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On free vector balleans

Igor Protasov and Ksenia Protasova

Abstract. A vector balleans is a vector space over $\mathbb{R}$ endowed with a coarse structure in such a way that the vector operations are coarse mappings. We prove that, for every ballean $(X, \mathcal{E})$, there exists the unique free vector ballean $\mathbb{V}(X, \mathcal{E})$ and describe the coarse structure of $\mathbb{V}(X, \mathcal{E})$. It is shown that normality of $\mathbb{V}(X, \mathcal{E})$ is equivalent to metrizability of $(X, \mathcal{E})$.

1. Introduction

Let X be a set. A family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* if

- each $\varepsilon \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) \colon x \in X\}$;
- if $\varepsilon, \delta \in \mathcal{E}$ then $\varepsilon \circ \delta \in \mathcal{E}$ and $\varepsilon^{-1} \in \mathcal{E}$, where $\varepsilon \circ \delta = \{(x, y) \colon \exists z((x, z) \in \varepsilon, (z, y) \in \delta)\}$, $\varepsilon^{-1} = \{(y, x) \colon (x, y) \in \varepsilon\}$;
- if $\varepsilon \in \mathcal{E}$ and $\triangle_X \subseteq \varepsilon' \subseteq \varepsilon$ then $\varepsilon' \in \mathcal{E}$;
- for any $x, y \in X$, there exists $\varepsilon \in \mathcal{E}$ such that $(x, y) \in \varepsilon$.

A subset $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $\varepsilon \in \mathcal{E}$, there exists $\varepsilon' \in \mathcal{E}'$ such that $\varepsilon \subseteq \varepsilon'$. For $x \in X$, $A \subseteq X$ and $\varepsilon \in \mathcal{E}$, we denote $\varepsilon[x] = \{y \in X \colon (x, y) \in \varepsilon\}$, $\varepsilon[A] = \cup_{a \in A} \ \varepsilon[a]$ and say that $\varepsilon[x]$ and $\varepsilon[A]$ are *balls of radius* ε around x and A.

The pair $(X, \mathcal{E})$ is called a *coarse space* [11] or a ballean [8], [10].

Each subset $Y \subseteq X$ defines the *subballean* $(Y, \mathcal{E}_Y)$, where $\mathcal{E}_Y$ is the restriction of $\mathcal{E}$ to $Y \times Y$. A subset Y is called *bounded* if $Y \subseteq \varepsilon[x]$ for some $x \in X$ and $\varepsilon \in \mathcal{E}$.

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Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be balleans. A mapping $f\colon X \to X'$ is called *coarse* or *macro-uniform* if, for every $\varepsilon \in \mathcal{E}$, there exists $\varepsilon' \in \mathcal{E}'$ such that $f(\varepsilon[x]) \subseteq \varepsilon'[f(x)]$ for each $x \in X$. If f is a bijection such that f and f^{-1} are coarse then f is called an *asymorphism*.

Every metric d on a set X defines the *metric ballean* $(X, \mathcal{E}_d)$, where $\mathcal{E}_d$ has the base $\{\{(x,y)\colon d(x,y) < r\}\colon r \in \mathbb{R}^+\}$. We say that a ballean $(X, \mathcal{E})$ is *metrizable* if there exists a metric d on X such that $\mathcal{E} = \mathcal{E}_d$. In what follows, we consider $\mathbb{R}$ as a ballean defined by the metric $d(x,y) = |x-y|$.

Given two balleans $(X_1, \mathcal{E}_1)$, $(X_2, \mathcal{E}_2)$, we define the product $(X_1 \times X_2, \mathcal{E})$, where $\mathcal{E}$ has the base $\mathcal{E}_1 \times \mathcal{E}_2$.

Let $\mathcal{V}$ be a vector space over $\mathbb{R}$ and let $\mathcal{E}$ be a coarse structure on $\mathcal{V}$. Following [5], we say that $(\mathcal{V}, \mathcal{E})$ is a *vector ballean* if the operations

$$\mathcal{V} \times \mathcal{V} \to \mathcal{V}, \quad (x,y) \mapsto x+y \quad \textit{and} \quad \mathbb{R} \times \mathcal{V}, \quad (\lambda, x) \mapsto \lambda x$$

are coarse.

A family $\mathcal{I}$ of subsets of $\mathcal{V}$ is called a *vector ideal* if

(1) if $U, U' \in \mathcal{I}$ and $W \subseteq U$ then $U \cup U' \in \mathcal{I}$ and $W \in \mathcal{I}$;
(2) for every $U \in \mathcal{I}$, $U + U \in \mathcal{I}$;
(3) for any $U \in \mathcal{I}$ and $\lambda \in \mathbb{R}^+$, $[-\lambda, \lambda]U \in \mathcal{I}$, where $[-\lambda, \lambda]U = \bigcup\{\lambda' U\colon \lambda' \in [-\lambda, \lambda]\}$;
(4) $\bigcup \mathcal{I} = \mathcal{V}$.

A family $\mathcal{I}' \subseteq \mathcal{I}$ is called a *base* for $\mathcal{I}$ if, for each $U \in \mathcal{I}$, there is $U' \in \mathcal{I}'$ such that $U \subseteq U'$.

If $(\mathcal{V}, \mathcal{E})$ is a vector ballean then the family $\mathcal{I}$ of all bounded subsets of $\mathcal{V}$ is a vector ideal. On the other hand, every vector ideal $\mathcal{I}$ on $\mathcal{V}$ defines the vector ballean $(\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is a coarse structure with the base $\{\{(x,y)\colon x - y \in U\}\colon U \in \mathcal{I}\}$. Thus, we have got a bijective correspondence between vector balleans on $\mathcal{V}$ and vector ideas. Following this correspondence, we write $(\mathcal{V}, \mathcal{I})$ in place of $(\mathcal{V}, \mathcal{E})$.

Let $(\mathcal{V}, \mathcal{I})$, $(\mathcal{V}', \mathcal{I}')$ be vector balleans. We note that a linear mapping $f\colon \mathcal{V} \to \mathcal{V}'$ is coarse if and only if $f(U) \in \mathcal{I}'$ for each $U \in \mathcal{I}$.

In Section 2, we show that, for every ballean $(X, \mathcal{E})$, there exists the unique vector ideal $\mathcal{I}_{(X,\mathcal{E})}$ on the vector space $\mathbb{V}(X)$ with the basis X such that

- $(X, \mathcal{E})$ is a subballean of $(\mathbb{V}(X), \mathcal{I}_{(X,\mathcal{E})})$;
- for every vector ballean $(\mathcal{V}, \mathcal{I})$, every coarse mapping $(X, \mathcal{E}) \to (\mathcal{V}, \mathcal{I})$ gives rise to the unique coarse linear mapping $(\mathbb{V}(X), \mathcal{I}_{(X,\mathcal{E})}) \to (\mathcal{V}, \mathcal{I})$.

We denote $\mathbb{V}(X, \mathcal{E}) = (\mathbb{V}(X), \mathcal{I}_{(X,\mathcal{E})})$ and say that $\mathcal{I}_{(X,\mathcal{E})}$ and $\mathbb{V}(X, \mathcal{E})$ are *free vector ideal* and *free vector ballean* over $(X, \mathcal{E})$.

Free vector balleans can be considered as counterparts of free vector spaces studied in many papers, for examples, [1], [3], [4]. It should be mentioned that the free activity in topological algebra was initiated by the famous paper of Markov on free topological groups [6]. For free coarse groups see [9].

2. Construction

Given a ballean $(X, \mathcal{E})$, we consider X as the basis of the vector space $\mathbb{V}(X)$. For each $\varepsilon \in \mathcal{E}$ and $n \in \mathbb{N}$, we set $\mathcal{D}_\varepsilon = \{x - y\colon (x, y) \in \varepsilon\}$ and denote by $\mathcal{S}_{n,\varepsilon}$ the sum of n copies of $[-n, n]\mathcal{D}_\varepsilon$.

Theorem 1. *Let $(X, \mathcal{E})$ be a ballean and let $z \in X$. Then the family*

$$\{\mathcal{S}_{n,\varepsilon} + [-n, n]z\colon \varepsilon \in \mathcal{E}, n \in \mathbb{N}\}$$

is a base of the free vector ideal $\mathcal{I}_{(X,\mathcal{E})}$.

Proof. We denote by $\mathcal{I}$ the family of all subsets U of $\mathbb{V}(X)$ such that U is contained in some $\mathcal{S}_{n,\varepsilon} + [-n, n]z$. Clearly, $\mathcal{I}$ satisfies (1), (2), (3) from the definition of a vector ideal. To see that $\bigcup \mathcal{I} = \mathbb{V}(X)$, we take an arbitrary $y \in X$ and choose $\varepsilon \in \mathcal{E}$ so that $(y, z) \in \varepsilon$. Then $y = (y - z) + z \in \mathcal{D}_\mathcal{E} + z$. In view of (2), (3), we conclude that $\bigcup \mathcal{I} = \mathbb{V}(X)$.

To show that $(X, \mathcal{E})$ is a subballean of $(\mathbb{V}(X), \mathcal{I})$, we denote by $\mathcal{E}'$ the coarse structure of the ballean $(\mathbb{V}(X), \mathcal{I})$. Since $\mathcal{D}_\mathcal{E} \in \mathcal{I}$ for each $\varepsilon \in \mathcal{E}$, $\mathcal{E} = \mathcal{E}^{-1}$ we have $\mathcal{E} \subseteq \mathcal{E}'|_X$. To verify the inclusion $\mathcal{E}' \mid_X \subseteq \mathcal{E}$ we take $x, y \in X$, assume that $x - y \in \mathcal{S}_{n,\mathcal{E}} + [-n, n]z$ and show that $(x, y) \in \varepsilon^n$.

We write $x - y = \lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n) + \lambda_{n+1}z$, $(x_i, y_i) \in \varepsilon$, $\lambda_i \in [-n, n]$. Since $(\lambda_1 - \lambda_1) + \ldots + (\lambda_n - \lambda_n) + \lambda_{n+1} = 0$, we have $\lambda_{n+1} = 0$ so $x - y = \lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n)$. If $(x_1, y_1), \ldots, x_n, y_n) \in \{x, y\}$ then the statement is evident because $(x_i, y_i) \in \varepsilon$. Assume that there exists $a \in \{x_1, y_1, \ldots, x_n, y_n\}$ such that $a \notin \{x, y\}$. We take all items $\lambda_i(x_i - y_i)$, $i \in I$ such that $a \in \{x_i, y_i\}$, and denote by s the sum of all these items. The coefficient before a in the canonical decomposition of s by the basis X must be 0. We take $\lambda_k(x_k - y_k)$, $k \in I$. If $x_k = a$ or $y_k = a$ then we replace each a in $\lambda_i(x_i - y_i)$, $i \in I$, to y_k or x_k respectively. Then we get $x - y = \lambda_1(x_1' - y_1') + \ldots + \lambda_n(x_n' - y_n')$, $a \notin \{x_1', y_1', \ldots, x_n' - y_n'\}$ and $(x_i', y_i') \in \varepsilon^2$. Repeating this trick, we run into the case $x_1, y_1, \ldots, x_n, y_n \in \{x, y\}$ and $(x_i, y_i) \in \varepsilon^n$.

To conclude the proof, we observe that $\mathcal{I}$ is the minimal vector ideal on $\mathbb{V}(X)$ such that $(X, \mathcal{E})$ is a subballean of $(\mathbb{V}(X), \mathcal{I})$. If $(\mathcal{V}, \mathcal{I}')$ is a ballean and $f\colon (X,\mathcal{E}) \to (\mathcal{V},\mathcal{I}')$ is a coarse mapping then the linear extension $h\colon (\mathbb{V}(X),\mathcal{I}) \to (\mathcal{V},\mathcal{I}')$ of f is coarse because $h^{-1}(\mathcal{I}')$ is a vector ideal on $\mathbb{V}(X)$ and $\mathcal{I} \subseteq h^{-1}(\mathcal{I}')$. $\square$

3. Metrizability and normality

Theorem 2. *A ballean $\mathbb{V}(X,\mathcal{E})$ is metrizable if and only if $(X,\mathcal{E})$ is metrizable.*

Proof. By [10, Theorem 2.1.1], $(X,\mathcal{E})$ is metrizable if and only if $\mathcal{E}$ has a countable base. Apply Theorem 1. $\square$

Let $(X,\mathcal{E})$ be a ballean, $A \subseteq X$. A subset U of X is called an *asymptotic neighbourhood* of A if, for every $\varepsilon \in \mathcal{E}$, the set $\varepsilon[A]\setminus U$ is bounded.

Two subsets A, B of X are called *asymptotically disjoint (asymptotically separated)* if, for every $\varepsilon \in \mathcal{E}$, the intersection $\varepsilon[A] \cap \varepsilon[B]$ is bounded (A, B have disjoint asymptotic neighbourhoods).

A ballean $(X,\mathcal{E})$ is called *normal* [7] if any two asymptotically disjoint subsets of X are asymptotically separated. Every metrizable ballean is normal.

Given an arbitrary ballean $(X,\mathcal{E})$, the family $\mathcal{B}_X$ of all bounded subsets of X is called a *bornology* of $(X,\mathcal{E})$. A subfamily $\mathcal{B}' \subseteq \mathcal{B}_X$ is called a *base* of $\mathcal{B}_X$ if, for every $B \in \mathcal{B}_X$, there exists $B' \in \mathcal{B}_X$ such that $B \subseteq B'$. The minimal cardinality of bases of $\mathcal{B}_X$ is denoted by $\operatorname{cof}\mathcal{B}_X$.

Theorem 3. *For every ballean $(X,\mathcal{E})$, the free vector ballean $\mathbb{V}(X,\mathcal{E})$ is normal if and only if $\mathbb{V}(X,\mathcal{E})$ is metrizable.*

Proof. For $|X| = 1$, the statement is evident. Let $|X| > 1$, $a \in X$, $L = \mathbb{R}a$, $Y = X \setminus \{a\}$. Applying Theorem 1, we conclude that the canonical isomorphism between $\mathbb{V}(X,\mathcal{E})$ and $L \times \mathbb{V}(Y,\mathcal{E}_Y)$ is an asymorphism. If $\mathbb{V}(X,\mathcal{E})$ is normal then, by Theorem 1.4 from [2], $\operatorname{cof}\mathcal{B}_L = \operatorname{cof}\mathcal{B}_Z$, where $Z = \mathbb{V}(Y,\mathcal{E}_Y)$. Since $\operatorname{cof}\mathcal{B}_L = \aleph_0$, $\mathcal{B}_Z$ has a countable base. To conclude the proof, it suffices to note that $\mathcal{B}_Z$ is the vector ideal such that $\mathbb{V}(Y,\mathcal{E}_Y) = (\mathbb{V}(Y),\mathcal{I})$. Hence $\mathcal{I}$ has a countable base and $\mathbb{V}(Y,\mathcal{E}_Y)$ is metrizable by Theorem 2.1.1. from [10]. $\square$

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Contact information

I. Protasov,
K. Protasova

Department of Computer Science and Cybernetics, Kyiv University, Volodymyrska 64, 01033, Kyiv, Ukraine
E-Mail(s): `i.v.protasov@gmail.com`, `ksuha@freenet.com.ua`

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