© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **32** (2021). Number 1, pp. 138–146
DOI:10.12958/adm1424

Cancellation ideals of a ring extension

S. Tchamna

Communicated by E. I. Zelmanov

Abstract. We study properties of cancellation ideals of ring extensions. Let $R \subseteq S$ be a ring extension. A nonzero S-regular ideal I of R is called a *(quasi)-cancellation ideal* of the ring extension $R \subseteq S$ if whenever $IB = IC$ for two S-regular (finitely generated) R-submodules B and C of S, then $B = C$. We show that a finitely generated ideal I is a cancellation ideal of the ring extension $R \subseteq S$ if and only if I is S-invertible.

1. Introduction and background

Throughout this article, we assume that all rings are commutative with identity. The notion of cancellation ideal for a ring has been studied in [1] and [2]. An ideal I of a ring R is called *cancellation ideal* if whenever $IB = IC$ for two ideals B and C of R, then $B = C$ [2]. A finitely generated ideal is a cancellation ideal if and only if for each maximal ideal M of R, I_M is a regular principal ideal of R_M [1, Theorem 1]. D.D Anderson and D.F Anderson used the notion of cancellation ideal to characterize Prüfer domain. A ring R is a Prüfer domain if and only if every finitely generated nonzero ideal of R is a cancellation ideal[1, Theorem 6]. In this paper, we study the notion of cancellation ideal for ring extensions; which is a generalization of the notion of cancellation ideal for rings. Let $R \subseteq S$ be a ring extension, and let A be an R-submodule of S. The R-submodule A is said to be *S-regular* if $AS = S$[5, Definition 1, p. 84]. For two R-submodules E, F of S, denote by $[E : F]$ the set of all $x \in S$ such that $xF \subseteq E$.

2020 MSC: 13A15, 13A18, 13B02.
Key words and phrases: ring extension, cancellation ideal, pullback diagram.

An R-submodule A of S is said to be S-*invertible*, if there exists an R-submodule B of S such that $AB = R$[5, Definition 3, p 90]. In this case, we write $B = A^{-1}$, and $A^{-1} = [R : A] = \{x \in S : xA \subseteq R\}$ [5, Remark 1.10, p. 90]. For the R-submodule A of S, and for a multiplicative subset τ of R, we denote by $A_{[\tau]}$ the set of all $x \in S$ such that $tx \in A$ for some $t \in \tau$. If $\mathfrak{p}$ is a prime ideal of R, and $\tau = R \setminus \mathfrak{p}$, then $A_{[\mathfrak{p}]}$ denotes the set of all $x \in S$ such that $tx \in A$ for some $t \in \tau$. The set $A_{[\tau]}$ is called the saturation of A by τ. Properties of the saturation of a submodule are studied in [5, p. 18] and [6].

An S-regular ideal I of R is called (quasi)-cancellation ideal of the ring extension $R \subseteq S$ if whenever $IB = IC$ for two S-regular (finitely generated) R-submodules B and C of S, then $B = C$. In section 2, we study properties of (quasi)-cancellation ideals of ring extensions. In Proposition 2.4, we prove that a finitely generated S-regular ideal I of R is a cancellation ideal if and only it is a quasi-cancellation ideal. In Theorem 2.12, we show that for an S-regular finitely generated ideal I of R, the followings are equivalent:

(1) I is a cancellation ideal of the ring extension $R \subseteq S$.
(2) I is an S-invertible ideal of R.
(3) $IR[X]$ is a cancellation ideal of the ring extension $R[X] \subseteq S[X]$.

Remark 1.1. Let $R \subseteq S$ be a ring extension, and let A, B be two R-submodules of S. Then $A = B$ if and only if $A_{[\mathfrak{m}]} = B_{[\mathfrak{m}]}$ for each maximal ideal $\mathfrak{m}$ of R. In fact, if $A = B$, then it clear that $A_{[\mathfrak{m}]} = B_{[\mathfrak{m}]}$ for each maximal ideal $\mathfrak{m}$ of R. Conversely, if $A_{[\mathfrak{m}]} = B_{[\mathfrak{m}]}$ for each $\mathfrak{m} \in \mathcal{M}$, where $\mathcal{M}$ is the set of all maximal ideals of R, then by [5, Remark 5.5, p. 50], we have $A = \cap_{\mathfrak{m}\in\mathcal{M}} A_{[\mathfrak{m}]} = \cap_{\mathfrak{m}\in\mathcal{M}} B_{\mathfrak{m}} = B$.

Let $R \subseteq S$ and $L \subseteq T$ be two ring extensions, and consider the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\ \alpha\ } & L \\ \big\downarrow & & \big\downarrow \\ S & \xrightarrow{\ \Psi\ } & T \end{array}$$

where $\ker \Psi$ is an ideal of R, $\Psi : S \longrightarrow T$ is surjective, the restriction $\alpha : R \longrightarrow L$ of Ψ is also surjective and the vertical mappings are inclusions. When $\ker \Psi$ is a maximal ideal of S, the previous commutative diagram is called *a pullback diagram a type* $\square$. Pullback diagrams of type $\square$ are studied by S. Gabelli and E. Houston in [4].

Lemma 1.2. *Consider the above pullback diagram of type* $\square$. *If* A, B *are two* S*-regular ideals of* R *such that* $\Psi(A) = \Psi(B)$, *then* $A = B$.

Proof. Let A, B be two S-regular ideals of R such that $\Psi(A) = \Psi(B)$. By [7, Remark 1.1], we have $\ker \Psi \subseteq A$ and $\ker \Psi \subseteq B$. Let $a \in A$. Then there exists $b \in B$ such that $\Psi(a) = \Psi(b)$. Hence $a - b \in \ker \Psi \subseteq B$. Thus $a \in B$. This shows that $A \subseteq B$. With the same argument, $B \subseteq A$. Thus $A = B$. $\square$

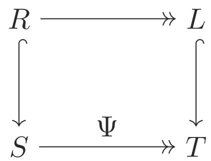
2. Cancellation ideals of ring extensions

In this section, we define and study properties of cancellation ideals of ring extensions.

Definition 2.1. *Let* $R \subseteq S$ *be a ring extension. A nonzero* S*-regular ideal* I *of* R *is called a (quasi)-cancellation ideal of the ring extension* $R \subseteq S$ *if whenever* $IB = IC$ *for two* S*-regular (finitely generated)* R*-submodules* B *and* C *of* S, *then* $B = C$.

The following proposition studies cancellation ideals in pullback diagram of type $\square$. In this article, the Jacobson radical of a ring is denoted $\mathrm{Jac}(R)$.

Proposition 2.2. *Suppose that the following diagram*



is a pullback diagram of type $\square$ *such that* $\ker \Psi \subseteq \mathrm{Jac}(R)$. *Then an* S*-regular ideal* I *of* R *is a cancellation ideal of the extension* $R \subseteq S$ *if and only if* $\Psi(I)$ *is a cancellation ideal of the extension* $L \subseteq T$.

Proof. Suppose that I is a cancellation ideal of the extension $R \subseteq S$. Since $IS = S$, we have $\Psi(I)\Psi(S) = \Psi(S)$. It follows that $\Psi(I)T = T$. Hence $\Psi(I)$ is a T-regular ideal of L. Let E and F be two T-regular L-submodules of T such that $\Psi(I)E = \Psi(I)F$. Let $B = \Psi^{-1}(E)$ and $C = \Psi^{-1}(F)$. Then by [7, Lemma 2.8(1)] B and C are two S-regular ideals of R. Furthermore, $E = \Psi(B)$ and $F = \Psi(C)$ since Ψ is surjective. It follows from the equality $\Psi(I)E = \Psi(I)F$ that $\Psi(I)\Psi(B) = \Psi(I)\Psi(C)$. Hence $\Psi(IB) = \Psi(IC)$. Furthermore, $(IB)S = IS = S$ and $(IC)S = IS = S$. Therefore, by

Lemma 1.2, we have $IB = IC$. Hence $B = C$ since I is a cancellation ideal of the extension $R \subseteq S$. It follows that $E = \Psi(B) = \Psi(C) = F$. This shows that $\Psi(I)$ is a cancellation ideal of the extension $L \subseteq T$.

Conversely, suppose that $\Psi(I)$ is a cancellation ideal of the extension $L \subseteq T$. Let B and C be two S-regular R-submodules of S such that $IB = IC$. Then $\Psi(I)\Psi(B) = \Psi(I)\Psi(C)$. Since $BS = S$, we have $\Psi(B)T = T$. Hence $\Psi(B)$ is a T-regular ideal of L. With the same argument, $\Psi(C)$ is a T-regular ideal of L. It follows that $\Psi(B) = \Psi(C)$ since $\Psi(I)$ is a cancellation ideal of the extension $L \subseteq T$. Therefore, by Lemma 1.2, we have $B = C$. This shows that I is a cancellation ideal of the extension $R \subseteq S$. □

In the next proposition, we give a characterization of a cancellation ideal of a ring extension. This result is an analogue of [3, Proposition 2.1, p. 10] in the case of cancellation ideal of a ring.

Proposition 2.3. *Let $R \subseteq S$ be a ring extension, and let I be an S-regular ideal of R. The following statements are equivalent.*

(1) *I is a (quasi)-cancellation ideal of the ring extension $R \subseteq S$.*
(2) *$[IJ : I] = J$ for any S-regular (finitely generated) R-submodule J of S.*
(3) *If $IJ \subseteq IK$ for two S-regular (finitely generated) R-submodules J and K of S, then $J \subseteq K$.*

Proof. (1) $\Rightarrow$ (2) Suppose that I is a cancellation ideal of the extension $R \subseteq S$, and let J be an S-regular R-submodule of S. The containment $J \subseteq [IJ : I]$ is always true. Let $x \in [IJ : I]$. Then $xI \subseteq IJ$. It follows that $(x, J)I \subseteq IJ$, where (x, J) is the R-submodule of S generated by x and J. Therefore, $(x, J)I = IJ$ since the containment $IJ \subseteq (x, J)I$ is always true. Furthermore, (x, J) is an S-regular R-submodule of S since $J \subseteq (x, J)$. It follows from the definition of a cancellation ideal that $(x, J) = J$. This shows that $x \in J$, and thus $[IJ : I] \subseteq J$. Therefore $[IJ : I] = J$.

(2) $\Rightarrow$ (3) Suppose that the statement (2) is true. Let J and K be two S-regular R-submodules of S. Then by (2), we have $[IK : I] = K$. If $IJ \subseteq IK$, then $J \subseteq [IK : I] = K$.

(3) $\Rightarrow$ (1) This implication is obvious. □

Proposition 2.4. *Let $R \subseteq S$ be a ring extension, and let I be a finitely generated S-regular ideal of R. Then I is a cancellation ideal of $R \subseteq S$ if and only if I is a quasi-cancellation ideal of $R \subseteq S$.*

Proof. Let I be a finitely generated S-regular ideal of R. If I is a cancellation ideal of the extension $R \subseteq S$, then obviously I is an quasi-cancellation ideal of the extension $R \subseteq S$. Conversely, suppose that I is a quasi-cancellation ideal of the extension $R \subseteq S$. Let $a_1, \dots, a_n \in R$ be a set of generators of I. Let B, C be two S-regular R-submodules of S such that $IB \subseteq IC$. Let $b \in B$. Then $bI \subseteq IC$. So, for $1 \leqslant i \leqslant n$, we have $ba_i = \sum_{j=1}^{k} a_j c_{ij}$ with $c_{ij} \in C$ for $1 \leqslant j \leqslant k$. Furthermore, since $CS = S$, there exist $u_1, \dots, u_\ell \in C$ and $s_1, \dots, s_\ell \in S$ such that $u_1 s_1 + \cdots + u_\ell s_\ell = 1$. Let C' be the R-submodule of S generated by the elements of the set $\{u_1, \dots, u_n, c_{ij} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k\}$. Let B_0 be the R-submodule of S generated by b. Then $(B_0 + (u_1, \dots, u_n)R)\, I \subseteq IC'$. It follows from the the equivalence (1) $\Leftrightarrow$ (3) of Proposition 2.3 that $B_0 + (u_1, \dots, u_n)R \subseteq C'$ since $B_0 + (u_1, \dots, u_n)$ and C' are finitely generated S-regular ideal of S. Therefore, $b \in C' \subseteq C$. Hence $B \subseteq C$ since b was arbitrary chosen in B. This shows that I is a cancellation ideal of the extension $R \subseteq S$. □

Lemma 2.5. *Let $R \subseteq S$ be a ring extension, and let $u_1, \dots, u_\ell \in S$. Define the sets $E = (u_1, \dots, u_\ell)R_{[\mathfrak{p}]}$ and $A = (u_1, \dots, u_\ell)R$, where $\mathfrak{p}$ is a prime ideal of R. For any ideal I of R, we have:*

1. *$(AI)_{[\mathfrak{p}]} = (EI)_{[\mathfrak{p}]}$. In particular, $A_{[\mathfrak{p}]} = E_{[\mathfrak{p}]}$.*
2. *$(EI)_{[\mathfrak{p}]} = (EI_{[\mathfrak{p}]})_{[\mathfrak{p}]}$.*

Proof. (1) First, observe that $AI \subseteq EI$. So $(AI)_{[\mathfrak{p}]} \subseteq (EI)_{[\mathfrak{p}]}$. Let $x \in (EI)_{[\mathfrak{p}]}$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $tx \in EI$. Therefore, $tx = \sum_{i=1}^{n} e_i x_i$ for some $e_i \in E$ and $x_i \in I$, $1 \leqslant i \leqslant n$. For each $1 \leqslant i \leqslant n$, write $e_i = \sum_{j=1}^{\ell} u_j y_{ij}$ with $y_{ij} \in R_{[\mathfrak{p}]}$ for $1 \leqslant j \leqslant \ell$. Let $s_{ij} \in R \setminus \mathfrak{p}$ such that $s_{ij} y_{ij} \in R$, $s_i = \prod_{j=1}^{\ell} s_{ij}$ and $s = \prod_{i=1}^{n}$. Then $s_i e_i \in A$. It follows that $(st)x = \sum_{i=1}^{\ell} (se_i) x_i \in AI$. Thus $x \in (AI)_{[\mathfrak{p}]}$ since $st \in R \setminus \mathfrak{p}$. This shows that $(EI)_{[\mathfrak{p}]} \subseteq (AI)_{[\mathfrak{p}]}$. Hence $(AI)_{[\mathfrak{p}]} = (EI)_{[\mathfrak{p}]}$. In particular, if we take $I = R$, then we get $A_{[\mathfrak{p}]} = E_{[\mathfrak{p}]}$.

(2) The containment $(EI)_{[\mathfrak{p}]} \subseteq \big(EI_{[\mathfrak{p}]}\big)_{[\mathfrak{p}]}$ is clear since $EI \subseteq EI_{[\mathfrak{p}]}$. Let $x \in \big(EI_{[\mathfrak{p}]}\big)_{[\mathfrak{p}]}$. Then $tx \in EI_{[\mathfrak{p}]}$ for some $t \in R \setminus \mathfrak{p}$. Thus $tx = \sum_{i=1}^{k} v_i y_i$ with $v_i \in E$ and $y_i \in I_{[\mathfrak{p}]}$ for $1 \leqslant i \leqslant k$. Let $s_i \in R \setminus \mathfrak{p}$ such that $s_i y_i \in I$, and let $s = \prod_{i=1}^{k} s_i$. Then $(st)x = \sum_{i=1}^{k} v_i (s y_i) \in EI$. It follows that $x \in (EI)_{[\mathfrak{p}]}$. Therefore, $(EI)_{[\mathfrak{p}]} = (EI_{[\mathfrak{p}]})_{[\mathfrak{p}]}$ □

Theorem 2.6. *Let $R \subseteq S$ be a ring extension, and let I be a finitely generated S-regular ideal of R. The following statements are equivalent.*

1. *I is a quasi-cancellation ideal of the extension $R \subseteq S$.*

(2) *For each prime ideal $\mathfrak{p}$ of R, and for each S-regular finitely generated $R_{[\mathfrak{p}]}$-submodule E of S, we have $[(EI)_{[\mathfrak{p}]} : I_{[\mathfrak{p}]}] = E_{[\mathfrak{p}]}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that I is a quasi-cancellation ideal of the extension $R \subseteq S$, and let $\mathfrak{p}$ be a prime ideal of R. Let E be a finitely generated S-regular $R_{[\mathfrak{p}]}$-submodule of S. Then $E = (u_1, \ldots, u_\ell)\, R_{[\mathfrak{p}]}$ for some elements $u_1, \ldots, u_\ell$ of S. Let A be the R-submodule of S generated by $u_1, \ldots, u_\ell$. Then by Proposition 2.3 and Proposition 2.4, we have $[AI : I] = A$. It follows from [6, Proposition 2.1(4)] that $[(AI)_\mathfrak{p} : I_{[\mathfrak{p}]}] = A_{[\mathfrak{p}]}$. Hence by Lemma 2.5, we have $[(EI_{[\mathfrak{p}]})_{[\mathfrak{p}]} : I_{[\mathfrak{p}]}] = [(EI)_\mathfrak{p} : I_{[\mathfrak{p}]}] = [(AI)_\mathfrak{p} : I_{[\mathfrak{p}]}] = A_{[\mathfrak{p}]} = E_{[\mathfrak{p}]}$.

(2) $\Rightarrow$ (1) Suppose that the statement (2) is true. Let A be an S-regular finitely generated R-submodule of S, and let $\mathfrak{p}$ be a prime ideal of R. Let $E = AR_{[\mathfrak{p}]}$. Then by Lemma 2.5, we have $(AI)_{[\mathfrak{p}]} = (EI)_{[\mathfrak{p}]}$ and $A_{[\mathfrak{p}]} = E_{[\mathfrak{p}]}$. So, by hypothesis we have $A_{[\mathfrak{p}]} = E_{[\mathfrak{p}]} = [(EI)_{[\mathfrak{p}]} : I_{[\mathfrak{p}]}] = [(AI)_{[\mathfrak{p}]} : I_{[\mathfrak{p}]}]$. But by [6, Proposition 2.1(4)], we have $[(AI)_{[\mathfrak{p}]} : I_{[\mathfrak{p}]}] = [(AI) : I]_{[\mathfrak{p}]}$. Therefore, $A_{[\mathfrak{p}]} = [(AI) : I]_{[\mathfrak{p}]}$ for each prime ideal $\mathfrak{p}$ of R. It follows from Remark 1.1 that $[AI : I] = A$. Therefore, by the equivalence (1) $\Leftrightarrow$ (2) of Proposition 2.3, I is a quasi-cancellation ideal of the extension $R \subseteq S$. □

In their book [5], Knebusch and Zhang defined the notion of Prüfer extension using valuation ring [5, Definition 1, p. 46]. Several characterizations of a Prüfer extension are given in [5, Theorem 5.2, p. 47]. For the purpose of this work, we will use the following: a ring extension $R \subseteq S$ is called Prüfer extension if R is integrally closed in S and $R[\alpha] = R[\alpha^n]$ for any $\alpha \in S$ and any $n \in \mathbb{N}$.

Lemma 2.7. *[5, Theorem 1.13, p. 91] If a ring extension $R \subseteq S$ is a Prüfer extension, then every finitely generated S-regular R-submodule of S is S-invertible.*

Proposition 2.8. *Let $R \subseteq S$ be a ring extension, and let I be an S-regular ideal of R.*

(1) *If I is a cancellation ideal of the extension $R \subseteq S$, then $[I : I] = R$.*
(2) *If the extension $R \subseteq S$ is Prüfer, then the converse of statement* (1) *is also true (i.e. in a Prüfer extension $R \subseteq S$, if I is an S-regular ideal satisfying $[I : I] = R$, then I is a quasi-cancellation ideal).*

Proof. (1) The proof follows directly from the equivalence (1) $\Leftrightarrow$ (2) of Theorem 2.3. It suffices to take $J = R$.

(2) Suppose that the extension $R \subseteq S$ is Prüfer, and let I be an S-regular ideal of R such that $[I : I] = R$. Let A be an S-regular finitely

generated R-submodule of S. Then by Lemma 2.7, A is S-invertible. We show that $A[I : I] = [AI : I]$. Let $x \in [AI : I]$. Then $xI \subseteq AI$. Hence $xIA^{-1} \subseteq I$. Thus $xA^{-1} \subseteq [I : I]$. It follows that $x \in A[I : I]$. On the other hand, let $y = \sum_{i=1}^{k} a_i v_i \in A[I : I]$ with $a_i \in A$ and $v_i \in [I : I]$ for $1 \leqslant i \leqslant k$. Then $v_i I \subseteq I$. Hence $a_i v_i I \subseteq AI$. Therefore, $a_i v_i \in [AI : I]$. So $y = \sum_{i=1}^{k} a_i v_i \in [AI : I]$. This shows that $[AI : I] = A[I : I]$. Hence $[AI : I] = A[I : I] = AR = A$. Hence, by the equivalence (1) $\Leftrightarrow$ (2) of Proposition 2.3, I is a quasi-cancellation ideal of the extension $R \subseteq S$. □

Let $R \subseteq S$ be a ring extension. A nonzero S-regular ideal I of R is called *m-canonical ideal* of the extension $R \subseteq S$ if $[I : [I : J]] = J$ for all S-regular ideal J of R. Properties of m-canonical ideals of a ring extension are studied in [7].

Corollary 2.9. *Any m-canonical ideal of a Prüfer extension is a quasi-cancellation ideal.*

Proof. If I is an m-canonical ideal of a Prüfer extension $R \subseteq S$, then by [7, Proposition 2.3], we have $[I : I] = R$. It follows from Proposition 2.8(2) that I is a quasi-cancellation ideal of the extension $R \subseteq S$. □

Lemma 2.10. *Let $R \subseteq S$ be a ring extension, and let I be an S-regular ideal of R which is a cancellation ideal of $R \subseteq S$. If $I = (x, y) + A$, where A is an ideal of R containing $\mathfrak{m}I$ for some maximal ideal $\mathfrak{m}$ of R, then $I = (x) + A$ or $I = (y) + A$.*

Proof. Let $J = (x^2 + y^2, xy, xA, yA, A^2)R$. Then $IJ = I^3$. Observe that I^2 is S-regular since $I^2S = I(IS) = IS = S$. Also, from the equality $IJ = I^3$ we have $(IJ)S = I^3S = I(IS) = IS = S$. So $JS = S$. This shows that J is an S-regular ideal of R. It follows from the equation $IJ = I^3$ and the fact that I is a cancellation ideal of the extension $R \subseteq S$ that $J = I^2$. Thus $x^2 = t(x^2 + y^2)$ + terms from (xy, xA, yA, A^2), with $t \in R$. Suppose that $t \in \mathfrak{m}$. Then $x^2 \in (y^2, xy, xA, yA, A^2)$, since $tx \in \mathfrak{m}I \in A$. Let $K = (y) + A$. Then $I^2 = IK$. Furthermore, from the equality $IK = I^2$, we have $K(IS) = I^2S$. Hence $KS = S$. Therefore, K is an S-regular ideal of S. It follows that $I = K$ since I is a cancellation ideal of the extension $R \subseteq S$. The rest of the proof is similar to the proof of [2, Lemma]. □

Proposition 2.11. *Let $R \subseteq S$ be a ring extension, and let I be a nonzero S-regular ideal of R. If I is a cancellation ideal of the extension $R \subseteq S$, then for each maximal ideal $\mathfrak{m}$ of R, there exists $a \in R$ such that $I_{[\mathfrak{m}]} = (a)_{[\mathfrak{m}]}$.*

Proof. Suppose that I is a cancellation ideal of the ring extension $R \subseteq S$, and let $\mathfrak{m}$ be a maximal ideal of R. Suppose that $I \subseteq \mathfrak{m}$. Then by Lemma 2.10, and the proof of [2, Theorem], there exists $a \in I$ such that for each $b \in I$, $(1-u)b = ra$ for some $u \in \mathfrak{m}$ and $r \in R$. Therefore, $(1-u)b \in (a)$. So $b \in (a)_{[\mathfrak{m}]}$. This shows that $I_{[\mathfrak{m}]} \in (a)_{[\mathfrak{m}]}$. On the other hand, the equality $(a)_{[\mathfrak{m}]} \subseteq I_{[\mathfrak{m}]}$ is always true. Thus $I_{[\mathfrak{m}]} = (a)_{[\mathfrak{m}]}$. If $I \nsubseteq \mathfrak{m}$, then $I_{[\mathfrak{m}]} = (1)_{[\mathfrak{m}]} = R_{[\mathfrak{m}]}$. In fact, for $x \in R_{[\mathfrak{m}]}$, there exists $s \in R \setminus \mathfrak{m}$ such that $sx \in R$. Thus $(st)x \in I$ for each $t \in I \setminus \mathfrak{m}$. It follows that $x \in I_{[\mathfrak{m}]}$. □

Theorem 2.12. *Let $R \subseteq S$ be a ring extension, and let I be a nonzero finitely generated S-regular ideal of R. The following statements are equivalent.*

(1) *I is a cancellation ideal of the extension $R \subseteq S$.*
(2) *I is a quasi-cancellation ideal of the extension $R \subseteq S$.*
(3) *I is an S-invertible ideal of R.*
(4) *$IR[X]$ is a cancellation ideal of the extension $R[X] \subseteq S[X]$.*

Proof. The equivalence (1) ⇔ (2) is the result of Theorem 2.4.

(1) ⇒ (3) Suppose that I is a cancellation ideal of the extension $R \subseteq S$, and let $\mathfrak{m}$ be a maximal ideal of R. By the previous proposition, $I_{[\mathfrak{m}]} = (a)_{[\mathfrak{m}]}$ for some $a \in R$. It follows that $\big(I_{[\mathfrak{m}]}\big)_{\mathfrak{m}_{[\mathfrak{m}]}} = \big((a)_{[\mathfrak{m}]}\big)_{\mathfrak{m}_{[\mathfrak{m}]}}$. But by [5, Lemma 2.9(b), p. 28], we have $I_{\mathfrak{m}} = \big(I_{[\mathfrak{m}]}\big)_{\mathfrak{m}_{[\mathfrak{m}]}}$ and $(a)_{\mathfrak{m}} = \big((a)_{[\mathfrak{m}]}\big)_{\mathfrak{m}_{[\mathfrak{m}]}}$. Hence $I_{\mathfrak{m}} = (a)_{\mathfrak{m}}$. This shows that I is locally principal. It follows from [5, Proposition 2.3, p. 97] that I is S-invertible.

(3) ⇒ (1) This implication is obvious.

(3) ⇒ (4) Suppose that I is an S-invertible ideal of the extension $R \subseteq S$. First, note that $(IR[X])(S[X]) = S[X]$ since $IS = S$. Hence $IR[X]$ is an $S[X]$-regular ideal of $R[X]$. Let J be the R-submodule of S such that $IJ = R$. Then $(IR[X])(JR[X]) = R[X]$. This shows that $IR[X]$ is an $S[X]$-invertible ideal of $R[X]$. It follows from the equivalence (1) ⇔ (3) that $IR[X]$ is a cancellation ideal of the extension $R[X] \subseteq S[X]$.

(4) ⇒ (1) Suppose that $IR[X]$ is a cancellation ideal of the extension $R[X] \subseteq S[X]$. Let J be an S-regular ideal of R. Then by the equivalence (1) ⇔ (2) of Proposition 2.3, we have $[(IR[X])(JR[X]) : IR[X]] = JR[X]$.

We show that $[IJ : I] = J$. First, note that the containment $J \subseteq [IJ : I]$ is always true. Let $u \in [IJ : I]$. Then $uI \subseteq IJ$. Therefore, $uIR[X] \subseteq (IJ)R[X] \subseteq (IR[X])(JR[X])$. Hence $u \in [(IR[X])(JR[X]) : IR[X]] = JR[X]$. It follows that $u \in JR[X] \cap S = J$. This shows that $[IJ : I] \subseteq J$. Hence $[IJ : I] = J$. It follows from the equivalence (1) ⇔ (2) of Proposition 2.3 that I is a cancellation ideal of the extension $R \subseteq S$. □

Corollary 2.13. *Let $R \subseteq S$ be a ring extension, and let I be a finitely generated S-regular ideal of R. If I is a cancellation ideal of the extension $R \subseteq S$, then $I_{[\mathfrak{m}]}$ is a cancellation ideal of the extension $R_{[\mathfrak{m}]} \subseteq S$ for each maximal ideal $\mathfrak{m}$ of R.*

Proof. Let I be a finitely generated S-regular ideal of R, and let $\mathfrak{m}$ be a maximal ideal of R. Suppose that I is a cancellation ideal of the extension $R \subseteq S$. Then by the previous theorem, I is S-invertible. Let J be an R-submodule of S such that $IJ = R$. Then $I_{[\mathfrak{m}]}J_{[\mathfrak{m}]} \subseteq (IJ)_{[\mathfrak{m}]} \subseteq R_{[\mathfrak{m}]}$. Furthermore, since $IS = S$, there exist $x_i \in I$ and $y_i \in J$, $1 \leqslant i \leqslant \ell$, such that $1 = \sum_{i=1}^{\ell} x_i y_i$. Let $u \in R_{[\mathfrak{m}]}$. There exists $t \in R \setminus \mathfrak{m}$ such that $tu \in R$ and $u = \sum_{i=1}^{\ell}(ux_i)y_i$. But for $1 \leqslant i \leqslant \ell$, $t(ux_i) = (tu)x_i \in I$ since $tu \in R$ and $x_i \in I$. It follows that $ux_i \in I_{[\mathfrak{m}]}$. Therefore, $u = \sum_{i=1}^{\ell}(ux_i)y_i \in I_{[\mathfrak{m}]}J \subseteq I_{[\mathfrak{m}]}J_{[\mathfrak{m}]}$. This shows that $R_{[\mathfrak{m}]} \subseteq I_{[\mathfrak{m}]}J_{[\mathfrak{m}]}$. Thus $I_{[\mathfrak{m}]}J_{[\mathfrak{m}]} = R_{[\mathfrak{m}]}$. Hence $I_{[\mathfrak{m}]}$ is an S-invertible $R_{[\mathfrak{m}]}$-submodule of S. It follows that $I_{[\mathfrak{m}]}$ is a cancellation ideal of the extension $R_{[\mathfrak{m}]} \subseteq S$, since an invertible ideal of ring extension is always a cancellation ideal. □

References

[1] Anderson, D.D., Anderson, D. F. (1984). Some remarks on cancellation ideals. Math Japonica. 29 (6), pp 879-886.

[2] Anderson, D.D., Roitman, M. (1997). A characterization of cancellation ideals. Proc. Amer. Math. Soc. No. 10, pp 2853 - 2854.

[3] Fuchs, L.; Salce, L. (2001) *Modules over non-Noetherian domains.* Mathematical Surveys and Monographs, Vol. 84. American Mathematical Society, Providence, RI. xvi+613 pp.

[4] Gabelli, S; Houston, E. (2000). Ideal theory in pullbacks. In Chapman S. T; Glaz S., eds. *Non-Noetherian commutative ring theory.* Math. Appl., Vol 520. Dordrecht: Kluwer Acad. Publ., pp. 199-227.

[5] Knebusch, M., Zhang, D. (2002). *Manis valuations and Prüfer extensions I.* Lecture Notes in Mathematics, Vol. 1791. Berlin: Springer-Verlag.

[6] Paudel, L., Tchamna, S. (2018). On the saturation of submodules. Algebra and Discrete Mathematics, Vol. 26, No. 1, pp. 110 - 123

[7] Tchamna, S. (2017). Multiplicative canonical ideals of ring extensions. Journal of Algebra and Its Appl. Vol. 16, No. 4:170069.

Contact information

Simplice Tchamna
Department of Mathematics
Georgia College, Milledgeville, GA, USA
E-Mail(s): simplice.tchamna@gcsu.edu

Received by the editors: 26.07.2019
and in final form 30.10.2020.