrm{Aut}(D,\vdash)$.

3. Interassociates of some non-inverse semigroups

In this section we characterize all interassociates of some non-inverse semigroups which we shall use in section 4 for describing all three-element (strong) (commutative) doppelsemigroups up to isomorphism.

In the following Propositions 3.1 and 3.2 we use Lemma 1.1 to recognize all (strong) interassociates of the semigroups O_X^{+0} and O_X^A.

Given a semigroup $(S,\cdot)$, let $\mathrm{Int}(S,\cdot)$ denote the set of all semigroups that are interassociates of $(S,\cdot)$.

Proposition 3.1. *Let* O_X^{+0} *be a semigroup obtained from a null semigroup* $O_X=(X,\dashv)$ *with zero* z *by adjoining an extra zero* $0\notin X$. *The set* $\mathrm{Int}(O_X^{+0})$ *consists of a null semigroup* $O_{X\cup\{0\}}$ *with zero* 0 *and semigroups* $(X,\vdash)^{+0}$ *for all semigroups* $(X,\vdash)$ *with zero* z. *All interassociates of* O_X^{+0} *are strong.*

Proof. The semigroup O_X^{+0} is an inflation of its subsemilattice $A=\{0,z\}$ with the associated retraction $r:O_X^{+0}\to A$,

$$r(x)=\begin{cases}0, & x=0,\\ z, & x\in X.\end{cases}$$

Let $l:A\to A$ be a left translation of the semilattice $(A,\dashv)$. Then $l(0)=l(0\dashv 0)=l(0)\dashv 0=0$. So, there are two left translations of A: $l_1(x)=0$ and $l_2(x)=x$ for all $x\in A$. Let $(X^0,\vdash)$ be any interassociate of O_X^{+0}, where $X^0=X\cup\{0\}$. By Lemma 1.1, A is an ideal of $(X^0,\vdash)$, r is a homomorphism from $(X^0,\vdash)$ onto $(A,\vdash)$, and the semigroup $(A,\vdash)$ is equal to $(A,\dashv_{l_1})$, where $x\vdash y=x\dashv_{l_1}y=x\dashv l_1(y)=0$ for all $x,y\in A$, or $(A,\vdash)$ is equal to $(A,\dashv_{l_2})$, where $x\vdash y=x\dashv_{l_2}y=x\dashv l_2(y)=x\dashv y$ for all $x,y\in A$. It follows that $(A,\vdash)$ is a null semigroup with zero 0 or $(A,\vdash)=(A,\dashv)$.

If $(A,\vdash)$ is a null semigroup then $r(x\vdash y)=r(x)\vdash r(y)=0$. Therefore, the definition of r implies $x\vdash y=0$ for all $x,y\in X^0$. Consequently, in this case $(X^0,\vdash)$ is a null semigroup with zero 0.

Let $(A,\vdash) = (A,\dashv)$. Taking into account that r is a homomorphic retraction and $A \ni 0$ is an ideal, we conclude that $0 \vdash x = r(0 \vdash x) = r(0) \vdash r(x) = 0 \dashv r(x) = 0$ and $x \vdash 0 = r(x \vdash 0) = r(x) \vdash r(0) = r(x) \dashv 0 = 0$ for all $x \in X^0$. If $x, y \in X$ then $r(x \vdash y) = r(x) \vdash r(y) = z \vdash z = z \dashv z = z$. Thus, the definition of r implies $x \vdash y \in X$ for all $x, y \in X$. Consequently, $(X,\vdash)$ is an interassociate of a null semigroup $(X,\dashv)$ with zero z. It follows that $(X,\vdash)$ is an arbitrary semigroup with zero z, and $(X^0,\vdash) = (X,\vdash)^{+0}$.

To show that all interassociates of O_X^{+0} are strong, it is sufficient to use the following two facts:

- if $(X,\vdash)$ is a strong interassociate of $(X,\dashv)$ then $(X,\vdash)^{+0}$ is a strong interassociate of $(X,\dashv)^{+0}$;
- all interassociates of a null semigroup are strong. □

Proposition 3.2. *A semigroup $(X,\vdash)$ with zero z is an interassociate of O_X^A together with the operation $\dashv$ if and only if the following conditions hold:*

1) *$(A^0,\vdash)$ coincides with $O_{A^0}^B$ for some $B \subset A$, where $A^0 = A \cup \{z\}$;*
2) *$A \vdash (X \setminus A) = (X \setminus A) \vdash A = \{z\}$;*
3) *$X \setminus A$ is a subsemigroup with zero z of $(X,\vdash)$.*

All interassociates of O_X^A are strong.

Proof. Note that O_X^A is an inflation of its subsemilattice A^0 with the associated retraction $r : O_X^A \to A^0$,

$$r(x) = \begin{cases} x, & x \in A, \\ z, & x \notin A. \end{cases}$$

Let $l : A^0 \to A^0$ be a left translation of $(A^0,\dashv)$. Then $l(z) = l(z \dashv z) = l(z) \dashv z = z$. If $a \in A$, $l(a) = b \in A^0$ and $b \neq a$ then $a \dashv b = z$. Therefore, $b = b \dashv b = l(a) \dashv b = l(a \dashv b) = l(z) = z$. It follows that $l(a) \in \{z, a\}$ for any $a \in A$. On the other hand, it is clear that for any $B \subset A$ the map $l_B : A^0 \to A^0$,

$$l_B(x) = \begin{cases} x, & x \in B, \\ z, & x \in A^0 \setminus B \end{cases}$$

is a left translation of A^0.

Let $(X,\vdash)$ be any interassociate of O_X^A. By Lemma 1.1, A^0 is an ideal of $(X,\vdash)$, r is a homomorphism from $(X,\vdash)$ onto $(A^0,\vdash)$, and $(A^0,\vdash)$ is equal to $(A^0,\dashv_{l_B})$, where

$$x \vdash y = x \dashv_{l_B} y = x \dashv l_B(y) = \begin{cases} x & \text{if } x = y \in B, \\ z & \text{otherwise} \end{cases}$$

for all $x, y \in A^0$. This implies $(A^0, \vdash)$ coincides with $O^B_{A^0}$ for $B \subset A$.

Since A^0 is an ideal of $(X, \vdash)$, $a \vdash x,\ x \vdash a \in A^0$ for all $a \in A^0, x \in X \setminus A$. Taking into account that $r(a \vdash x) = r(a) \vdash r(x) = r(a) \vdash z = z$ and $r(x \vdash a) = r(x) \vdash r(a) = z \vdash r(a) = z$ for all $a \in A^0, x \in X \setminus A$, we conclude that $x \vdash a, a \vdash x \in (X \setminus A) \cap A^0 = \{z\}$. Therefore, $A \vdash (X \setminus A) = (X \setminus A) \vdash A = \{z\}$.

Let us show that $X \setminus A$ is a subsemigroup of $(X, \vdash)$. Indeed, since r is a homomorphism, $r((X \setminus A) \vdash (X \setminus A)) = r(X \setminus A) \vdash r(X \setminus A) = \{z\} \vdash \{z\} = \{z\}$, and the definition of r implies $(X \setminus A) \vdash (X \setminus A) \subset X \setminus A$.

Since $(X \setminus A, \dashv)$ is a null semigroup with zero z, $(X \setminus A, \vdash)$ is any semigroup with the same zero z.

To show that a semigroup $(X, \vdash)$ for which the conditions 1)-3) hold is a strong interassociate of O^A_X, it is sufficient to note the following two facts:

- an element $s \in \{x \dashv (y \vdash z), (x \dashv y) \vdash z, x \vdash (y \dashv z), (x \vdash y) \dashv z\}$ is non-zero if and only if $x = y = z \in B$ for some $B \subset A$;
- $b \dashv (b \vdash b) = (b \dashv b) \vdash b = b \vdash (b \dashv b) = (b \vdash b) \dashv b = b$ for any $b \in B$ for some $B \subset A$. □

In the following Proposition 3.3 we recognize all interassociates of the semigroup $(M, \dashv)^{\tilde{1}}$ for any monoid $(M, \dashv)$.

Proposition 3.3. *Let $(M, \dashv)$ be a monoid with identity e, and $M^{\tilde{1}} = M \cup \{\tilde{1}\}$, where $\tilde{1} \notin M$. If $(M^{\tilde{1}}, \vdash)$ is an interassociate of $(M, \dashv)^{\tilde{1}}$ then $(M^{\tilde{1}}, \vdash) = (M, \dashv)^{+1}$ or $(M^{\tilde{1}}, \vdash)$ is a variant of $(M, \dashv)^{\tilde{1}}$ with the sandwich operation $x \vdash y = x \dashv a \dashv y$, where $a = \tilde{1} \vdash \tilde{1} \in M$. If $(M, \dashv)$ is a commutative monoid then all interassociates of $(M, \dashv)^{\tilde{1}}$ are strong interassociate with each other.*

Proof. Let $(M^{\tilde{1}}, \vdash)$ be an interassociate of the semigroup $(M, \dashv)^{\tilde{1}}$. Then for any $x, y \in M$ we have the following equalities:

$$x \vdash y = (x \dashv \tilde{1}) \vdash (\tilde{1} \dashv y) = x \dashv (\tilde{1} \vdash \tilde{1}) \dashv y = x \dashv a \ \dashv y,$$

where $a = \tilde{1} \vdash \tilde{1} \in M^{\tilde{1}}$.

Consider two cases.

(1) Let $a = \tilde{1}$. Then $x \vdash y = x \dashv \tilde{1} \dashv y = x \dashv y$ for all $x, y \in M$. Taking into account that $\tilde{1} \vdash \tilde{1} = \tilde{1}$ and for any $x \in M$ the following equalities hold:

$$x \vdash \tilde{1} = (x \dashv \tilde{1}) \vdash \tilde{1} = x \dashv (\tilde{1} \vdash \tilde{1}) = x \dashv \tilde{1} = x,$$
$$\tilde{1} \vdash x = \tilde{1} \vdash (\tilde{1} \dashv x) = (\tilde{1} \vdash \tilde{1}) \dashv x = \tilde{1} \dashv x = x,$$

we conclude that in this case $(M^{\tilde{1}}, \vdash) = (M, \dashv)^{+1}$.

(2) Let $a \neq \tilde{1}$, and thus $a \in M$. We claim that $\tilde{1} \vdash x, x \vdash \tilde{1} \in M$ for any $x \in M^{\tilde{1}}$. Suppose that $\tilde{1} \vdash c = \tilde{1}$ for some $c \in M$. Then $e = \tilde{1} \dashv \tilde{1} = (\tilde{1} \vdash c) \dashv \tilde{1} = \tilde{1} \vdash (c \dashv \tilde{1}) = \tilde{1} \vdash c = \tilde{1}$, and we have a contradiction. By analogy, $x \vdash \tilde{1} \in M$ for any $x \in M^{\tilde{1}}$.

For any $x \in M^{\tilde{1}}$ we have that

$$x \vdash \tilde{1} = (x \vdash \tilde{1}) \dashv \tilde{1} = x \vdash (\tilde{1} \dashv \tilde{1}) = x \vdash e,$$
$$\tilde{1} \vdash x = \tilde{1} \dashv (\tilde{1} \vdash x) = (\tilde{1} \dashv \tilde{1}) \vdash x = e \vdash x.$$

Taking into account that for $a = \tilde{1} \vdash \tilde{1} \in M$

$$\tilde{1} \vdash \tilde{1} = \tilde{1} \dashv (\tilde{1} \vdash \tilde{1}) \dashv \tilde{1} = \tilde{1} \dashv a \dashv \tilde{1}$$

and for any $x \in M$

$$\tilde{1} \vdash x = e \vdash x = e \dashv a \dashv x = \tilde{1} \dashv a \dashv x,$$
$$x \vdash \tilde{1} = x \vdash e = x \dashv a \dashv e = x \dashv a \dashv \tilde{1},$$

we conclude that $(M^{\tilde{1}}, \vdash)$ is a variant of $(M, \dashv)^{\tilde{1}}$ with the sandwich operation $x \vdash y = x \dashv a \dashv y$, where $a = \tilde{1} \vdash \tilde{1} \in M$.

Let $(M, \dashv)$ be a commutative monoid. Taking into account that for each $a \in M$ the variants with respect to a of $(M, \dashv)^{\tilde{1}}$ and $(M, \dashv)^{+1}$ coincide, and the set of interassociates of $(M, \dashv)^{\tilde{1}}$ consists of $(M, \dashv)^{+1}$ and variants of $(M, \dashv)^{\tilde{1}}$ with respect to all $a \in M$, we conclude that each interassociate of $(M, \dashv)^{\tilde{1}}$ is an interassociate of $(M, \dashv)^{+1}$. Since $(M, \dashv)^{+1}$ is a monoid, all of its interassociates are variants. Consequently, $\mathrm{Int}((M, \dashv)^{\tilde{1}}) = \mathrm{Int}((M, \dashv)^{+1})$. Let $(M^{\tilde{1}}, \vdash_1)$ and $(M^{\tilde{1}}, \vdash_2)$ be any two interassociate of $(M, \dashv)^{+1}$. Then $x \vdash_1 y = x \dashv a_1 \dashv y$ and $x \vdash_2 y = x \dashv a_2 \dashv y$ for some $a_1, a_2 \in M^{\tilde{1}}$ and any $x, y \in M^{\tilde{1}}$. Taking into account that $(M, \dashv)^{+1}$ is commutative and hence $x \vdash_1 (y \vdash_2 z) = x \vdash_1 (y \dashv a_2 \dashv z) = x \dashv a_1 \dashv y \dashv a_2 \dashv z = x \dashv a_2 \dashv y \dashv a_1 \dashv z = x \vdash_2 (y \dashv a_1 \dashv z) = x \vdash_2 (y \vdash_1 z)$, we conclude that $\vdash_1$ and $\vdash_2$ are strong interassociate. ☐

In the following Propositions 3.4 and 3.5 we recognize all (strong) interassociates of the semigroups LO_X^{+0} and RO_X^{+0}.

Proposition 3.4. *The set* $\mathrm{Int}(LO_X^{+0})$ *consists of all semigroups* $LO_{A \leftarrow X}^{\sim 0}$*, where* $A \subset X$*. Any two interassociates of* LO_X^{+0} *are interassociate with each other. The semigroup* $LO_{A \leftarrow X}^{\sim 0}$ *is a strong interassociate of the semigroup* $LO_{B \leftarrow X}^{\sim 0}$ *if and only if* $A = B$ *or* $A = \varnothing$ *or* $B = \varnothing$*.*

Proof. Let $(X^0, \vdash)$ be an interassociate of the semigroup LO_X^{+0} with operation $\dashv$.

If $a \vdash b = 0$ for some $a, b \in X$ then

$$x \vdash b = (x \dashv a) \vdash b = x \dashv (a \vdash b) = x \dashv 0 = 0$$

for any $x \in X^0$.

If $c \vdash d \neq 0$ for some $c, d \in X$ then

$$x \vdash d = (x \dashv c) \vdash d = x \dashv (c \vdash d) = x$$

for any $x \in X^0$.

Let $A = \{a \in X \mid x \vdash a \neq 0 \text{ for any } x \in X\}$. It follows that $(X^0, \vdash)$ coincides with $LO_{A\leftarrow X}^{\sim 0}$.

Let us show that for any $A, B \subset X$ the semigroups $LO_{A\leftarrow X}^{\sim 0}$ with operation $\dashv_A$ and $LO_{B\leftarrow X}^{\sim 0}$ with operation $\vdash_B$ are interassociate with each other.

To prove $x \vdash_B (y \dashv_A z) = (x \vdash_B y) \dashv_A z$ consider the following two cases:

- if $z \in A$ then $x \vdash_B (y \dashv_A z) = x \vdash_B y = (x \vdash_B y) \dashv_A z$ for any $x, y \in X^0$;
- if $z \in X^0 \setminus A$ then $x \vdash_B (y \dashv_A z) = x \vdash_B 0 = 0 = (x \vdash_B y) \dashv_A z$ for any $x, y \in X^0$.

To prove $x \dashv_A (y \vdash_B z) = (x \dashv_A y) \vdash_B z$ consider the following two cases:

- if $z \in B$ then $x \dashv_A (y \vdash_B z) = x \dashv_A y = (x \dashv_A y) \vdash_B z$ for any $x, y \in X^0$;
- if $z \in X^0 \setminus B$ then $x \dashv_A (y \vdash_B z) = x \dashv_A 0 = 0 = (x \dashv_A y) \vdash_B z$ for any $x, y \in X^0$.

Let us prove that a semigroup $LO_{A\leftarrow X}^{\sim 0}$ is a strong interassociate of a semigroup $LO_{B\leftarrow X}^{\sim 0}$ if and only if $A = B$ or $A = \varnothing$ or $B = \varnothing$.

If $A = B$ then $LO_{A\leftarrow X}^{\sim 0} = LO_{B\leftarrow X}^{\sim 0}$. So, $LO_{A\leftarrow X}^{\sim 0}$ is a strong interassociate of a semigroup $LO_{B\leftarrow X}^{\sim 0}$.

If $A = \varnothing$ or $B = \varnothing$ then $LO_{A\leftarrow X}^{\sim 0}$ or $LO_{B\leftarrow X}^{\sim 0}$ is a null semigroup. Since a null semigroup is a strong interassociate of any semigroup with zero, in this case, $LO_{A\leftarrow X}^{\sim 0}$ and $LO_{B\leftarrow X}^{\sim 0}$ are strong interassociate with each other.

Let A and B are different non-empty subsets of X. Show that $LO_{A\leftarrow X}^{\sim 0}$ and $LO_{B\leftarrow X}^{\sim 0}$ are not strong interassociate with each other. For this, it is sufficient to consider the following two cases.

- There are exist $a \in A$ and $b \in B \setminus A$. Then $a \vdash_B (a \dashv_A b) = a \vdash_B 0 = 0$ while $a \dashv_A (a \vdash_B b) = a \dashv_A a = a \neq 0$.
- There are exist $b \in B$ and $a \in A \setminus B$. Then $b \dashv_A (b \vdash_B a) = b \dashv_A 0 = 0$ while $b \vdash_B (b \dashv_A a) = b \vdash_B b = b \neq 0$. □

Taking into account that $(X, \dashv)$ is an interassociate of $(X, \vdash)$ if and only if $(X, \dashv^d)$ is an interassociate of $(X, \vdash^d)$, and for each $A \subset X$ the semigroup $LO^{\sim 0}_{A \leftarrow X}$ is dual to $RO^{\sim 0}_{A \leftarrow X}$, we conclude the following proposition.

Proposition 3.5. *The set* $\mathrm{Int}(RO^{+0}_X)$ *consists of all semigroups* $RO^{\sim 0}_{A \leftarrow X}$, *where* $A \subset X$. *Any two interassociates of* RO^{+0}_X *are interassociate with each other. The semigroup* $RO^{\sim 0}_{A \leftarrow X}$ *is a strong interassociate of the semigroup* $RO^{\sim 0}_{B \leftarrow X}$ *if and only if* $A = B$ *or* $A = \varnothing$ *or* $B = \varnothing$.

Let a and c be different elements of a set X. Consider the semigroup $LOB_X = (X, \dashv^a_c)$, where the binary operation $\dashv^a_c$ on X is defined in the following way:

$$x \dashv^a_c y = \begin{cases} x & \text{if } x \neq c \\ a & \text{if } x = c \text{ and } y \neq c \\ c & \text{if } x = y = c. \end{cases}$$

Proposition 3.6. *If* $(X, \vdash)$ *is an interassociate of* $(X, \dashv^a_c)$ *then* $(X, \vdash) = (X, \dashv^a_c)$ *or* $(X, \vdash) = LO_{X \setminus \{c\} \leftarrow X} = (X, \vdash^a_c)$, *where*

$$x \vdash^a_c y = \begin{cases} x, & x \neq c, \\ a, & x = c. \end{cases}$$

All interassociates of $(X, \dashv^a_c)$ *are strong.*

Proof. Since each element $z \in X \setminus \{c\}$ is a left zero of the semigroup $(X, \dashv^a_c)$, z is a left zero of $(X, \vdash)$.

For each $x \in X$ we have

$$c \vdash x = (c \dashv^a_c c) \vdash x = c \dashv^a_c (c \vdash x) \in \{a, c\}.$$

If $x \neq c$ then for each $y \in X$ the following equalities hold:

$$c \vdash x = c \vdash (x \dashv^a_c y) = (c \vdash x) \dashv^a_c y.$$

It follows that $c \vdash x$ is a left zero, and therefore, $c \vdash x \in X \setminus \{c\}$ for all $x \neq c$. Consequently, $c \vdash x = a$ for all $x \neq c$.

If $c \vdash c = c$ then $(X, \vdash) = (X, \dashv^a_c)$. If $c \vdash c = a$ then $(X, \vdash) = (X, \vdash^a_c)$.

Let us show that $(X, \vdash_c^a)$ is a strong interassociate of $(X, \dashv_c^a)$. Since each element $x \in X \setminus \{c\}$ is a left zero of $(X, \vdash_c^a)$ and $(X, \dashv_c^a)$, $x \dashv_c^a (y \vdash_c^a z) = x = x \vdash_c^a (y \dashv_c^a z)$ for any $x \in X \setminus \{c\}$ and $y, z \in X$. Taking into account that $c \dashv_c^a (y \vdash_c^a z) \in c \dashv_c^a (X \setminus \{c\}) = \{a\}$ and $c \vdash_c^a (y \dashv_c^a z) = a$, we conclude that $c \dashv_c^a (y \vdash_c^a z) = c \vdash_c^a (y \dashv_c^a z)$ for any $y, z \in X$. □

Dually one can characterize all interassociates of the semigroup ROB_X.

4. Three-element doppelsemigroups and their automorphism groups

In this section we describe up to isomorphism all (strong) doppelsemigroups with at most three elements and their automorphism groups.

Firstly, recall some useful facts which we shall often use in this section. In fact, each semigroup $(S, \dashv)$ can be consider as a (strong) doppelsemigroup $(S, \dashv, \dashv)$ with the automorphism group $\mathrm{Aut}(S, \dashv, \dashv) = \mathrm{Aut}(S, \dashv)$, and we denote this *trivial* doppelsemigroup by S. As always, we denote by $(S, \dashv_a)$ a variant of a semigroup $(S, \dashv)$, where $x \dashv_a y = x \dashv a \dashv y$. If the semigroups $(S, \dashv_a)$ and $(S, \dashv_b)$ are variants of a commutative semigroup $(S, \dashv)$ then the doppelsemigroup $(S, \dashv_a, \dashv_b)$ is strong. If semigroup is a monoid then all of its interassociates are variants. A semigroup coincides with each of its interassociates if and only if it is a rectangular band, see [1, Lemma 5.5]. Every group is isomorphic to each of its interassociates, see [4]. Following the algebraic tradition, we take for a model of the class of cyclic groups of order n the multiplicative group $C_n = \{z \in \mathbb{C} : z^n = 1\}$ of n-th roots of 1.

Let $(D_1, \dashv_1, \vdash_1)$ be such a doppelsemigroup that for each doppelsemigroup $(D_2, \dashv_2, \vdash_2)$ the isomorphisms $(D_2, \dashv_2) \cong (D_1, \dashv_1)$ and $(D_2, \vdash_2) \cong (D_1, \vdash_1)$ imply $(D_2, \dashv_2, \vdash_2) \cong (D_1, \dashv_1, \vdash_1)$. If $\mathbb{S}$ and $\mathbb{T}$ are model semigroups of classes of semigroups isomorphic to $(D_1, \dashv_1)$ and $(D_1, \vdash_1)$, respectively, then by $\mathbb{S} \between \mathbb{T}$ we denote a model doppelsemigroup of the class of doppelsemigroups isomorphic to $(D_1, \dashv_1, \vdash_1)$.

Note that if $(D, \dashv, \vdash)$ is a (strong) doppelsemigroup then $(D, \vdash, \dashv)$ is a (strong) doppelsemigroup as well. In general case, the doppelsemigroups $(D, \dashv, \vdash)$ and $(D, \vdash, \dashv)$ are not isomorphic. It is clear that $\mathrm{Aut}(D, \dashv, \vdash) = \mathrm{Aut}(D, \vdash, \dashv)$.

It is well-known that there are exactly five pairwise non-isomorphic semigroups having two elements: C_2, L_2, O_2, LO_2, RO_2.

Consider the cyclic group $C_2 = \{-1, 1\}$ and find up to isomorphism all doppelsemigroups $(D, \dashv, \vdash)$ with $(D, \dashv) \cong C_2$. Because C_2 is a monoid, all of its interassociates are variants. Since every group is isomorphic to each of its interassociates, in this case there are two (strong) doppelsemigroups up to isomorphism: C_2 and $C_2 \between C_2^{-1} = (\{-1,1\},\cdot,\cdot_{-1})$. These doppelsemigroups are not isomorphic. Indeed, let ψ is an isomorphism from $(\{-1,1\},\cdot,\cdot_{-1})$ to $(\{-1,1\},\cdot,\cdot)$. Taking into account that -1 is a neutral element of the group $(\{-1,1\},\cdot_{-1})$ and ψ must preserve the neutral elements of both groups $(\{-1,1\},\cdot)$ and $(\{-1,1\},\cdot_{-1})$ of the doppelsemigroup $(\{-1,1\},\cdot,\cdot_{-1})$, we conclude that $\psi(1) = 1$ and $\psi(-1) = 1$, which contradicts the assertion that ψ is an isomorphism. Since $\mathrm{Aut}(C_2) \cong C_1$, $\mathrm{Aut}(C_2 \between C_2^{-1}) \cong C_1$.

Since LO_2 and RO_2 are rectangular bands, all their interassociates coincide with them, and therefore, in this case there are only two doppelsemigroups: LO_2 and RO_2.

It is well-known that a null semigroup O_X is an interassociate of each semigroup on X with the same zero. Consequently, O_2 has two non-isomorphic interassociates: O_2 and L_2. Taking into account that the semilattice L_2 is the monoid $(\{0,1\},\min)$, we conclude that L_2 has two non-isomorphic interassociates: L_2 and $(\{0,1\},\min_0) = O_2$. By Propositions 2.1 and 2.2, it follows that the last four non-isomorphic doppelsemigroups are O_2, $O_2 \between L_2$, L_2 and $L_2 \between O_2$. Note that commutativity of L_2 implies that all these doppelsemigroups are strong. By Proposition 2.3, $\mathrm{Aut}(L_2 \between O_2) = \mathrm{Aut}(O_2 \between L_2) = \mathrm{Aut}(L_2) \cong C_1$.

Consequently, there exist 6 pairwise non-isomorphic commutative two-element doppelsemigroups and 2 non-isomorphic non-commutative doppelsemigroups of order 2. All two-element doppelsemigroups are strong.

In the following table we present up to isomorphism all two-element doppelsemigroups and their automorphism groups.

D	C_2	O_2	L_2	$C_2 \between C_2^{-1}$	$O_2 \between L_2$	$L_2 \between O_2$	LO_2	RO_2
$\mathrm{Aut}(D)$	C_1	C_1	C_1	C_1	C_1	C_1	C_2	C_2

TABLE 1. Two-element doppelsemigroups and their automorphism groups

In the remaining part of the paper we concentrate on describing up to isomorphism all three-element (strong) doppelsemigroups.

Among 19683 different binary operations on a three-element set S there are exactly 113 operations which are associative. In other words, there exist exactly 113 three-element semigroups, and many of these are isomorphic so that there are essentially only 24 pairwise non-isomorphic semigroups of order 3, see [2, 5, 6].

Among 24 pairwise non-isomorphic semigroups of order 3 there are 12 commutative semigroups. The rest 12 pairwise non-isomorphic non-commutative three-element semigroups are divided into the pairs of dual semigroups that are antiisomorphic. The automorphism groups of dual semigroups coincide.

List of all pairwise non-isomorphic semigroups of order 3 and their automorphism groups are presented in Table 2 and Table 3 taken from [6].

S	C_3	O_3	$\mathrm{M}_{2,2}$	C_2^{+1}	$C_2^{\tilde{1}}$	$\mathrm{M}_{3,1}$	O_2^{+1}	O_2^{+0}	L_3	C_2^{+0}	O_3^2	O_3^1
$\mathrm{Aut}(S)$	C_2	C_2	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_2	C_1

Table 2. Commutative semigroups S of order 3 and their automorphism groups

S	LO_3, RO_3	LO_2^{+0}, RO_2^{+0}	$LO_{1\leftarrow 2}^{\sim 0}$, $RO_{1\leftarrow 2}^{\sim 0}$	LO_2^{+1}, RO_2^{+1}	LOB_3, ROB_3	$LO_{2\leftarrow 3}$, $RO_{2\leftarrow 3}$
$\mathrm{Aut}(S)$	S_3	C_2	C_1	C_2	C_1	C_2

Table 3. Non-commutative three-element semigroups and their automorphism groups

In the sequel, we divide our investigation into cases. In the case of a semigroup S we shall find all doppelsemigroups $(D, \dashv, \vdash)$ such that $(D, \dashv)$ is isomorphic to S.

Case C_3. Up to isomorphism, the multiplicative group $C_3 = \{1, a, a^{-1}\}$, where $a = e^{2\pi i/3}$, is a unique group of order 3. Since C_3 is a monoid, all of its interassociates are variants. Because C_3 is commutative, all of its interassociates are strong. Since every group is isomorphic to each of its interassociates, in this case there are exactly three (strong) doppelsemigroups: C_3, $(C_3, \cdot, \cdot_a)$ and $(C_3, \cdot, \cdot_{a^{-1}})$. It is easy to check the map $\psi : C_3 \to C_3$, $\psi(g) = g^{-1}$ (where g^{-1} is the inverse of g in the group $(C_3, \cdot)$), is an isomorphism from $(C_3, \cdot, \cdot_a)$ to $(C_3, \cdot, \cdot_{a^{-1}})$. We denote by $C_3 \between C_3^{-1}$ the doppelsemigroup $(C_3, \cdot, \cdot_{a^{-1}})$. By the same arguments as for the group C_2, we conclude that the doppelsemigroups C_3 and $C_3 \between C_3^{-1}$ are non-isomorphic. Let $\psi : C_3 \to C_3$ is an automorphism of the doppelsemigroup $(C_3, \cdot, \cdot_{a^{-1}})$. Taking into account that 1 is the identity of the group $(C_3, \cdot)$ and a is the identity of the group $(C_3, \cdot_{a^{-1}})$, we conclude that $\psi(1) = 1$ and $\psi(a) = a$. Consequently, $\psi(a^{-1}) = a^{-1}$, and ψ is the identity automorphism. It follows that $\mathrm{Aut}(C_3 \between C_3^{-1}) \cong C_1$.

Case O_3. A null semigroup O_3 is a (strong) interassociate of each three-element semigroup with the same zero. Thus, up to isomorphism there are the following 12 (strong) doppelsemigroups: O_3, $O_3 \between \mathrm{M}_{3,1}$, $O_3 \between O_2^{+1}$,

$O_3 \between O_2^{+0}$, $O_3 \between L_3$, $O_3 \between C_2^{+0}$, $O_3 \between O_3^2$, $O_3 \between O_3^1$, $O_3 \between LO_2^{+0}$, $O_3 \between RO_2^{+0}$, $O_3 \between LO^{\sim 0}_{1\leftarrow 2}$, $O_3 \between RO^{\sim 0}_{1\leftarrow 2}$. According to Proposition 2.1, up to isomorphism there are no other doppelsemigroups $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong O_3$. By Proposition 2.3, $\mathrm{Aut}(O_3 \between S) \cong \mathrm{Aut}(S)$ for any three-element semigroup S with zero.

Case $\mathrm{M}_{2,2}$. Consider the monogenic semigroup $\mathrm{M}_{2,2} = \{a, a^2, a^3 \mid a^4 = a^2\}$. There are three interassociates of this semigroup: $(\mathrm{M}_{2,2}, *_k)$, where $a^x *_k a^y = a^{x+y+k-2}$ for every $a^x, a^y \in \mathrm{M}_{2,2}$ and $k \in \{1, 2, 3\}$, see [7, Theorem 1.1]. It easy to check that $(\mathrm{M}_{2,2}, *_1) = \{a^2, a^3\}^{+1} \cong C_2^{+1}$, $(\mathrm{M}_{2,2}, *_2) = (\mathrm{M}_{2,2}, *)$ and $(\mathrm{M}_{2,2}, *_3) = \{a^2, a^3\}^{\tilde{1}} \cong C_2^{\tilde{1}}$. So, in this case there are three doppelsemigroups: $\mathrm{M}_{2,2}$, $\mathrm{M}_{2,2} \between C_2^{+1}$ and $\mathrm{M}_{2,2} \between C_2^{\tilde{1}}$. Since all three interassociates of $\mathrm{M}_{2,2}$ are pairwise non-isomorphic, according to Proposition 2.2 we conclude that up to isomorphism there are no other doppelsemigroups $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong \mathrm{M}_{2,2}$. Since $\mathrm{Aut}(\mathrm{M}_{2,2}) \cong C_1$, $\mathrm{Aut}(\mathrm{M}_{2,2} \between C_2^{+1}) \cong C_1$ and $\mathrm{Aut}(\mathrm{M}_{2,2} \between C_2^{\tilde{1}}) \cong C_1$.

Case C_2^{+1}. Since C_2^{+1} is a monoid, all of its interassociates are variants. Let e be an extra identity adjoined to $C_2 = \{-1, 1\}$. Then $(\{-1, 1, e\}, \cdot_e) = C_2^{+1}$, $(\{-1, 1, e\}, \cdot_1) \cong C_2^{\tilde{1}}$ and $(\{-1, 1, e\}, \cdot_{-1}) \cong \mathrm{M}_{2,2}$. Therefore, there are three doppelsemigroups: C_2^{+1}, $C_2^{+1} \between C_2^{\tilde{1}}$ and $C_2^{+1} \between \mathrm{M}_{2,2}$. Since C_2^{+1} is a commutative monoid, all these doppelsemigroups are strong. Taking into account that all three interassociates of C_2^{+1} are pairwise non-isomorphic, by Proposition 2.2 we conclude that up to isomorphism there are no other doppelsemigroups $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong C_2^{+1}$. Since $\mathrm{Aut}(C_2^{+1}) \cong C_1$, $\mathrm{Aut}(C_2^{+1} \between C_2^{\tilde{1}}) \cong C_1$ and $\mathrm{Aut}(C_2^{+1} \between \mathrm{M}_{2,2}) \cong C_1$.

Case $C_2^{\tilde{1}}$. According to Proposition 3.3 the semigroup $C_2^{\tilde{1}}$ has three interassociates. As we have seen in previous cases, these interassociates must be isomorphic to $C_2^{\tilde{1}}$, C_2^{+1} and $\mathrm{M}_{2,2}$. Taking into account that, by Proposition 3.3, all interassociates of $C_2^{\tilde{1}}$ are strong, we conclude that in this case there are three pairwise non-isomorphic strong doppelsemigroups: $C_2^{\tilde{1}}$, $C_2^{\tilde{1}} \between C_2^{+1}$ and $C_2^{\tilde{1}} \between \mathrm{M}_{2,2}$. Since $\mathrm{Aut}(C_2^{\tilde{1}}) \cong C_1$, $\mathrm{Aut}(C_2^{\tilde{1}} \between C_2^{+1}) \cong C_1$ and $\mathrm{Aut}(C_2^{\tilde{1}} \between \mathrm{M}_{2,2}) \cong C_1$.

Case $\mathrm{M}_{3,1}$. Consider the monogenic semigroup $\mathrm{M}_{3,1} = \{a, a^2, a^3 \mid a^4 = a^3\}$. There are three interassociates of this semigroup: $(\mathrm{M}_{3,1}, *_k)$, where $a^x *_k a^y = a^{x+y+k-2}$ for every $a^x, a^y \in \mathrm{M}_{3,1}$ and $k \in \{1, 2, 3\}$, see [7, Theorem 1.1]. It easy to check that $(\mathrm{M}_{3,1}, *_1) = \{a^2, a^3\}^{+1} \cong O_2^{+1}$, $(\mathrm{M}_{3,1}, *_2) = (\mathrm{M}_{3,1}, *)$ and $(\mathrm{M}_{3,1}, *_3) \cong O_3$. So, in this case we have three doppelsemigroups: $\mathrm{M}_{3,1}$, $\mathrm{M}_{3,1} \between O_2^{+1}$ and $\mathrm{M}_{3,1} \between O_3$. Since all three interassociates of $\mathrm{M}_{3,1}$ are pairwise non-isomorphic, according to Proposition 2.2 we conclude that up to isomorphism there are no other doppelsemigroups $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong \mathrm{M}_{3,1}$. Since $\mathrm{Aut}(\mathrm{M}_{3,1}) \cong C_1$, $\mathrm{Aut}(\mathrm{M}_{3,1} \between O_2^{+1}) \cong C_1$ and $\mathrm{Aut}(\mathrm{M}_{3,1} \between O_3) \cong C_1$.

Case O_2^{+1}. Because the semigroup O_2^{+1} is a monoid, all of its interassociates are variants. Since there are only three variants of a three-element semigroup, previous cases imply that these interassociates isomorphic to O_2^{+1}, $\mathrm{M}_{3,1}$ and O_3. Taking into account that O_2^{+1} is a commutative monoid, we conclude that in this case there are three pairwise non-isomorphic strong doppelsemigroups: O_2^{+1}, $O_2^{+1} \between \mathrm{M}_{3,1}$ and $O_2^{+1} \between O_3$. Since $\mathrm{Aut}(O_2^{+1}) \cong C_1$, $\mathrm{Aut}(O_2^{+1} \between \mathrm{M}_{3,1}) \cong C_1$ and $\mathrm{Aut}(O_2^{+1} \between O_3) \cong C_1$.

Case O_2^{+0}. Proposition 3.1 implies that there are three interassociates of semigroup O_2^{+0}, and all these interassociates are strong. They are isomorphic to O_2^{+0}, L_3 and O_3. So, in this case there are three strong doppelsemigroups O_2^{+0}, $O_2^{+0} \between L_3 \cong (O_2 \between L_2)^{+0}$ and $O_2^{+0} \between O_3$. Since all three interassociates of O_2^{+0} are pairwise non-isomorphic, according to Proposition 2.2 we conclude that up to isomorphism there are no other doppelsemigroups $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong O_2^{+0}$. Since $\mathrm{Aut}(O_2^{+0}) \cong C_1$, $\mathrm{Aut}(O_2^{+0} \between L_3) \cong C_1$ and $\mathrm{Aut}(O_2^{+0} \between O_3) \cong C_1$.

Case L_3. Since the linear semilattice L_3 is a monoid, all of its interassociates are variants. Three-element semigroup has only three variants, thus previous cases imply that these interassociates isomorphic to L_3, O_2^{+0} and O_3. Therefore, in this case we have the trivial doppelsemigroup L_3 and two (strong) doppelsemigroups $L_3 \between O_2^{+0} \cong (L_2 \between O_2)^{+0}$ and $L_3 \between O_3$. Since $\mathrm{Aut}(L_3) \cong C_1$, $\mathrm{Aut}(L_3 \between O_2^{+0}) \cong C_1$ and $\mathrm{Aut}(L_3 \between O_3) \cong C_1$.

Case C_2^{+0}. Consider the semigroup C_2^{+0} isomorphic to a commutative monoid $(\{-1, 1, 0\}, \cdot)$ with zero 0. Except a null semigroup O_3, this monoid has two isomorphic variants $(\{-1, 1, 0\}, \cdot)$ and $(\{-1, 1, 0\}, \cdot_{-1})$. In this case there are three (strong) doppelsemigroups: $C_2^{+0} \between O_3$, C_2^{+0} and $(\{-1, 1, 0\}, \cdot, \cdot_{-1})$. These doppelsemigroups are not isomorphic. Indeed, let ψ is an isomorphism from $(\{-1, 1, 0\}, \cdot, \cdot_{-1})$ to $(\{-1, 1, 0\}, \cdot, \cdot)$. Taking into account that -1 is a neutral element of the semigroup $(\{-1, 1, 0\}, \cdot_{-1})$ and ψ must preserve the neutral elements of both semigroups $(\{-1, 1, 0\}, \cdot)$ and $(\{-1, 1, 0\}, \cdot_{-1})$ of the doppelsemigroup $(\{-1, 1, 0\}, \cdot, \cdot_{-1})$, we conclude that $\psi(-1) = 1$ and $\psi(1) = 1$, which contradicts the assertion that ψ is an isomorphism. Taking into account that $(\{-1, 1, 0\}, \cdot_{-1}) \cong (C_2^{-1})^{+0}$, where $C_2^{-1} = (\{-1, 1\}, \cdot_{-1})$, we denote by $C_2^{+0} \between (C_2^{-1})^{+0}$ the doppelsemigroup $(\{-1, 1, 0\}, \cdot, \cdot_{-1})$. It is easy to see that $C_2^{+0} \between (C_2^{-1})^{+0} \cong (C_2 \between C_2^{-1})^{+0}$, and hence $\mathrm{Aut}(C_2^{+0} \between (C_2^{-1})^{+0}) \cong \mathrm{Aut}((C_2 \between C_2^{-1})^{+0}) \cong \mathrm{Aut}(C_2 \between C_2^{-1}) \cong C_1$. Since $\mathrm{Aut}(C_2^{+0}) \cong C_1$, $\mathrm{Aut}(C_2^{+0} \between O_3) \cong C_1$.

Case O_3^2. Consider the non-linear semilattice O_3^2 isomorphic to the semigroup $\{a, b, 0\}$ with the operation $\dashv$:

$$x \dashv y = \begin{cases} x & \text{if } y = x \in \{a, b\}, \\ 0 & \text{otherwise.} \end{cases}$$

According to Proposition 3.2, this semigroup has four (strong) interassociates: O_3^2, O_3, $(\{a, b, 0\}, \vdash_a)$ and $(\{a, b, 0\}, \vdash_b)$, where for $i \in \{a, b\}$

$$x \vdash_i y = \begin{cases} x & \text{if } y = x = i, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that the map $\psi : \{a, b, 0\} \to \{a, b, 0\}$, $\psi(a) = b$, $\psi(b) = a$ and $\psi(0) = 0$, is a doppelsemigroup isomorphism from $(\{a, b, 0\}, \dashv, \vdash_a)$ to $(\{a, b, 0\}, \dashv, \vdash_b)$.

Therefore, in this case there are three pairwise non-isomorphic (strong) doppelsemigroups: O_3^2, $O_3^2 \between O_3$ and $O_3^2 \between O_3^1$. Since $\mathrm{Aut}(O_3^1) \cong C_1$, $\mathrm{Aut}(O_3^2 \between O_3^1) \cong C_1$. By Proposition 2.3, $\mathrm{Aut}(O_3^2 \between O_3) \cong \mathrm{Aut}(O_3 \between O_3^2) \cong \mathrm{Aut}(O_3^2) \cong C_2$.

Case O_3^1. Consider the last commutative semigroup O_3^1 isomorphic to the semigroup $(\{a, b, 0\}, \vdash_a)$ from the previous case. By Proposition 3.2, this semigroup has the same four (strong) interassociates as O_3^2. Show that the doppelsemigroups $(\{a, b, 0\}, \vdash_a, \vdash_a)$ and $(\{a, b, 0\}, \vdash_a, \vdash_b)$ are not isomorphic. Suppose that ψ is an isomorphism from $(\{a, b, 0\}, \vdash_a, \vdash_b)$ to $(\{a, b, 0\}, \vdash_a, \vdash_a)$. Then ψ must preserve a unique non-zero idempotent of these doppelsemigroups. Therefore, $\psi(a) = a$ and $\psi(b) = a$, which contradicts the assertion that ψ is an isomorphism. Denote by $O_3^a \between O_3^b$ the doppelsemigroup $(\{a, b, 0\}, \vdash_a, \vdash_b)$. Thus, in this case we have four non-isomorphic (strong) doppelsemigroups: O_3^1, $O_3^a \between O_3^b$, $O_3^1 \between O_3^2$ and $O_3^1 \between O_3$. Since $\mathrm{Aut}(O_3^a) \cong \mathrm{Aut}(O_3^1) \cong C_1$, $\mathrm{Aut}(O_3^a \between O_3^b) \cong C_1$, $\mathrm{Aut}(O_3^1 \between O_3^2) \cong C_1$ and $\mathrm{Aut}(O_3^1 \between O_3) \cong C_1$.

Let $(D, \dashv, \vdash)$ be a doppelsemigroup. Denote by $(D, \dashv, \vdash)^d$ its *dual* doppelsemigroup $(D, \dashv^d, \vdash^d)$, where $x \dashv^d y = y \dashv x$ and $x \vdash^d y = y \vdash x$. In fact, $(D, \dashv, \vdash)^d$ is a (strong) doppelsemigroup if and only if $(D, \dashv, \vdash)$ is a (strong) doppelsemigroup. So, non-commutative doppelsemigroups are divided into the pairs of dual doppelsemigroups. A map $\psi : D_1 \to D_2$ is a isomorphism from a doppelsemigroup $(D_1, \dashv_1, \vdash_1)$ to $(D_2 \dashv_2, \vdash_2)$ if and only if ψ is a isomorphism from a doppelsemigroup $(D_1, \dashv_1, \vdash_1)^d$ to $(D_2 \dashv_2, \vdash_2)^d$. Thus, $\mathrm{Aut}((D, \dashv, \vdash)^d) = \mathrm{Aut}(D, \dashv, \vdash)$.

It follows that it is sufficient to consider non-commutative three-element semigroups LO_3, LO_2^{+0}, $LO^{\sim 0}_{1\leftarrow 2}$, LO_2^{+1}, LOB_3, $LO_{2\leftarrow 3}$. The cases of semigroups RO_3, RO_2^{+0}, $RO^{\sim 0}_{1\leftarrow 2}$, RO_2^{+1}, ROB_3, $RO_{2\leftarrow 3}$ we shall get using the duality.

Case LO_3. Since LO_3 is a rectangular band, all its interassociates coincide with LO_3, and therefore, in this case there is a unique doppelsemigroup LO_3.

Case LO_2^{+0}. Consider the semigroup LO_2^{+0} isomorphic to $\{a, b, 0\}$ with the operation $\dashv$:

$$x \dashv y = \begin{cases} x & \text{if } y \in \{a, b\}, \\ 0 & \text{if } y = 0. \end{cases}$$

According to Proposition 3.4, this semigroup has four interassociates: LO_2^{+0}, O_3, $(\{a, b, 0\}, \vdash_a)$ and $(\{a, b, 0\}, \vdash_b)$, where for $i \in \{a, b\}$

$$x \vdash_i y = \begin{cases} x & \text{if } y = i, \\ 0 & \text{if } y \neq i. \end{cases}$$

It is easy to check that the map $\psi : \{a, b, 0\} \to \{a, b, 0\}$, $\psi(a) = b$, $\psi(b) = a$ and $\psi(0) = 0$, is a doppelsemigroup isomorphism from $(\{a, b, 0\}, \dashv, \vdash_a)$ to $(\{a, b, 0\}, \dashv, \vdash_b)$. Since $(\{a, b, 0\}, \vdash_a) \cong (\{a, b, 0\}, \vdash_b) \cong LO^{\sim 0}_{1\leftarrow 2}$, denote by $LO_2^{+0} \between LO^{\sim 0}_{1\leftarrow 2}$ the doppelsemigroup $(\{a, b, 0\}, \dashv, \vdash_a) \cong (\{a, b, 0\}, \dashv, \vdash_b)$.

Thus, in this case we have three pairwise non-isomorphic doppelsemigroups: LO_2^{+0}, $LO_2^{+0} \between O_3$ and $LO_2^{+0} \between LO^{\sim 0}_{1\leftarrow 2}$. Consequently, up to isomorphism there are no other doppelsemigroups $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong LO_2^{+0}$. By Proposition 3.4, the doppelsemigroups LO_2^{+0} and $LO_2^{+0} \between O_3$ are strong while $LO_2^{+0} \between LO^{\sim 0}_{1\leftarrow 2}$ is not strong.

According to Proposition 2.4,

$$\mathrm{Aut}(LO_2^{+0} \between LO^{\sim 0}_{1\leftarrow 2}) \cong \mathrm{Aut}(LO^{\sim 0}_{1\leftarrow 2}) \cong C_1.$$

By Proposition 2.3,

$$\mathrm{Aut}(LO_2^{+0} \between O_3) \cong \mathrm{Aut}(O_3 \between LO_2^{+0}) \cong \mathrm{Aut}(LO_2^{+0}) \cong C_2.$$

Case $LO^{\sim 0}_{1\leftarrow 2}$. Consider the semigroup $LO^{\sim 0}_{1\leftarrow 2}$ isomorphic to the semigroup $(\{a, b, 0\}, \vdash_a)$ from the previous case. Since this semigroup is the last semigroup with zero, the previous cases imply that it has the following interassociates: O_3, LO_2^{+0}, and interassociates that isomorphic to $(\{a, b, 0\}, \vdash_a)$.

Consider interassociates of $(\{a,b,0\},\vdash_a)$ that isomorphic to $(\{a,b,0\},\vdash_a)$. Since an isomorphism ψ must preserve a unique right identity a and zero 0, we conclude that $\psi(a)$ must be a right identity and $\psi(0)=0$. Thus, $(\{a,b,0\},\vdash_b)$ is a unique different from $(\{a,b,0\},\vdash_a)$ interassociate isomorphic to $(\{a,b,0\},\vdash_a)$. Show that the doppelsemigroups $(\{a,b,0\},\vdash_a,\vdash_a)$ and $(\{a,b,0\},\vdash_a,\vdash_b)$ are not isomorphic. Suppose that ψ is an isomorphism from $(\{a,b,0\},\vdash_a,\vdash_b)$ to $(\{a,b,0\},\vdash_a,\vdash_a)$. Then ψ must preserve right identities a and b of the semigroups $(\{a,b,0\},\vdash_a)$ and $(\{a,b,0\},\vdash_b)$, respectively. Therefore, $\psi(a)=a$ and $\psi(b)=a$, which contradicts the assertion that ψ is an isomorphism. Denote by $LO^{\sim 0}_{a\leftarrow 2}\between LO^{\sim 0}_{b\leftarrow 2}$ the doppelsemigroup $(\{a,b,0\},\vdash_a,\vdash_b)$. Thus, up to isomorphism, $LO^{\sim 0}_{1\leftarrow 2}$, $LO^{\sim 0}_{1\leftarrow 2}\between O_3$, $LO^{\sim 0}_{1\leftarrow 2}\between LO^{+0}_{2}$ and $LO^{\sim 0}_{a\leftarrow 2}\between LO^{\sim 0}_{b\leftarrow 2}$ are the last four doppelsemigroups with zero. Since $\mathrm{Aut}(LO^{\sim 0}_{a\leftarrow 2})\cong \mathrm{Aut}(LO^{\sim 0}_{1\leftarrow 2})\cong C_1$, $\mathrm{Aut}(LO^{\sim 0}_{1\leftarrow 2}\between LO^{+0}_{2})\cong C_1$, $\mathrm{Aut}(LO^{\sim 0}_{1\leftarrow 2}\between O_3)\cong C_1$ and $\mathrm{Aut}(LO^{\sim 0}_{a\leftarrow 2}\between LO^{\sim 0}_{b\leftarrow 2})\cong C_1$. By Proposition 3.4, the doppelsemigroups $LO^{\sim 0}_{1\leftarrow 2}$ and $LO^{\sim 0}_{1\leftarrow 2}\between O_3$ are strong while $LO^{\sim 0}_{1\leftarrow 2}\between LO^{+0}_{2}$ and $LO^{\sim 0}_{a\leftarrow 2}\between LO^{\sim 0}_{b\leftarrow 2}$ are not strong.

Case LO^{+1}_2. Consider a monoid LO^{+1}_2 with operation $\dashv$ and identity 1, where $LO_2=\{a,b\}$ is a two-element left zero semigroup. Since each interassociate of LO^{+1}_2 is a variant, we conclude that except LO^{+1}_2 there two interassociates: $(\{a,b,1\},\vdash_a)$ and $(\{a,b,1\},\vdash_b)$ isomorphic to $LO_{2\leftarrow 3}$, where for $i\in\{a,b\}$

$$x\vdash_i y=\begin{cases} x, & x\neq 1,\\ i, & x=1.\end{cases}$$

It is easy to check that the map $\psi:\{a,b,1\}\to\{a,b,1\}$, $\psi(a)=b$, $\psi(b)=a$ and $\psi(0)=0$, is a doppelsemigroup isomorphism from $(\{a,b,1\},\dashv,\vdash_a)$ to $(\{a,b,1\},\dashv,\vdash_b)$. Since $1\dashv(b\vdash_a b)=1\dashv b=b$ while $1\vdash_a(b\dashv b)=1\vdash_a b=a\neq b$, the doppelsemigroup $(\{a,b,1\},\dashv,\vdash_a)$ is not strong. Therefore, in this case there are two pairwise non-isomorphic doppelsemigroups: LO^{+1}_2 and $LO^{+1}_2\between LO_{2\leftarrow 3}$. The semigroup LO^{+1}_2 is strong while $LO^{+1}_2\between LO_{2\leftarrow 3}$ is not strong. By Proposition 2.4, $\mathrm{Aut}(LO^{+1}_2\between LO_{2\leftarrow 3})\cong\mathrm{Aut}(LO_{2\leftarrow 3})\cong C_2$.

Case LOB_3. Consider a non-commutative band LOB_3 isomorphic to the semigroup $\{a,b,c\}$ with the operation $\dashv^a_c$, where

$$x\dashv^a_c y=\begin{cases} x & \text{if } x\neq c,\\ a & \text{if } x=c \text{ and } y\neq c,\\ c & \text{if } x=y=c.\end{cases}$$

By Proposition 3.6, LOB_3 has two interassociates isomorphic to LOB_3 and $LO_{2\leftarrow 3}$. According to Proposition 2.1, up to isomorphism there are no other doppelsemigroups $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong LOB_3$. Thus, in this case there are two non-isomorphic doppelsemigroups: LOB_3 and $LOB_3 \between LO_{2\leftarrow 3}$. By Proposition 3.6, these doppelsemigroups are strong. Since $\mathrm{Aut}(LOB_3) \cong C_1$, $\mathrm{Aut}(LOB_3 \between LO_{2\leftarrow 3}) \cong C_1$.

Case $LO_{2\leftarrow 3}$. Finally, consider the last three-element semigroup $LO_{2\leftarrow 3}$ isomorphic to the semigroup $\{a, b, c\}$ with operation $\dashv$ defined as follows:

$$x \dashv y = \begin{cases} x, & x \neq c, \\ a, & x = c. \end{cases}$$

Since this semigroup is the last semigroup, the previous cases imply that it has the following interassociates: LO_2^{+1}, LOB_3, and interassociates that isomorphic to $(\{a, b, c\}, \dashv)$. Consider interassociates of $(\{a, b, c\}, \dashv)$ that isomorphic to $(\{a, b, c\}, \dashv)$. Since a and b are left zeros of $(\{a, b, c\}, \dashv)$, they must be left zeros of each interassociate of $(\{a, b, c\}, \dashv)$. It is clear that there exists only one different from $(\{a, b, c\}, \dashv)$ its interassociate $(\{a, b, c\}, \vdash) \cong (\{a, b, c\}, \dashv)$, where

$$x \vdash y = \begin{cases} x, & x \neq c, \\ b, & x = c. \end{cases}$$

It is easy to check that the map $\psi : \{a, b, c\} \to \{a, b, c\}$, $\psi(a) = b$, $\psi(b) = a$ and $\psi(c) = c$, is a doppelsemigroup isomorphism from $(\{a, b, c\}, \dashv, \dashv)$ to $(\{a, b, c\}, \dashv, \vdash)$. Consequently, $LO_{2\leftarrow 3}$, $LO_{2\leftarrow 3} \between LO_2^{+1}$ and $LO_{2\leftarrow 3} \between LOB_3$ are the last three doppelsemigroups of order 3.

It follows that $\mathrm{Aut}(LO_{2\leftarrow 3} \between LO_2^{+1}) \cong \mathrm{Aut}(LO_2^{+1} \between LO_{2\leftarrow 3}) \cong C_2$ and $\mathrm{Aut}(LO_{2\leftarrow 3} \between LOB_3) \cong \mathrm{Aut}(LOB_3 \between LO_{2\leftarrow 3}) \cong C_1$. Since $LOB_3 \between LO_{2\leftarrow 3}$ is strong, $LO_{2\leftarrow 3} \between LOB_3$ is strong as well. By analogy, $LO_{2\leftarrow 3} \between LO_2^{+1}$ is not strong.

We summarize the obtained results on the pairwise non-isomorphic nontrivial three-element (strong) doppelsemigroups and their automorphism groups in the following Tables 4, 5 and 6.

It follows that we have proved the following theorem.

Theorem 4.1. *There exist* 75 *pairwise non-isomorphic three-element doppelsemigroups among which* 41 *doppelsemigroups are commutative. Non-commutative doppelsemigroups are divided into* 17 *pairs of dual doppelsemigroups. Also up to isomorphism there are* 65 *strong doppelsemigroups of order* 3, *and all non-strong doppelsemigroups are not commutative. There exist exactly* 24 *pairwise non-isomorphic three-element trivial doppelsemigroups.*

D	$C_3 \between C_3^{-1}$	$O_3 \between \mathrm{M}_{3,1}$	$O_3 \between O_2^{+1}$	$O_3 \between O_2^{+0}$	$O_3 \between L_3$	$O_3 \between C_2^{+0}$
Aut(D)	C_1	C_1	C_1	C_1	C_1	C_1
D	$O_3 \between O_3^2$	$O_3 \between O_3^1$	$\mathrm{M}_{2,2} \between C_2^{+1}$	$\mathrm{M}_{2,2} \between C_2^{\tilde{1}}$	$C_2^{+1} \between C_2^{\tilde{1}}$	$C_2^{+1} \between \mathrm{M}_{2,2}$
Aut(D)	C_2	C_1	C_1	C_1	C_1	C_1
D	$C_2^{\tilde{1}} \between \mathrm{M}_{2,2}$	$C_2^{\tilde{1}} \between C_2^{+1}$	$\mathrm{M}_{3,1} \between O_2^{+1}$	$\mathrm{M}_{3,1} \between O_3$	$O_2^{+1} \between \mathrm{M}_{3,1}$	$O_2^{+1} \between O_3$
Aut(D)	C_1	C_1	C_1	C_1	C_1	C_1
D	$(O_2 \between L_2)^{+0}$	$O_2^{+0} \between O_3$	$L_3 \between O_3$	$(L_2 \between O_2)^{+0}$	$(C_2 \between C_2^{-1})^{+0}$	$C_2^{+0} \between O_3$
Aut(D)	C_1	C_1	C_1	C_1	C_1	C_1
D	$O_3^2 \between O_3^1$	$O_3^2 \between O_3$	$O_3^a \between O_3^b$	$O_3^1 \between O_3^2$	$O_3^1 \between O_3$	
Aut(D)	C_1	C_2	C_1	C_1	C_1	

TABLE 4. Three-element (strong) non-trivial commutative doppelsemigroups and their automorphism groups

D	$O_3 \between LO_2^{+0}$ $O_3 \between RO_2^{+0}$	$O_3 \between LO^{\sim 0}_{1\leftarrow 2}$ $O_3 \between RO^{\sim 0}_{1\leftarrow 2}$	$LO_2^{+0} \between O_3$ $RO_2^{+0} \between O_3$	$LO^{\sim 0}_{1\leftarrow 2} \between O_3$ $RO^{\sim 0}_{1\leftarrow 2} \between O_3$	$LOB_3 \between LO_{2\leftarrow 3}$ $ROB_3 \between RO_{2\leftarrow 3}$	$LO_{2\leftarrow 3} \between LOB_3$ $RO_{2\leftarrow 3} \between ROB_3$
Aut(D)	C_2	C_1	C_2	C_1	C_1	C_1

TABLE 5. Three-element non-trivial non-commutative strong doppelsemigroups and their automorphism groups

D	$LO_2^{+0} \between LO^{\sim 0}_{1\leftarrow 2}$ $RO_2^{+0} \between RO^{\sim 0}_{1\leftarrow 2}$	$LO^{\sim 0}_{1\leftarrow 2} \between LO_2^{+0}$ $RO^{\sim 0}_{1\leftarrow 2} \between RO_2^{+0}$	$LO^{\sim 0}_{a\leftarrow 2} \between LO^{\sim 0}_{b\leftarrow 2}$ $RO^{\sim 0}_{a\leftarrow 2} \between RO^{\sim 0}_{b\leftarrow 2}$	$LO_2^{+1} \between LO_{2\leftarrow 3}$ $RO_2^{+1} \between RO_{2\leftarrow 3}$	$LO_{2\leftarrow 3} \between LO_2^{+1}$ $RO_{2\leftarrow 3} \between RO_2^{+1}$
Aut(D)	C_1	C_1	C_1	C_2	C_2

TABLE 6. Three-element (non-commutative) non-strong doppelsemigroups and their automorphism groups

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