

On the *le*-semigroups whose semigroup of bi-ideal elements is a normal band

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ABSTRACT. It is well known that the semigroup $\mathcal{B}(S)$ of all bi-ideal elements of an *le*-semigroup S is a band if and only if S is both regular and intra-regular. Here we show that $\mathcal{B}(S)$ is a band if and only if it is a normal band and give a complete characterization of the *le*-semigroups S for which the associated semigroup $\mathcal{B}(S)$ is in each of the seven nontrivial subvarieties of normal bands. We also show that the set $\mathcal{B}_m(S)$ of all minimal bi-ideal elements of S forms a rectangular band and that $\mathcal{B}_m(S)$ is a bi-ideal of the semigroup $\mathcal{B}(S)$.

1. Introduction

In the ideal theory of commutative rings, it was observed by W. Krull [15] that several results do not depend on the fact that the ideals are composed of elements. The same is true for the ideal theory of semigroups also. Consequently, these results can be formulated in a more general setting of lattice-ordered semigroups where an element represents an ideal of the ring or semigroup as an undivided entity. There are series of articles dealing with lattice-ordered semigroups, generalizing theorems from commutative ideal theory [1], [3], [5] and from the ideal theory of semigroups [12], [13], [21], [22]. Presently, lattice ordered semigroups are providing us a general setting not only for ‘abstract ideal theory’, but also

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for order-preserving transformations of a finite chain, power semigroups of an arbitrary semigroup, and for many other areas of algebra where the objects form similar kinds of lattice-ordered semigroups.

In the present paper we study le -semigroups globally; our aim here is to find out to what extent properties of the subsemigroup $\mathcal{B}(S)$ of all bi-ideal elements of an le -semigroup S affect the structure of the le -semigroup as a whole. In 1952, R.A. Good and D.R. Hughes [6] introduced the notion of bi-ideals of a semigroup; these have been generalized again and again to rings, semirings, ternary semirings, Γ -semigroups, etc [4], [8]–[11], [14], [23]. It has also been proved that this notion is very useful for characterizing different types of regularity of rings, semirings, and semigroups [2], [16]–[19]. In [12], N. Kehayopulu defined bi-ideal elements of an le -semigroup as a generalization of bi-ideals. Here we introduce the notion of minimal bi-ideal elements and show that the product of any two bi-ideal elements is a bi-ideal element, and that the product of any two minimal bi-ideal elements is a minimal bi-ideal element. Thus the set $\mathcal{B}(S)$ of all bi-ideal elements and the set $\mathcal{B}_m(S)$ of all minimal bi-ideal elements are subsemigroups of S . It is well known that S is both regular and intra-regular if and only if $b^2 = b$ for every bi-ideal element b of S , equivalently $\mathcal{B}(S)$ is a band. Here we show that $\mathcal{B}(S)$ is a locally testable semigroup and hence a normal band (since a band is locally testable if and only if it is a normal band) if S is both regular and intra-regular. The variety of normal bands has exactly eight subvarieties. Here we have characterized the le -semigroups S such that $\mathcal{B}(S)$ is in each of these subvarieties of normal bands.

This introduction is followed by preliminaries. In Section 3, we characterize the le -semigroups S such that $\mathcal{B}(S)$ is in each of the subvarieties of normal bands. In the last section, we show that the semigroup $\mathcal{B}_m(S)$ of all minimal bi-ideal elements of S is a bi-ideal of the semigroup $\mathcal{B}(S)$ whereas the set $\mathcal{L}_m(S)$ of all minimal left ideal elements is a left ideal of the semigroup $\mathcal{L}(S)$ of all left ideal elements of S .

2. Preliminaries and foundations

An le -semigroup S is an algebra $(S, \cdot, \vee, \wedge, e)$ such that (S, \cdot) is a semigroup, (S, \vee, \wedge, e) is a lattice with a greatest element which is denoted by e , and for all $a, b, c \in S$,

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

For different examples and relevance, both classical and modern, of the le -semigroups we refer to [21]. Throughout the paper S will stand for an le -semigroup $(S, \cdot, \vee, \wedge, e)$.

The usual order relation \leq on the set S is defined by: for $a, b \in S$

$$a \leq b \text{ if } a \vee b = b.$$

Since the multiplication is distributive over the lattice join, it follows that the order \leq is compatible with the multiplication in S , that is, for all $a, b, c \in S$,

$$a \leq b \implies ac \leq bc \text{ and } ca \leq cb.$$

Let A be a nonempty subset of S . We denote $(A) = \{x \in S \mid x \leq a \text{ for some } a \in A\}$. A nonempty subset L is called a *left (right) ideal* of S if $SL \subseteq L$ ($LS \subseteq L$) and $(L) \subseteq L$. A subset I is called an *ideal* if it is both a left and a right ideal of S . For $a \in S$, the left ideal generated by a is given by

$$(a)_l = \{x \in S \mid x \leq sa \text{ for some } s \in S \cup \{1\}\}.$$

An element $a \in S$ is called *regular* if $a \leq aea$; and *intra-regular* if $a \leq ea^2e$. If every element of S is regular (intra-regular) then the le -semigroup S is defined to be *regular (intra-regular)*. We also say that a is

- (i) a *subsemigroup element* if $a^2 \leq a$;
- (ii) a *left ideal element* if $ea \leq a$;
- (iii) a *right ideal element* if $ae \leq a$;
- (iv) a *bi-ideal element* if it is a subsemigroup element and $aea \leq a$.

From the above definitions it is evident that every left and right ideal element is also a subsemigroup element. The definition of bi-ideal elements that we have given here is a little bit different from that of bi-ideal elements considered by Kehayopulu [12], Pasku and Petro [22]. According to these authors, a bi-ideal element b needs not satisfy $b^2 \leq b$, i.e. needs not be a subsemigroup element, and is actually an abstraction of the generalized bi-ideals (of a semigroup) and not of the bi-ideals.

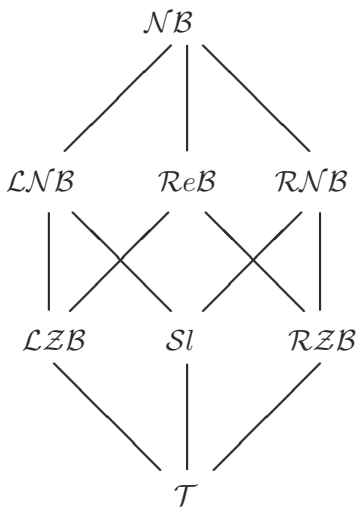
Let $a \in S$. Then $b = a \vee a^2 \vee aea$ is the least bi-ideal element in S such that $a \leq b$. We call $a \vee a^2 \vee aea$ the *bi-ideal element generated by a* , and denote this by $\beta(a)$. Thus $a \in S$ is a bi-ideal element if and only if $\beta(a) = a$.

Now we recall some notions of semigroups (without order). A semigroup F is called *regular* if for every $a \in F$ there is $x \in F$ such

that $a = axa$. By a *band* we mean a semigroup B such that $b^2 = b$ for all $b \in B$. A band S is *normal* if for all $a, b, c \in S$, $abca = acba$. A subsemigroup B of a semigroup F is called a *bi-ideal* of F if $BFB \subseteq B$.

In the diagram below, we use the following symbols to denote the different subvarieties of normal bands.

Normal band	\mathcal{NB}	$abcd = acbd,$
Rectangular band	\mathcal{ReB}	$aba = a,$
Left normal band	\mathcal{LNB}	$abc = acb,$
Right normal band	\mathcal{RNB}	$abc = bac,$
Left zero band	\mathcal{LZB}	$ab = a,$
Right zero band	\mathcal{RZB}	$ab = b,$
Semilattice	\mathcal{Sl}	$ab = ba,$
Trivial semigroup	\mathcal{T}	$a = b.$



A semigroup is called *locally finite* if every finitely generated subsemigroup is finite. A *locally testable semigroup* [24] is a semigroup which is locally finite and in which fSf is a semilattice for all idempotent $f \in S$. Nambooripad [20] proved that a regular semigroup S is locally testable if and only if fSf is a semilattice for all idempotent $f \in S$.

We refer the reader to [7] for the fundamentals of semigroup theory.

3. Subsemigroup of all bi-ideal elements

We denote the set of all left, right, and bi-ideal elements of S by $\mathcal{L}(S)$, $\mathcal{R}(S)$, and $\mathcal{B}(S)$, respectively. Then $\mathcal{L}(S)$, $\mathcal{R}(S)$, and $\mathcal{B}(S)$ are all nonempty, since e is a left ideal, a right ideal, and a bi-ideal element of S . Now for any two bi-ideal elements a and b of S , $(ab)^2 = (aba)b \leq ab$ and $abeab = (abea)b \leq ab$, since a is a bi-ideal element of S , which shows that the product of any two bi-ideal elements is a bi-ideal element. Thus $\mathcal{B}(S)$ is a subsemigroup of S . Similarly both $\mathcal{L}(S)$ and $\mathcal{R}(S)$ are subsemigroups of S .

Now we show that the regularity of an le -semigroup is equivalent to the regularity of the semigroup $\mathcal{B}(S)$. This, we think, is well known. But as we have seen the sufficient part nowhere, for the sake of completeness, we include a proof.

Proposition 3.1. *Let S be an le -semigroup. Then S is regular if and only if the semigroup $\mathcal{B}(S)$ of all bi-ideal elements is regular.*

Proof. First assume that S is regular and that $b \in \mathcal{B}(S)$. Since b is a bi-ideal element, $beb \leq b$. On the other hand, $b \leq beb$ by the regularity of S . Thus we have $b = beb$ which shows that b is a regular element in $\mathcal{B}(S)$, since e is also a bi-ideal element of S .

Conversely, suppose that $\mathcal{B}(S)$ is a regular semigroup. Consider $a \in S$. Then $\beta(a) = a \vee a^2 \vee aea \in \mathcal{B}(S)$ and so there is $b \in \mathcal{B}(S)$ such that $a \vee a^2 \vee aea = (a \vee a^2 \vee aea)b(a \vee a^2 \vee aea) \leq (a \vee a^2 \vee aea)e(a \vee a^2 \vee aea) \leq aea$. This implies that $a \leq aea$. Thus S is a regular le -semigroup. \square

If S is a regular le -semigroup, then for every $a \in S$, $a \leq aea$ implies that $a^2 \leq aaea \leq aea$. Hence the bi-ideal element $\beta(a)$ generated by a reduces to the form $\beta(a) = aea$. Thus in a regular le -semigroup the notions of bi-ideal elements as we have defined and that defined by N. Kehayopulu [12] are the same. Therefore in a regular le -semigroup S , an element $b \in S$ is a bi-ideal element if and only if $b = ca$ for some right ideal element c and left ideal element a [12, Lemma 2]. This can be reframed as:

Theorem 3.2. *Let S be an le -semigroup. Then $\mathcal{R}(S)\mathcal{L}(S) \subseteq \mathcal{B}(S)$. If moreover, S is a regular le -semigroup, then $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$.*

We also omit the proof of the following result, since this can be proved easily:

Proposition 3.3. *Let S be a regular le -semigroup. Then $\mathcal{R}(S)$ and $\mathcal{L}(S)$ are bands.*

The following important result can be proved similarly to that in [13] for the quasi-ideal elements.

Theorem 3.4. *An le -semigroup S is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a band.*

Now we show that $\mathcal{B}(S)$ is in fact a normal band if S is both a regular and intra-regular le -semigroup.

Theorem 3.5. *Let S be an le -semigroup. Then S is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a normal band.*

Proof. Let $a, b, c \in \mathcal{B}(S)$. Then $(bab)(bcb) = ba(bccb) \leq ba(beb) \leq bab$. Similarly, $(bab)(bcb) \leq bcb$. Thus $(bab)(bcb) \leq (bab) \wedge (bcb)$. Now let $u = (bab) \wedge (bcb)$. Then $u \leq bab$ and $u \leq bcb$. Since S is both regular and intra-regular, so $\mathcal{B}(S)$ is a band. Now $ueu = (bab \wedge bcb)e(bab \wedge bcb) = babebab \wedge babebcb \wedge bcbebab \wedge bcbcbcb \leq bab \wedge babebcb \wedge bcbebab \wedge bcb \leq bab \wedge bcb = u$ shows that $u \in \mathcal{B}(S)$ which implies that $u = u^2 \leq (bab)(bcb)$. Thus $(bab) \wedge (bcb) \leq (bab)(bcb)$ and hence $(bab)(bcb) = (bab) \wedge (bcb)$.

Then $b\mathcal{B}(S)b = \{bab \mid a \in \mathcal{B}(S)\}$ is a semilattice for every $b \in \mathcal{B}(S)$. Thus $\mathcal{B}(S)$ is a locally testable semigroup. Since a locally testable semigroup is a band if and only if it is a normal band [24, Theorem 5], so $\mathcal{B}(S)$ is a normal band.

The converse follows from the Theorem 3.4. □

An ordered semigroup S is said to be *left (right) duo* if every left (right) ideal of S is a right (left) ideal of S ; and S is said to be *duo* if S is both left and right duo.

Lemma 3.6. *An le -semigroup S is left duo if and only if $ae \leq ea$ for all $a \in S$.*

Proof. First assume that S is left duo and let $a \in S$. Then the left ideal $(a]_l = \{x \in S \mid x \leq sa \text{ for some } s \in S\}$ generated by a is a right ideal also. Then $ae \in (a]_l$ implies that there is some $s \in S$ such that $ae \leq sa$ and this implies that $ae \leq ea$.

Conversely let L be a left ideal of S and $a \in L$. Then for every $s \in S$, $as \leq ae \leq ea \in L$ implies that $as \in L$. Thus L is a right ideal of S and hence S is left duo. □

Immediately we have:

Proposition 3.7. *An le-semigroup S is duo if and only if $ae = ea$ for all $a \in S$.*

Let S be a regular left duo le-semigroup. Then for every $a \in S$, $a \leq aea \leq (ae)aea \leq ea^2ea$ shows that S is intra-regular. Hence $\mathcal{B}(S)$ is a band. In fact we have:

Theorem 3.8. *An le-semigroup S is regular left duo if and only if $\mathcal{B}(S)$ is a left normal band.*

Proof. First assume that S is regular left duo. Then $\mathcal{B}(S)$ is a band. Let $a, b, c \in \mathcal{B}(S)$. Then $abc = (abc)(abc) = aabcabc \leq a(ae)cabc \leq a(ea)cabc \leq acabc = aca(bc)(bc) \leq acab(cb)e \leq acabecb \leq acb$. Similarly $acb \leq abc$. Thus $abc = acb$ and hence $\mathcal{B}(S)$ is a left normal band.

Conversely, assume that $\mathcal{B}(S)$ is a left normal band. Then S is regular. Also for every $a \in S$, both ea and aea are bi-ideal elements of S , and hence $ae = (ae)(ae)(ae) = (aea)(ea)e = (aea)e(ea)$ [since $\mathcal{B}(S)$ is a normal band] $= (aeae^2)a \leq ea$ which shows that S is left duo. \square

The left-right dual of this theorem is as follows:

Theorem 3.9. *An le-semigroup S is regular right duo if and only if $\mathcal{B}(S)$ is a right normal band.*

A band is a semilattice if and only if it is both a left and a right normal band. Hence it follows immediately that:

Theorem 3.10. *An le-semigroup S is regular duo if and only if $\mathcal{B}(S)$ is a semilattice.*

Theorem 3.11. *Let S be an le-semigroup. Then $\mathcal{B}(S)$ is a rectangular band if and only if S is regular and $eae = ebe$ for all $a, b \in S$.*

Proof. First assume that $\mathcal{B}(S)$ is a rectangular band and that $a, b \in S$. Since $\mathcal{B}(S)$ is a band, so S is regular and hence $\beta(a) = aea$ and $\beta(b) = beb$. Then $\beta(a) = \beta(a)\beta(b)\beta(a)$ implies that $a \leq aea = (aea)(beb)(aea) \leq ebe$. Then $eae \leq e^2be^2 \leq ebe$. Similarly $\beta(b) = \beta(b)\beta(a)\beta(b)$ implies that $ebe \leq eae$. Thus $eae = ebe$ for all $a, b \in S$.

Conversely let $a \in S$. Since S is regular, so $a \leq aea \leq aeaea \leq aea^2ea$, by the given condition. Thus $a \leq ea^2e$, and hence S is intra-regular. Therefore $\mathcal{B}(S)$ is a band, by Theorem 3.4. Now let a, b be two bi-ideal elements of S . Since a is a bi-ideal element and S is already known to be regular, then $aea = a$, and so $a = aea = aeaea = aeabaea = aba$; and hence $\mathcal{B}(S)$ is a rectangular band. \square

Theorem 3.12. *Let S be an le -semigroup. Then $\mathcal{B}(S)$ is a left zero band if and only if S is regular and $ae \leq eb$ for all $a, b \in S$.*

Proof. First assume that $\mathcal{B}(S)$ is a left zero band and $a, b \in S$. Since $\mathcal{B}(S)$ is band, so S is regular and hence $\beta(ae) = ae^2ae$ and $\beta(b) = beb$. Then $\beta(ae) = \beta(ae)\beta(b)$ implies that $ae \leq ae^2ae = (ae^2ae)(beb) \leq eb$. Thus $ae \leq eb$ for all $a, b \in S$.

Conversely let $a \in S$. Since S is regular, so $a \leq aea \leq aeaea \leq ae^2a^2a$, by the given condition. Thus $a \leq ea^2e$, and hence S is intra-regular. Therefore $\mathcal{B}(S)$ is a band, by Theorem 3.4. Now let a, b be two bi-ideal elements of S . Since S is regular, $a = aea$, so that $ab \leq ae = ae(ae) \leq ae^2a \leq aea = a$ and $a = aeaea \leq ae(ae) \leq ae(eab) \leq (aea)b = ab$. Thus $a = ab$ and hence $\mathcal{B}(S)$ is a left zero band. \square

The left-right dual of this theorem is as follows:

Theorem 3.13. *Let S be an le -semigroup. Then $\mathcal{B}(S)$ is a right zero band if and only if S is regular and $ea \leq be$ for all $a, b \in S$.*

4. Subsemigroup of all minimal bi-ideal elements

In this section we introduce minimal bi-ideal elements and minimal left ideal elements, and show that the set of all minimal bi-ideal elements of S is a subsemigroup of $\mathcal{B}(S)$.

Definition 4.1. Let S be an le -semigroup. A bi-ideal element b is said to be minimal if for every bi-ideal element a of S ,

$$a \leq b \text{ implies that } a = b.$$

Minimal left (right) ideal elements are defined similarly.

We denote the set of all minimal bi-ideal, left ideal, and right ideal elements of S by $\mathcal{B}_m(S)$, $\mathcal{L}_m(S)$, and $\mathcal{R}_m(S)$, respectively.

Now we show that $\mathcal{B}_m(S)$ is a subsemigroup of $\mathcal{B}(S)$. For this consider $a, b \in \mathcal{B}_m(S)$. Then ab is a bi-ideal element. To check the minimality, let c be a bi-ideal element such that $c \leq ab$. Then ca and bc are bi-ideal elements such that $ca \leq aba \leq a$. Then by minimality of a we have $ca = a$. Similarly, $bc = b$. Then $ab = cabc \leq cec \leq c$ and hence $c = ab$. Thus $ab \in \mathcal{B}_m(S)$.

Similarly, it can be proved that both $\mathcal{L}_m(S)$ and $\mathcal{R}_m(S)$ are subsemigroups of $\mathcal{B}(S)$.

We also have:

Theorem 4.2. *If S is an le-semigroup then $\mathcal{B}_m(S) = \mathcal{R}_m(S)\mathcal{L}_m(S)$.*

Proof. First consider $a \in \mathcal{R}_m(S)$ and $c \in \mathcal{L}_m(S)$, and denote $b = ac$. Then b is a bi-ideal element, by Theorem 3.2. To show the minimality of b , let $p \leq b$ be a bi-ideal element of S . Then pe is a right ideal element of S and $pe \leq be = ace \leq ae \leq a$ implies by the minimality of a as a right ideal element that $pe = a$. Similarly we have $ep = c$, since c is a minimal left ideal element. Then $p \leq b = ac = peep \leq p$ implies that $p = b$; and so b becomes a minimal bi-ideal element. Thus $\mathcal{R}_m(S)\mathcal{L}_m(S) \subseteq \mathcal{B}_m(S)$.

Now consider $b \in \mathcal{B}_m(S)$. Then be and eb are a right ideal element and a left ideal element, respectively. Let $a \leq be$ be a right ideal element of S . Then ab is a bi-ideal element of S such that $ab \leq beb \leq b$, and so $ab = b$, since b is a minimal bi-ideal element. Then $a \leq be = abe \leq ae \leq a$ implies that $a = be$. Thus be is a minimal right ideal element of S . Similarly eb is a minimal left ideal element of S . Then $beeb$ is a bi-ideal element, by Theorem 3.2. Now $beeb \leq b$ implies that $b = beeb$; and so $b \in \mathcal{R}_m(S)\mathcal{L}_m(S)$. Thus $\mathcal{B}_m(S) \subseteq \mathcal{R}_m(S)\mathcal{L}_m(S)$. Hence $\mathcal{B}_m(S) = \mathcal{R}_m(S)\mathcal{L}_m(S)$. \square

Theorem 4.3. *a) Let S be an le-semigroup such that the set $\mathcal{L}_m(S)$ of all minimal left ideal elements is non-empty. Then $\mathcal{L}_m(S)$ is a left ideal of the semigroup $\mathcal{L}(S)$. Moreover, $\mathcal{L}_m(S)$ is a right zero band.*

b) Let S be an le-semigroup such that the set $\mathcal{R}_m(S)$ of all minimal right ideal elements is non-empty. Then $\mathcal{R}_m(S)$ is a right ideal of the semigroup $\mathcal{R}(S)$. Moreover, $\mathcal{R}_m(S)$ is a left zero band.

Proof. a) Let $l \in \mathcal{L}(S)$ and $a \in \mathcal{L}_m(S)$. Then la is a left ideal element such that $la \leq ea \leq a$. This implies that $la = a$, since a is a minimal left ideal element. Hence $la \in \mathcal{L}_m(S)$ and so $\mathcal{L}(S)\mathcal{L}_m(S) \subseteq \mathcal{L}_m(S)$. Thus $\mathcal{L}_m(S)$ is a left ideal of $\mathcal{L}(S)$.

Now $la = a$ for every $l \in \mathcal{L}(S)$ and $a \in \mathcal{L}_m(S)$ implies that $ab = b$ for every $a, b \in \mathcal{L}_m(S)$; and hence $\mathcal{L}_m(S)$ is a right zero band.

b) Follows as the left-right dual of a). \square

Now we characterize the semigroup $\mathcal{B}_m(S)$ of all minimal bi-ideal elements of S .

Theorem 4.4. *Let S be an le-semigroup such that the set $\mathcal{B}_m(S)$ of all minimal bi-ideal elements is non-empty. Then $\mathcal{B}_m(S)$ is a bi-ideal of the semigroup $\mathcal{B}(S)$. Moreover, $\mathcal{B}_m(S)$ is a rectangular band.*

Proof. We have already shown that $\mathcal{B}_m(S)$ is a subsemigroup of $\mathcal{B}(S)$. Now consider $a, c \in \mathcal{B}_m(S)$ and $b \in \mathcal{B}(S)$. Then abc is a bi-ideal element

of S . To show the minimality of abc , let $d \leq abc$ be a bi-ideal element of S . Then da is a bi-ideal element of S and $da \leq abca \leq a$ implies that $a = da$, since a is a minimal bi-ideal element. Similarly minimality of c implies that $c = cd$. Then $abc = dabcd \leq d$ and so $d = abc$ which shows that abc is a minimal bi-ideal element of S . Thus $\mathcal{B}_m(S)\mathcal{B}(S)\mathcal{B}_m(S) \subseteq \mathcal{B}_m(S)$ and hence $\mathcal{B}_m(S)$ is a bi-ideal of $\mathcal{B}(S)$.

If $b \in \mathcal{B}_m(S)$, then b is a subsemigroup element of S and so $b^2 \leq b$. Now minimality of b implies that $b^2 = b$. Thus $\mathcal{B}_m(S)$ is a band. Let $a, b \in \mathcal{B}_m(S)$. Then aba is a bi-ideal element such that $aba \leq aea \leq a$ which implies that $aba = a$, since a is a minimal bi-ideal element of S . Thus $\mathcal{B}_m(S)$ is a rectangular band. \square

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