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Clean coalgebras and clean comodules of finitely generated projective modules*

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Abstract. Let R be a commutative ring with multiplicative identity and P is a finitely generated projective R-module. If P^* is the set of R-module homomorphism from P to R, then the tensor product $P^* \otimes_R P$ can be considered as an R-coalgebra. Furthermore, P and P^* is a comodule over coalgebra $P^* \otimes_R P$. Using the Morita context, this paper give sufficient conditions of clean coalgebra $P^* \otimes_R P$ and clean $P^* \otimes_R P$-comodule P and P^*. These sufficient conditions are determined by the conditions of module P and ring R.

Introduction

In this paper a commutative ring with the identity is denoted by R. A ring R is said to be a clean ring if every element of R can be express as a sum of a unit and an idempotent element [1]. Moreover, a clean ring is one of the subclasses of exchange rings [2, 3]. The previous authors have given some notions of clean rings and exchange rings for example [4–9].

Some authors have been studied the endomorphism structure of R-modules M. It is proved that the ring of a linear transformation of a countable linear vector space is clean [10] and the result is also true for arbitrary vector spaces over a field and any vector space over a division

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ring, it is has been proved in [11] and [12]. An R-module M is called a clean module if $\mathrm{End}_R(M)$ is a clean ring [13]. We recall the important result of [13], i.e., necessary and sufficient conditions of clean elements in an endomorphism ring (see Proposition 2.2 and Proposition 2.3). Furthermore in [14] the authors prove this property in a shorter way by proving that every non M-singular self-injective module M is clean (see Lemma 4).

The structure of comodules and coalgebras has been introduced in 1969 by Sweedler. He introduced a coalgebra over a field as the dualization of algebras over a field. Later, this ground field has been generalized to any ring with multiplicative identity [15]. Furthermore, a comodule over a coalgebra is well-known as a dualization of a module over a ring. For any R-coalgebra C we can construct $C^* = \mathrm{Hom}_R(C, R)$, where C^* is an algebra (ring) over convolution product. We called C^* as a dual algebra of C. Hence, we have an important result, i.e. if M is a right C-comodule, then M is a left module over the dual algebra C^*. Moreover, for any $M, N \in \mathbf{M^C}$ and $\mathrm{End}^C(M, N) \subseteq_{C^*} \mathrm{End}(M)$. Thus, the category of right C-comodule ($\mathbf{M^C}$) is a subcategory of left C^*-module ($_{C^*}\mathbf{M^C}$). In [15], the R-coalgebra C satisfies the α-condition if and only if $\mathbf{M^C}$ is a full subcategory of $_{C^*}\mathbf{M^C}$. Moreover, the $\mathrm{End}^C(M) =_{C^*} \mathrm{End}(M)$ if and only if C is locally projective as an R-module (see [15]).

Recall the structure of comodules and coalgebras [15]. We applied the notions of clean modules to comodule and coalgebra, and introduced the following definition.

Definition 1. Let R be a ring and (C, Δ, ε) an R-coalgebra. A right (left) C-comodule M is called a clean comodule if the endomorphism ring of right (or left) C-comodule M (denoted by $\mathrm{End}^C(M)$ (or $^C\mathrm{End}(M)$) is a clean ring.

Definition 1 means that if C satisfies the α-condition, the right C-comodule M is a clean comodule if and only if the ring $_{C^*}\mathrm{End}(M)$ is a clean ring, since $\mathrm{End}^C(M) =_{C^*} \mathrm{End}(M)$. Since every R-coalgebra C is a right and left comodule over itself, based on Definition 1 we introduce a clean coalgebra.

Definition 2. Let R be a ring. An R-colagebra C is called a clean coalgebra if C is a clean comodule over itself.

If C satisfies the α-condition, Definition 2 means that C is a clean coalgebra if C is clean as a C^*-module. We present the trivial of clean coalgebra. Consider any ring R as an R-coalgebra with the trivial comultiplication $\Delta_T : R \to R \otimes_R R$, $r \mapsto r \otimes r$, and counit $\varepsilon_T : R \to R$,

$r \mapsto r$, for any $r \in R$. Hence, the dual algebra of R, i.e., $(R^*, +, *)$ where $R^* = \mathrm{End}_R(R)$ is isomorphic to the ring R by mapping $f \mapsto f(1)$ for all $f \in R^*$. Then we have a trivial R-coalgebra $(R, \Delta_T, \varepsilon_T)$ is a clean if and only if R is a clean ring.

Furthermore, since every ring R can be considered as the trivial R-coalgebra $(R, \Delta_T, \varepsilon_T)$, any R-module M is a (right and left)comodule over coalgebra $(R, \Delta_T, \varepsilon_T)$ with coaction

$$\varrho^M : M \mapsto M \otimes_R R, m \mapsto m \otimes 1.$$

It implies any R-module M is clean if and only if (M, ϱ^M) is a clean (right and left) R-comodule, since $R \simeq R^*$.

Throughout P is a finitely generated (f.g) projective R-module and P^* is a set of all R-module homomorphism from P to R. In [15] we have already know that for any f.g projective module P and P^*, we can construct tensor product of P and P^*, i.e., $P^* \otimes_R P$. An R-module $P^* \otimes_R P$ is an R-coalgebra by a comultiplication Δ and counit ε as below:

Lemma 1. *[15] Let P be a finitely generated projective R-module with dual basis $p_1, p_2, ..., p_n \in P$ and $\pi_1, \pi_2, ..., \pi_n \in P^*$. The R-module $P^* \otimes_R P$ is an R-coalgebra with the comultiplication and counit defined by*

$$\Delta \colon P^* \otimes_R P \to (P^* \otimes_R P) \otimes_R (P^* \otimes_R P); \quad f \otimes p \mapsto \Sigma_i f \otimes p_i \otimes \pi_i \otimes p$$

and

$$\varepsilon \colon P^* \otimes_R P \to R, \qquad f \otimes p \mapsto f(p).$$

By the properties of the dual basis,

$$(I_{P^* \otimes_R P} \otimes \Delta)\varepsilon(f \otimes p) = \Sigma_i f \otimes p_i \pi_i(p) = f \otimes p,$$

that is ε is a counit and the coassociativity of Δ is proved by the following equality

$$\begin{aligned}(I_{P^* \otimes_R P} \otimes \Delta)\Delta(f \otimes p) &= \Sigma_{i,j} f \otimes p_i \otimes \pi_i \otimes p_j \otimes \pi_j \otimes p \\ &= (\Delta \otimes I_{P^* \otimes_R P})\Delta(f \otimes p).\end{aligned}$$

Furthermore, consider P and P^* as an R-module, then P and P^* respectively can be consider as a right and left comodule over R-coalgebra $P^* \otimes_R P$. By using the Morita context, which is we refer to [16], in this paper we investigate the sufficient conditions of clean R-coalgebra $P^* \otimes_R P$

and the cleanness of P and P^* as a $P^* \otimes_R P$-comodule. In Morita context we already know there are relationship between the structure of P, P^*, R and $S = \mathrm{End}_R(P)$ [16]. The following theorem explain the relationship between P and its dual, in which it is important to prove our main result.

Theorem 1. *[16] Let R be a ring, P be a right R-module, $S = \mathrm{End}_R(P)$ and $Q = P^* = \mathrm{Hom}_R(P, R)$. If P is a generator in R-MOD, then*

1) *$\alpha : Q \otimes_S P \to R$ is an (R, R)-isomorphism;*
2) *$Q \simeq \mathrm{Hom}_S({}_SP, {}_SS)$ as (R, S)-bimodules;*
3) *$P \simeq \mathrm{Hom}_S(Q_S, S_S)$ as (S, R)-bimodules;*
4) *$R \simeq \mathrm{End}({}_SP) \simeq \mathrm{End}(Q_S)$ as rings.*

Theorem 2. *[16] Let R be a ring, P be a right R-module, $S = \mathrm{End}_R(P)$ and $Q = P^* = \mathrm{Hom}_R(P, R)$. If P is finitely generated projective in R-MOD, then*

1) *$\beta : P \otimes_R Q \to S$ is an (S, S)-isomorphism;*
2) *$Q \simeq \mathrm{Hom}_R(P_R, R_R)$ as (R, S)-bimodules;*
3) *$P \simeq \mathrm{Hom}_R({}_RQ, {}_RR)$ as (S, R)-bimodules;*
4) *$S \simeq \mathrm{End}(P_R) \simeq \mathrm{End}({}_RQ)$ as rings.*

Since the cleanness of coalgebra and comodule are determined by the structure of its endomorphism, using Theorem 2 and Theorem 1, we observe when $\mathrm{End}_{(P^* \otimes_R P)^*}(P^* \otimes_R P), \mathrm{End}_{(P^* \otimes_R P)^*}(P)$ and $\mathrm{End}_{(P^* \otimes_R P)^*}(P^*)$ are clean.

1. The clean R-coalgebra $P^* \otimes_R P$

Let P be an R-module. Here, we can construct tensor product of P and P^*. Furthermore, since R is a commutative ring, $P^* \otimes_R P \cong P \otimes_R P^*$ as an R-module. In this section we give some results which are related to some conditions when the R-coalgebra $P^* \otimes_R P$ is clean. Let P be a finitely generated projective R-module with basis $p_1, p_2, ..., p_n \in P$ and dual basis $\pi_1, \pi_2, ..., \pi_n \in P^*$. Based on Theorem 2 we have $P \otimes_R P^* \cong \mathrm{End}_R(P)$ as an (S, S)-bimodule where

$$P \otimes_R P^* \to \mathrm{End}_R(P), p \otimes f \mapsto [a \mapsto pf(a)].$$

Now, consider the R-module $P^* \otimes_R P$ as an R-coalgebra, using the Morita Context we have the following proposition.

Theorem 3. *Let P be a finitely generated projective R-module with dual basis $p_1, p_2, ..., p_n \in P$ $\pi_1, \pi_2, ..., \pi_n \in P^*$. If P is a clean R-module, then the R-coalgebra $P^* \otimes_R P$ is clean.*

Proof. Let P be a finitely generated R-module and $P^* = \mathrm{Hom}_R(P, R)$ is an R-module. Suppose that P is a clean R-module. Since P and R is a finitely generated projective R-module, $P^* = \mathrm{Hom}_R(P, R)$ is also a finitely generated projective R-module [17]. Here, we need to prove weather R-coalgebra $P^* \otimes_R P$ satisfies the α-condition by proving the tensor product of $P^* \otimes_R P$ is a projective R-module.

To show that $P^* \otimes_R P$ is projective as an R-module, we must show that for any surjective map $f : A \to B$ of R-module, the map

$$f_* : \mathrm{Hom}_R(P^* \otimes_R P, A) \to \mathrm{Hom}_R(P^* \otimes_R P, B)$$

is also surjective. Since P^* is a projective R-module so that

$$h : \mathrm{Hom}_R(P^*, A) \to \mathrm{Hom}_R(P^*, B)$$

is surjective. By projectivity of P we obtain

$$h_* : \mathrm{Hom}_R(P, \mathrm{Hom}_R(P^*, A)) \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(P^*, B))$$

is also surjective. Put $C = A$ or B, then by [17] (see page 425) we have

$$\mathrm{Hom}_R(P, \mathrm{Hom}_R(P^*, C)) \simeq \mathrm{Hom}_R(P^* \otimes_R P, C)$$

It implies that f is isomorphic to h_*, and moreover f is a surjective map. Thus, $P^* \otimes_R P$ is a projective R-module. Therefore as an R-coalgebra, $P^* \otimes_R P$ satisfies the α-condition. Then we have

$$_{(P^* \otimes_R P)^*}\mathrm{End}(P^* \otimes_R P) \simeq (P^* \otimes_R P)^*.$$

We are going to show that $P^* \otimes_R P$ is a clean R-coalgebra, it means we need to prove that $(P^* \otimes_R P)^*$ is a clean ring (see Proposition 4.1.8). Based on [17], we have a relationship between tensor product and R-module homomorphism. Furthermore, since P is finitely generated, the dual algebra $P^* \otimes_R P$ is isomorphic to the ring $\mathrm{End}_R(P)$ by the bijective map as below:

$$\begin{aligned}(P^* \otimes_R P)^* = \mathrm{Hom}_R(P^* \otimes_R P, R) &\simeq \mathrm{Hom}_R(P, \mathrm{Hom}_R(P^*, R)) \\ &\simeq \mathrm{Hom}_R(P, P^{**}) \simeq \mathrm{End}_R(P) (\text{since } P^{**} \simeq P).\end{aligned}$$

Hence, if P is a clean R-module, then $\mathrm{End}_R P$ is a clean ring. It means $(P^* \otimes_R P)^* \simeq \mathrm{End}_R(P)$ is a clean ring. Since $P^* \otimes_R P^* \simeq \mathrm{End}_{(P^* \otimes_R P)^*}(P^* \otimes_R P)$ is a clean ring, $P^* \otimes_R P$ is a clean R-coalgebra. □

For $P = R$ we obtain $P^* = R^* = \mathrm{End}_R(R) \simeq R$ and $R^* \otimes_R R \simeq R$ is a coassociative R-coalgebra with counital. Thus, if R is a clean R-module (i.e., R is clean as a ring), then (R, Δ, ε) is a clean coalgebra over itself.

Recall the example of R-coalgebra $M_n(R)$ (see [15]). The matrix ring $M_n(R)$ is an R-coalgebra by the coproduct and counit as below

$$\Delta : M_n(R) \to M_n(R) \otimes_R M_n(R), e_{ij} \mapsto \Sigma_{i,j} e_{i,k} \otimes e_{kj}, \tag{1}$$

and

$$\varepsilon : M_n(R) \to R, e_{ij} \mapsto \delta_{i,j}. \tag{2}$$

It is called the (n, n)-matrix coalgebra over R. Throughout, the R-coalgebra $M_n(R)$ with the comultiplication (1) and the counit (2) denoted by $M_n^C(R)$. Furthermore, we will show that the R-coalgebra $M_n^C(R)$ can be identified as an R-coalgebra $P^* \otimes_R P$ when $P = R^n$.

Lemma 2. *Let $P = R^n$. Then the comultiplication and counit on R-coalgebra $(R^n)^* \otimes_R R^n$ is equivalent to the comultiplication and counit of R-coalgebra $M_n^C(R)$. It means $(R^n)^* \otimes_R R^n \approx M_n^C(R)$.*

Proof. Suppose that the canonical basis of R^n is $\{(0, 0, .., 1_i, 0, ..0)\}_{i\in\mathbb{N}}$ and basis of $(R^n)^*$ is $\{\pi_i\}_{i\in\mathbb{N}}$ where $\pi_i((0, 0, .., 1_j, 0, ..0)) = 1$ for $i = j$ and 0 for $i \neq j$. Therefore

1) The comultiplication

$$\begin{gathered}\Delta : (R^n)^* \otimes_R R^n \to ((R^n)^* \otimes_R R^n) \otimes_R (R^n)^* \otimes_R R^n \\ f \otimes p \mapsto \textstyle\sum_i f \otimes p_i \otimes \pi_i \otimes p\end{gathered}$$

For any $f = \sum_i a_i\pi_i$ and $p = \sum_i b_i p_i \in R^n$ we have

$$\begin{aligned}\Delta(f \otimes p) &= \sum_k (\sum_i a_i\pi_i) \otimes p_k \otimes \pi_k \otimes (\sum_j b_j p_j) \\ &= \sum_k (\sum_i a_i\pi_i(p_k)) \otimes (\sum_j b_j\pi_k(p_j))\end{aligned}$$

Since $(R^n)^* \otimes_R R^n \approx M_n(R)$ as an R-module by mapping $\pi_i \otimes p_j \mapsto e_{ij}$ for any i, j, we have

$$\Delta(f \otimes p) \simeq \sum_k (\sum_i a_i e_{ik} \otimes \sum_j b_j e_{kj}).$$

It implies the case $f \otimes p = \pi_i \otimes p_j \approx e_{ij}$, we have

$$\Delta(f \otimes p) = \Delta(e_{ij}) = \Delta(\pi_i \otimes p_j) = \sum_k \pi_i \otimes p_k \otimes \pi_k \otimes p_j$$
$$= \sum_k \pi_i(p_k) \otimes \pi_k(p_j) = \sum_k e_{ik} \otimes e_{kj}$$

Therefore,

$$\Delta(\pi_i \otimes p_j) \approx \Delta(e_{ij}) = \sum_k e_{ik} \otimes e_{kj}.$$

Consequently, this result similar to the comultiplication on R-coalgebra $M_n^C(R)$.

2) The counit of $(R^n)^* \otimes_R R^n$ is $\varepsilon(f \otimes p) = f(p)$. For any $f \otimes p \in (R^n)^* \otimes_R R^n$ where $f = \sum_i a_i \pi$ and $p = \sum_j b_j p_j \in R^n$ we have

$$\varepsilon(f \otimes p) = \varepsilon(\sum_i a_i \pi_i \otimes \sum_j b_j p_j) = \sum_i a_i \pi_i(\sum_j b_j p_j)$$
$$= \varepsilon(f \otimes p) = \sum_i a_i \sum_j b_j \pi_i(p_j) = a_i b_i.$$

Related with an R-coalgebra $M_n^C(R)$, for canonical basis $e_{ij} \approx \pi_i \otimes p_j$ (see Lemma 2). Putting $f \otimes p = \pi_i \otimes p_j \in (R^n)^* \otimes_R R^n$ (see Lemma 2), then

$$\varepsilon(e_{ij}) = \varepsilon(\pi_i \otimes p_j) = \pi_i(p_j) = \delta_{i,j}.$$

It is analogue to the counit of $M_n^C(R)$. □

It is clear that every ring is a trivial coalgebra over itself ([15]). On the other hand, we have already known that a ring R is clean if and only if $(R, \Delta_T, \varepsilon_T)$ is a clean R-coalgebra. Furthermore, if R is a clean ring, then the ring $M_n(R)$ is a clean ring [1]. Now, let consider the matrix ring $M_n(R)$ as a coalgebra over itself by the trivial comultiplication $(\Delta_T)$and counit (ε_T), denoted by $(M_n(R), \Delta_T, \varepsilon_T)$. Hence, if $M_n(R)$ is a clean ring, then $(M_n(R), \Delta_T, \varepsilon_T)$ is a clean coalgebra over itself. The following corollary explains the cleanness of R-coalgebra $M_n^C(R)$ with Δ and ε in Equation (1) and (2).

Corollary 1. *If R is a clean ring, then the R-coalgebra $M_n^C(R)$ is clean.*

Proof. By Lemma 2 $M_n^C(R)$ is a special case of $P^* \otimes_R P$ when $P = R^n$. Suppose that $P = R^n$. Since R is a clean ring, then R^n is a clean R-module [13]. By the Theorem 3 $(R^n)^* \otimes_R R^n = M_n^C(R)$ is a clean R-coalgebra, since R^n is a clean R-module. □

2. The cleanness of P and P^* as a $P^* \otimes_R P$-comodule

Let R be a commutative ring with multiplicative identity and P be a finitely generated projective R-module. In [15] if P is a clean R-module, then the R-coalgebra $P^* \otimes_R P$ is clean. If P is a finitely generated projective R-module with basis $p_1, p_2, ..., p_n \in P$ and dual basis $\pi_1, \pi_2, ..., \pi_n \in P^*$, then P is a right $P^* \otimes_R P$-comodule with the coaction

$$\varrho^P : P \to P \otimes_R (P^* \otimes_R P), p \mapsto \Sigma_i p_i \otimes \pi_i \otimes p.$$

P is a subgenerator in $\mathbf{M}^{P^* \otimes_R P}$ and there is a category isomorphism

$$\mathbf{M}^{P^* \otimes_R P} \simeq \mathbf{M}_{\mathrm{End}_R(P)}.$$

The dual P^* is a left $P^* \otimes_R P$-comodule with the coaction

$$^{P^*}\varrho : P^* \to (P^* \otimes_R P) \otimes_R P^*, f \mapsto \Sigma_i f \otimes p_i \otimes \pi_i.$$

Here, we will investigate the conditions under which P and P^* are clean comodules over $P^* \otimes_R P$.

Theorem 4. *Let P be a finitely generated projective R-module with basis $p_1, p_2, ..., p_n \in P$ and dual basis $\pi_1, \pi_2, ..., \pi_n \in P^*$. If R is a clean ring, then P is a right clean $P^* \otimes_R P$-comodule and P^* is a left clean $P^* \otimes_R P$-comodule.*

Proof. 1) Suppose that P is a projective R-module. Consider P as a right $P^* \otimes_R P$-comodule. We want to prove that P is a right clean $P^* \otimes_R P$-comodule, i.e., $_{(P^* \otimes_R P)^*} \mathrm{End}(P)$ is a clean ring.

Based on [15], since $P^* \otimes_R P$ is a finitely generated projective R-module, R-coalgebra $P^* \otimes_R P$ satisfies the α-condition and we have the following condition:

$$_{(P^* \otimes_R P)^*}\mathbf{M} \simeq \mathbf{M}^{P^* \otimes_R P}. \tag{3}$$

On the other hand, it is true that the ring $(P^* \otimes_R P)^* \simeq \mathrm{End}_R(P)$. Therefore,

$$\mathbf{M}^{P^* \otimes_R P} \simeq_{(P^* \otimes_R P)^*} \mathbf{M} \simeq_{\mathrm{End}_R(P)} \mathbf{M}.$$

We are going to prove that the ring $_{(P^* \otimes_R P)^*} \mathrm{End}(P) \in_{(P^* \otimes_R P)^*} \mathbf{M}$ is clean. Based on Equation (3) and using the Morita Context (see Theorem 1), since P is a generator, $R \simeq \mathrm{End}(_{\mathrm{End}_R(P)}P)$ as a ring. Therefore,

$$_{(P^* \otimes_R P)^*} \mathrm{End}(P) \simeq \mathrm{End}_{\mathrm{End}_R(P)}(P) \text{ and } \mathrm{End}_{\mathrm{End}_R(P)}(P) \simeq R$$

as a ring. Hence, ${}_{(P^*\otimes_R P)^*}\operatorname{End}(P) \simeq R$ as an R-module. Noted that R is a clean ring if and only if R is a clean R-module. Then

$$ {}_{(P^*\otimes_R P)^*}\operatorname{End}(P) \simeq R $$

is a clean ring. Consequently, P is a clean $P^* \otimes_R P$-comodule.

2) Consider P^* as a left $P^* \otimes_R P$-comodule. We want to prove that P^* is a left $P^* \otimes_R P$-comodule, i.e., $\operatorname{End}_{(P^*\otimes_R P)^*}(P^*)$ is a clean ring. Analogue with point (1) we have

$$ {}^{P^*\otimes_R P}\mathbf{M} \simeq \mathbf{M}_{(P^*\otimes_R P)^*} \simeq \mathbf{M}_{\operatorname{End}_R(P)}. $$

We going to prove that the ring $\operatorname{End}_{(P^*\otimes_R P)^*}(P^*) \in \mathbf{M}_{(P^*\otimes_R P)^*}$ is clean. From Equation (3) we have

$$ \operatorname{End}_{(P^*\otimes_R P)^*}(P^*) \simeq \operatorname{End}_{\operatorname{End}_R P}(P^*). $$

Furthermore, from Theorem 1 we have $R \simeq \operatorname{End}_{\operatorname{End}_R(P)}(P^*)$.
Therefore,

$$ \operatorname{End}_{(P^*\otimes_R P)^*}(P^*) \simeq R. $$

Consequently, if R is a clean ring then ${}_{(P^*\otimes_R P)^*}\operatorname{End}(P) \simeq R$ is a clean ring. Hence, P^* is a left clean $P^* \otimes_R P$-comodule. □

Remark 1. Let $P = R^n$. As a special case for any $n \in \mathbb{N}$ then R^n is a comodule over the coalgebra $M_n^C(R)$. On the other hand, if R is a clean ring (i.e., a clean R-module), then R^n is a clean R-module. Therefore, if R is a clean R-module, then R^n is a right clean $M_n^C(R)$-comodule.

This paper gives the sufficient conditions of clean R-coalgebra $P^* \otimes_R P$ and the cleanness of P and P^* as a $P^* \otimes_R P$-comodule. We already get some conclusions i.e., if P is a clean R-module, then the R-coalgebra $P^* \otimes_R P$ is clean and if R is a clean ring, then P (resp. P^*) is a right (resp. left) clean $P^* \otimes_R P$-comodule. We see that the cleanness of $P^* \otimes_R P$ depends on R if P is a finitely generated projective R-module (i.e., it is very closed to free R-module).

References

[1] Nicholson, W.K., *Lifting Idempotents and Exchange Rings*, Trans. Amer. Math. Soc., **229**, 1977, 269–278.

[2] Warfield, Jr., R. B., *Exchange rings and decompositions of modules*, Math. Ann. **199**, 1972, 31–36.

[3] Crawley, P., and Jónnson, B., *Refinements for Infinite Direct Decompositions Algebraic System*, Pacific J. Math., **14**, 1964, 797–855.

[4] Camillo, V.P., and Yu, H.P., *Exchange Rings, Units and Idempotents*, Comm. Algebra, **22(12)**, 1994, 4737–4749.

[5] Han, J., and Nicholson, W.K., *Extension of Clean Rings*, Comm. Algebra, **29(6)**, 2001, 2589–2595.

[6] Anderson, D.D., and Camillo, V.P., *Commutative Rings Whose Element are a sum of a Unit and Idempotent*, Comm. Algebra, **30(7)**, 2002, 3327–3336.

[7] Tousi, M., and Yassemi, S., *Tensor Product of Clean Rings*, Glasgow Math. J, **47**. 2005, 501–503.

[8] McGovern, W. Wm., *Characterization of commutative clean rings*, Int. J. Math. Game Theory Algebra, **15(40)**, 2006, 403–413.

[9] Chen, H., and Chen, M., *On Clean Ideals*, IJMMS, **62**, 2002, 3949–3956.

[10] Nicholson, W. K., and Varadarajan, K., *Countable Linear Transformations are Clean*, Proceedings of American Mathematical Socienty, **126**, 1998, 61–64.

[11] Scarcoid, M.O., *Perturbation the Linear Transformation By Idempotent*, Irish Math. Soc. Bull., **39**, 1997, 10–13.

[12] Nicholson, W.K., Varadarajan, K. and Zhou, Y., *Clean Endomorphism Rings*, Archiv der Mathematik, **83**, 2004, 340–343.

[13] Camillo, V.P., Khurana, D., Lam, T.Y., Nicholson, W.K. and Zhou, Y., *Continous Modules are Clean*, J. Algebra, **304**, 2006, 94–111.

[14] Camillo, V.P., Khurana, D., Lam, T.Y., Nicholson, W.K. and Zhou, Y., *A Short Proof that Continous Modules are Clean*, Contemporary Ring Theory 2011, Proceedings of the Sixth China-Japan-Korea International Conference on Ring Theory, 2012 165–169.

[15] Brzeziński, T., and Wisbauer, R., *Corings and Comodules*, Cambridge University Press, United Kingdom, 2003.

[16] Lam, T.Y., *Graduated Texts in Mathematics: Lectures on Modules and Rings*, Springer-Verlag, New York, inc, 1994.

[17] Adkins, W.A., and Weintraub, S. H., *Algebra "An Approach via Module Theory"*, Springer-Verlag New York, Inc., USA, 1992.

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