© Algebra and Discrete Mathematics
Volume **30** (2020). Number 2, pp. 282–289
DOI:10.12958/adm1396
RESEARCH ARTICLE

On a product of two formational tcc-subgroups*

A. Trofimuk

Communicated by L. A. Kurdachenko

For the 70th anniversary of L. A. Kurdachenko

Abstract. A subgroup A of a group G is called tcc-*subgroup* in G, if there is a subgroup T of G such that $G = AT$ and for any $X \leqslant A$ and $Y \leqslant T$ there exists an element $u \in \langle X, Y\rangle$ such that $XY^u \leqslant G$. The notation $H \leqslant G$ means that H is a subgroup of a group G. In this paper we consider a group $G = AB$ such that A and B are tcc-subgroups in G. We prove that G belongs to $\mathfrak{F}$, when A and B belong to $\mathfrak{F}$ and $\mathfrak{F}$ is a saturated formation of soluble groups such that $\mathfrak{U} \subseteq \mathfrak{F}$. Here $\mathfrak{U}$ is the formation of all supersoluble groups.

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [1, 2]. The monographs [2, 3] contain the necessary information of the theory of formations.

We say that the subgroups A and B of a group G are *totally permutable* if every subgroup of A is permutable with every subgroup of B.

Asaad and Shaalan [4] proved the supersolubility of a group $G = AB$ such that the subgroups A and B are totally permutable and supersoluble, see [4, Theorem 3.1]. Following Maier [5] such factorization is called

*This work was supported by the BRFFR (grant No. F19RM-071).
2010 MSC: 20D10.
Key words and phrases: supersoluble group, totally permutable product, saturated formation, tcc-permutable product, tcc-subgroup.

totally permutable product of the subgroups A and B. In the same paper Maier showed that this statement is also true for the saturated formations containing the formation $\mathfrak{U}$ of all supersoluble groups.

Theorem 1 ([5, Theorem]). *Let $G = HK$ be the totally permutable product of the subgroups H and K. Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{L} \subseteq \mathfrak{F}$. If H and K lie in F, then so does G. Here $\mathfrak{L}$ denote the class of groups all of whose Sylow subgroups are cyclic.*

In [5] Maier also proposes the following question: «Does the above result extend to non-saturated formations which contain all supersoluble groups?»

Ballester-Bolinches and Perez-Ramos gave a positive answer to this question in [6].

Theorem 2 ([6, Theorem]). *Let $\mathfrak{F}$ be a formation containing the class $\mathfrak{U}$ of all supersoluble groups. Suppose that $G = HK$ be the totally permutable product of the subgroups H and K. If H and K belong to $\mathfrak{F}$, then G belongs to $\mathfrak{F}$.*

In works [7], [8] the authors extended a previous Maier's result by considering an arbitrary number of pairwise totally permutable factors.

In the articles [9]–[13] (see also the references from [13]) we can see that the supersolubility of a group can also be obtained for other generalizations of totally permutable product.

The notation $H \leqslant G$ means that H is a subgroup of a group G. So, for example, the product $G = AB$ is said to be tcc-*permutable* [13], if for any $X \leqslant A$ and $Y \leqslant B$ there exists an element $u \in \langle X, Y\rangle$ such that $XY^u \leqslant G$. The subgroups A and B in this product are called tcc-*permutable*.

One of the main results of [12] for two tcc-permutable factors is formulated as follows.

Theorem 3 ([12, Theorem 5]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let the group $G = HK$ be the* tcc*-permutable product of subgroups H and K. If $H \in \mathfrak{F}$ and $K \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Besides, in [12] the authors gave examples showing that Theorem 3 not remains true for arbitrary non-saturated formations containing $\mathfrak{U}$ even in the universe of soluble groups.

Now we introduce the following concept

Definition 1. A subgroup A of a group G is called tcc-*subgroup* in G, if it satisfies the following conditions:

1) there is a subgroup T of G such that $G = AT$;

2) for any $X \leqslant A$ and $Y \leqslant T$ there exists an element $u \in \langle X, Y\rangle$ such that $XY^u \leqslant G$.

Clear that by condition 2 of Definition 1, $G = AT$ is the tcc-permutable product of the subgroups A and T. In this case, we say that the subgroup T is a tcc-*supplement* to A in G.

If $G = AB$ is the tcc-permutable product of subgroups A and B, then the subgroups A and B are tcc-subgroups in G. The converse is false.

Example 1. Let Z_n be a cyclic group of order n. Dihedral group $G = \langle a\rangle \rtimes \langle c\rangle$, $|a| = 12$, $|c| = 2$ ([14], IdGroup=[24,6]) is the product of tcc-subgroups $A = \langle a^3 c\rangle$ of order 2 and $B = \langle a^{10}\rangle \rtimes \langle c\rangle$ of order 12. But A and B are not tcc-permutable. Indeed, there are the subgroups $X = A$ and $Y = \langle c\rangle$ of A and B respectively such that doesn't exist $u \in \langle X, Y\rangle = \langle a^3\rangle \rtimes \langle c\rangle$ such that $XY^u \leqslant G$.

In the present paper we prove the following theorem.

Theorem A. *Let $G = AB$, where A and B are* tcc-*subgroups in G. Let $\mathfrak{F}$ be a saturated formation of soluble groups such that $\mathfrak{U} \subseteq \mathfrak{F}$. Suppose that A and B belong to $\mathfrak{F}$. Then G belongs to $\mathfrak{F}$.*

1. Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel.

A group whose chief factors have prime orders is called *supersoluble*. If $H \leqslant G$ and $H \neq G$, we write $H < G$. The notation $H \trianglelefteq G$ means that H is a normal subgroup of a group G. Denote by $Z(G)$, $F(G)$ and $\Phi(G)$ the centre, Fitting and Frattini subgroups of G respectively, and by $O_p(G)$ the greatest normal p-subgroup of G. Denote by $\pi(G)$ the set of all prime divisors of order of G. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$. If H is a subgroup of G, then $H_G = \bigcap_{x\in G} H^x$ is called *the core* of H in G.

A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. Let $\mathbb{P}$ be the set of all prime numbers. A *formation function* is a function f defined on $\mathbb{P}$ such that $f(p)$ is a, possibly empty, formation. A formation $\mathfrak{F}$ is said to be *local* if there exists a formation function f such that $G \in \mathfrak{F}$ if and only if for any chief factor H/K of G and any $p \in \pi(H/K)$, one has $G/C_G(H/K) \in f(p)$. We write $\mathfrak{F} = LF(f)$ and f is a local definition of $\mathfrak{F}$. By [3, Theorem IV.3.7], among all possible local definitions of a local formation $\mathfrak{F}$ there exists a unique f such that f is integrated (i.e. $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$) and full (i.e. $f(p) = \mathfrak{N}_p f(p)$ for all $p \in \mathbb{P}$). Here $\mathfrak{N}_p$ is the formation of all p-groups. Such local definition f is said to be *canonical local definition* of $\mathfrak{F}$. By [3, Theorem IV.4.6], a formation is saturated if and only if it is local.

If a group G contains a maximal subgroup M with trivial core, then G is said to be *primitive* and M is its *primitivator*.

Lemma 1. *Let $\mathfrak{F}$ be a saturated formation and G be a group. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all non-trivial normal subgroups N of G. Then G is a primitive group.*

Proof. Since $\mathfrak{F}$ is a saturated formation, it follows that $\Phi(G) = 1$ and G contains a unique minimal normal subgroup N. For some maximal subgroup M of G, we have $G = NM$, because $\Phi(G) = 1$. It is obvious that the core $M_G = 1$. Hence G is a primitive group. □

Lemma 2 ([1, Theorem II.3.2])**.** *Let G be a soluble primitive group and M is a primitivator of G. Then the following statements hold:*

(1) $\Phi(G) = 1$;

(2) $F(G) = C_G(F(G)) = O_p(G)$ *and* $F(G)$ *is an elementary abelian subgroup of order* p^n *for some prime* p *and some positive integer* n*;*

(3) G *contains a unique minimal normal subgroup* N *and moreover,* $N = F(G)$*;*

(4) $G = F(G) \rtimes M$ *and* $O_p(M) = 1$*;*

Lemma 3 ([13, Theorem 1, Propositions 1, 2])**.** *Let $G = AB$ be the* tcc-*permutable product of subgroups A and B and N be a minimal normal subgroup of G. Then the following statements hold:*

(1) $\{A \cap N, B \cap N\} \subseteq \{1, N\}$*;*

(2) *if* $N \leqslant A \cap B$ *or* $N \cap A = N \cap B = 1$*, then* $|N| = p$*, where* p *is a prime.*

Lemma 4 ([12, Theorem 4])**.** *Let $G = AB$ be the* tcc-*permutable product of subgroups A and B. Then $[A, B] \leqslant F(G)$.*

Lemma 5. *Let A be a* tcc-*subgroup in G and Y be a* tcc-*supplement to A in G. Then the following statements hold:*

(1) A *is a* tcc-*subgroup in* H *for any subgroup* H *of* G *such that* $A \leqslant H$*;*

(2) AN/N *is a* tcc-*subgroup in* G/N *for any* $N \trianglelefteq G$*;*

(3) *for every* $A_1 \trianglelefteq A$ *and* $X \leqslant Y$ *there exists an element* $y \in Y$ *such that* $A_1X^y \leqslant G$*. In particular,* $A_1M \leqslant G$ *for some maximal subgroup* M

of Y *and* $A_1H \leqslant G$ *for some Hall* π*-subgroup* H *of soluble* Y *and any* $\pi \subseteq \pi(G)$*;*

(4) $A_1K \leqslant G$ *for every subnormal subgroup* K *of* Y *and for every* $A_1 \trianglelefteq A$*;*

(5) *if* $T \trianglelefteq G$ *such that* $T \leqslant A$ *and* $T \cap Y = 1$*, then* $T_1 \trianglelefteq G$ *for every* $T_1 \trianglelefteq A$ *such that* $T_1 \leqslant T$*;*

(6) *if* $T \trianglelefteq G$ *such that* $T \cap A = 1$ *and* $T \leqslant Y$*, then* $A_1 \leqslant N_G(T_1)$ *for every* $T_1 \trianglelefteq T$ *and for every* $A_1 \trianglelefteq A$*.*

Proof. 1. Since Y is a tcc-supplement to A in G, it follows that $G = AY$, A and Y are tcc-permutable subgroups of G. By Dedekind's identity, $H = H \cap AY = A(H \cap Y)$. Since $H \cap Y \leqslant Y$, then for any $X \leqslant A$ and $Z \leqslant H \cap Y$ there exists an element $u \in \langle X, Z\rangle$ such that $XZ^u \leqslant G$. Hence A and $H \cap Y$ are tcc-permutable and therefore A is a tcc-subgroup in H.

2. Since $G = AY$, we have $G/N = (AN/N)(YN/N)$. Let B/N be an arbitrary subgroup of AN/N and X/N be an arbitrary subgroup of YN/N. Since $N \leqslant B \leqslant AN$, it follows that by Dedekind's identity, $B = B \cap AN = (B \cap A)N$. Similarly, $X = X \cap YN = (X \cap Y)N$. Since $B \cap A \leqslant A$ and $X \cap Y \leqslant Y$, we have $(B \cap A)(X \cap Y)^u \leqslant G$ for some $u \in \langle B \cap A, X \cap Y\rangle$. Hence

$$(B/N)(X/N)^{uN} = (B \cap A)(X \cap Y)^u N/N \leqslant G/N$$

for $uN \in \langle B \cap A, X \cap Y\rangle N/N \subseteq \langle B, X\rangle N/N = \langle B/N, X/N\rangle$. Thus AN/N is a tcc-subgroup in G/N.

3. Since A is a tcc-subgroup in G, by definition, for every $A_1 \trianglelefteq A$ and $X \leqslant Y$ there exists an element $u \in \langle A_1, X\rangle$ such that $A_1X^u \leqslant G$. Because $u \in G = AY = YA$, it follows that $u = ya$ for some $y \in Y$ and $a \in A$. Then

$$A_1X^u = A_1X^{ya} = A_1(X^y)^a = A_1^a(X^y)^a = (A_1X^y)^a \leqslant G.$$

Hence there is a subgroup A_1X^y in G for some $y \in Y$. Clearly, that if X is a Hall π-subgroup of Y, then $H = X^y$ is a Hall π-subgroup of Y. Thus $A_1H \leqslant G$. Similarly, for maximal subgroup X of Y. Then $M = X^y$ is a maximal subgroup of Y and $A_1M \leqslant G$.

4. Since K is subnormal in Y, there is a chain of subgroups $Y = K_0 \geqslant K_1 \geqslant \ldots \geqslant K_{n-1} \geqslant K_n = K$ such that K_{i+1} is normal in K_i for all i. We use induction by n. By (3), there exists an element $y \in Y$ such that $A_1K_1^y = A_1K_1 \leqslant G$. Hence the statement holds for $n = 0$ and $n = 1$. Therefore $n \geqslant 2$. By (1), A is a tcc-subgroup in AK_1 and K_1 is a

tcc-supplement to A in AK_1. Since the length of subnormal chain between K and K_1 less than n, it follows that by induction, there is a subgroup A_1K of AK_1. Consequently $A_1K \leqslant G$.

5. By (3), there is a subgroup T_1Y of G. Since $T_1 = T \cap T_1Y$ is normal in T_1Y, we have $Y \leqslant N_G(T_1)$ and T_1 is normal in $G = AY$.

6. Since T_1 is subnormal in Y, it follows that by (4), there is a subgroup A_1T_1 of G for any $A_1 \trianglelefteq A$. Because $T_1 = T \cap A_1T_1$ is normal in A_1T_1, we have $A_1 \leqslant N_G(T_1)$. □

Lemma 6. *Let A be a* tcc*-subgroup in a soluble primitive group G and Y be a* tcc*-supplement to A in G. Suppose that N is a minimal normal subgroup of G. If $N \cap A = 1$ and $N \leqslant Y$, then A is cyclic of order dividing $p-1$.*

Proof. Since $N \cap A = 1$ and $N \leqslant Y$, by Lemma 5 (6), $A \leqslant N_G(K)$ for any $K \trianglelefteq N$. Since N is an elementary abelian group, it follows that $|N| = p^s$ for some prime p and some integer s, and $N = N_1 \times N_2 \times \ldots \times N_s$, where $N_i = \langle x_i \rangle$ and $|N_i| = p$. Because $A \leqslant N_G(N_i)$, we have A induces a power automorphism group on N. Indeed, let $a \in A$ and $x_i \in N_i$, $x_j \in N_j$, $i \neq j$. Suppose that $x_i^a = x_i^t$, $x_j^a = x_j^r$ and $(x_ix_j)^a = (x_ix_j)^m = x_i^m x_j^m$, because N is abelian. Then $x_i^m x_j^m = x_i^t x_j^r$. Consequently $m = t = r$ and a transforms every element of N to the same power. By [2, Theorem 2.3], $A/C_A(N) \simeq\leqslant \mathrm{P}(N)$, where $\mathrm{P}(N)$ is the power automorphism group of N. Since $C_G(N) = N$, it follows that $C_A(N) = 1$. On the other hand, $\mathrm{P}(N)$ is a cyclic group of order $p-1$. Really $\mathrm{P}(N)$ is a group of scalar matrices over the field $\mathbf{P}$ consisting of p elements. Hence $\mathrm{P}(N)$ is isomorphic to the multiplicative group $\mathbf{P}^*$ of $\mathbf{P}$ and besides, $\mathbf{P}^*$ is a cyclic group of order $p-1$. Therefore A is a cyclic group of order dividing $p-1$. □

Lemma 7 ([15, Lemma 2.16])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

2. Proof of Theorem A

Assume that the claim is false and let G be a minimal counterexample. Let N be a non-trivial normal subgroup of G. The quotients $AN/N \simeq A/A\cap N$ and $BN/N \simeq B/B\cap N$ are tcc-subgroup in G/N by Lemma 5 (2), $AN/N \simeq A/A \cap N \in \mathfrak{F}$ and $BN/N \simeq B/B \cap N \in \mathfrak{F}$, because $\mathfrak{F}$ is a formation. Hence the quotient $G/N = (AN/N)(BN/N) \in \mathfrak{F}$ by induction.

If $F(G) \neq 1$, then $G/F(G) \in \mathfrak{F}$ and consequently G is soluble, since $\mathfrak{F}$ is a formation of soluble groups. Hence $F(G) = 1$ and by Lemma 4, $Y \leqslant C_G(A)$. Then A is normal in G and G is soluble.

Since $\mathfrak{F}$ is saturated, it follows that G is primitive by Lemma 1. Hence $\Phi(G) = 1$, $N = C_G(N) = F(G) = \mathrm{O}_p(G)$ is a unique minimal normal subgroup of G by Lemma 2 and $G = N \rtimes M$, where $|N| = p^n$ and M is a primitivator.

By Lemma 3, is either $|N| = p$, or $N \leqslant A$ and $N \cap Y = 1$, or $N \cap A = 1$ and $N \leqslant Y$. In the first case, by Lemma 7, $G \in \mathfrak{F}$. Suppose that $N \leqslant A$ and $N \cap Y = 1$. Since Y is a tcc-subgroup in G, it follows that by Lemma 6, Y is a cyclic group of order dividing $p - 1$. Then $Y \in g(p)$, where g is the canonical local definition of $\mathfrak{U}$. Since $\mathfrak{U} \subseteq \mathfrak{F}$, we have by [3, Proposition IV.3.11], $g(p) \subseteq f(p)$, where f is the canonical local definition of $\mathfrak{F}$. Hence $Y \in f(p)$.

Because Y is a cyclic group of order dividing $p-1$, it follows that Y is contained in some Hall p'-subgroup H of G. Hence there exists an element $g \in G$ such that $Y \leqslant H = H_1^g \leqslant M^g$, where H_1 is a Hall p'-subgroup of G such that $H_1 \leqslant M$, because $|G : M| = p^n$. Let $M_1 = M^g$. Then $G = N \rtimes M_1$ and M_1 is a primitivator. Clearly that $M_1 = (A \cap M_1)Y$.

Since $N \leqslant A$, we have $A = N \rtimes (A \cap M_1)$. Because $A \in \mathfrak{F}$, it follows that $A/C_A(N_1) \in f(p)$, where N_1 is a minimal normal subgroup of A such that $N_1 \leqslant N$. Since A is a tcc-subgroup in G, by Lemma 5 (5), N_1 is normal in G. Hence $N = N_1$ and $C_A(N_1) = C_A(N) = N$. Then $A \cap M_1 \simeq A/N \in f(p)$.

We consider the direct product $(A \cap M_1) \times Y = \{(a, b), a \in A \cap M_1, b \in Y\}$. Let $\varphi : (A \cap M_1) \times Y \to M_1 = (A \cap M_1)Y$ be a function and $\varphi(a, b) = ab$. Since by Lemma 4, $[A, Y] \leqslant F(G) = N$, it follows that $[A \cap M_1, Y] \leqslant N$. Because $[A \cap M_1, Y] \leqslant M_1$, we have $[A \cap M_1, Y] \leqslant M_1 \cap N = 1$. Hence $A \cap M_1 \leqslant C_{M_1}(Y)$ and φ is an epimorphism. Then by [2, Theorem 2.3],

$$(A \cap M_1) \times Y/\mathrm{Ker}\ \varphi \simeq \mathrm{Im}\ \varphi = M_1.$$

Since $f(p)$ is a formation, $A \cap M_1 \in f(p)$ and $Y \in f(p)$, it follows that $M_1 \in f(p)$. Because $N \in \mathfrak{N}_p$, we have $G \in \mathfrak{N}_p f(p) = f(p) \subseteq \mathfrak{F}$.

So, we assume that $N \cap A = 1$ and $N \leqslant Y$. Similarly, we can show that $N \cap B = 1$ and $N \leqslant X$, where X is a tcc-supplement to B in G. By Lemma 6, A and B are cyclic. Hence G is supersoluble and therefore $G \in \mathfrak{F}$.

The theorem is proved.

Corollary 1. *Let A and B be* tcc*-subgroups in G and $G = AB$. If A and B are supersoluble, then G is supersoluble.*

Corollary 2 ([4, Theorem 3.1])**.** *Suppose that A and B are supersoluble subgroups of G and $G = AB$. Suppose further that A and B are totally permutable. Then G is supersoluble.*

References

[1] B. Huppert, *Endliche Gruppen I.* Berlin-Heidelberg-New York, Springer, 1967.

[2] V. S. Monakhov, *Introduction to the Theory of Final Groups and Their Classes* [in Russian]. Vysh. Shkola, Minsk, 2006.

[3] K. Doerk and T. Hawkes, *Finite soluble groups.* Berlin-New York: Walter de Gruyter, 1992.

[4] M. Asaad, A. Shaalan, *On the supersolubility of finite groups*, Arch. Math., **53**, 1989, pp.318–326.

[5] R. Maier, *A completeness property of certain formations*, Bull. Lond. Math. Soc. **24**, 1992, pp.540–544.

[6] A. Ballester-Bolinches, M. D. Perez-Ramos, *A question of R. Maier concerning formations*, J. Algebra, **182**, 1996, pp.738–747.

[7] A. Carocca, *A note on the product of F-subgroups in a finite group*, Proc. Edinb. Math. Soc., **39**, 1996, pp.37–42.

[8] A. Ballester-Bolinches, M. C. Pedraza-Aguilera, M. D. Pérez-Ramos, *Finite groups which are products of pairwise totally permutable subgroups*, Proc. Edinb. Math. Soc., **41**, 1998, pp. 567–572.

[9] W. Guo, K. P. Shum, A. N. Skiba, *Criterions of supersolubility for products of supersoluble groups*, Publ. Math. Debrecen, **68(3-4)**, 2006, pp.433–449.

[10] M. Arroyo-Jorda, P. Arroyo-Jorda, A. Martinez-Pastor, M. D. Perez-Ramos, *On finite products of groups and supersolubility*, J. Algebra, **323**, 2010, pp.2922–2934.

[11] M. Arroyo-Jorda, P. Arroyo-Jorda, M. D. Perez-Ramos, *On conditional permutability and saturated formations*, Proc. Edinb. Math. Soc., **54**, 2011, pp.309–319.

[12] M. Arroyo-Jorda, P. Arroyo-Jorda, A. Martinez-Pastor and M. D. Perez-Ramos, *On conditional permutability and factorized groups*, Annali di Matematica Pura ed Applicata, **193**, 2014, pp.1123–1138.

[13] M. Arroyo-Jorda, P. Arroyo-Jorda, *Conditional permutability of subgroups and certain classes of groups*, Journal of Algebra, **476**, 2017, pp.395–414.

[14] GAP, *Groups, Algorithms, and Programming, Version 4.10.1. www.gap-system.org*, 2019.

[15] A. N. Skiba, *On weakly s-permutable subgroups of finite groups*, J. Algebra, **315**, 2007, pp.192–209.

Contact information

Alexander Trofimuk Department of Mathematics, Gomel Francisk Skorina State University, Gomel 246019, Belarus
E-Mail(s): alexander.trofimuk@gmail.com

Received by the editors: 03.06.2019.