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RESEARCH ARTICLE

On hereditary reducibility of 2-monomial matrices over commutative rings

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Abstract. A 2-monomial matrix over a commutative ring R is by definition any matrix of the form $M(t,k,n) = \Phi \begin{pmatrix} I_k & 0 \\ 0 & tI_{n-k} \end{pmatrix}$, $0 < k < n$, where t is a non-invertible element of R, Φ the companion matrix to $\lambda^n - 1$ and I_k the identity $k \times k$-matrix. In this paper we introduce the notion of hereditary reducibility (for these matrices) and indicate one general condition of the introduced reducibility.

Introduction

This paper is devoted to one class of monomial matrices over commutative rings which first arose in studying indecomposable representations of finite p-groups over local rings ([1]). They were studied more extensively (in a more generally) in [2]–[6].

Let R be a commutative ring with Jacobson radical $J(R) \neq 0$ and t a non-zero element from $J(R)$. An $n \times n$ matrix over R is called 2-*monomial*

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concerning t, if it is a permutation similar to a matrix of the following form:

$$M(t,k,n) := \Phi_n \begin{pmatrix} I_k & 0 \\ 0 & tI_{n-k} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & t \\ 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \end{pmatrix},$$

where $0 < k < n$, Φ_n is the companion matrix to the polynomial $x^n - 1$ and I_s is the identity $s \times s$ matrix. Such a matrix $M = M(t,k,n)$ is said to be *hereditary reducible* if it similar to a matrix

$$M' = \begin{pmatrix} M(t,k',n') & * \\ 0 & N \end{pmatrix}, \quad n' \neq n,$$

and *hereditary irreducible* if otherwise.

The aim of this paper is to prove the following result.

Theorem 1. *A 2-monomal matrix $M(t,k,n)$ is hereditary reducible if k and n are not coprime.*

In the next section, we indicate a more detailed interpretation of the idea of this statement.

1. Generalization of Theorem 1: formulation and proof

In this section we prove a more general theorem (from which Theorem 1 follows). Instead of R we consider the ring $\mathbb{Z}[\lambda]$ (of integer polynomials). Let (n,k) denote the greatest common divisor of the natural numbers n and k.

Theorem 2. *Let $n > k$ be positive integers, such that $(n,k) > 1$. Then for any positive divisors $d > 1$ of the number (n,k), the matrix $M(\lambda,k,n) \in M(n,\mathbb{Z}[\lambda])$ similar to a matrix of the following form*

$$\begin{pmatrix} M(\lambda,k',n') & B \\ 0 & A \end{pmatrix} \in M(n,\mathbb{Z}[\lambda]),$$

where $k' = \frac{k}{d}$ and $n' = \frac{n}{d}$.

Through this section $0 < k < n$, $0 < k' < n'$, and $0 < n' < n$. Before we prove Theorem 2, we provide four other important results which we need for the proof.

Proposition 1. *Let $n'|n$. Then there exists an $n \times n'$-matrix*

$$S = \begin{pmatrix} \lambda^{s_1} & 0 & \dots & 0 \\ 0 & \lambda^{s_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{n'}} \\ \lambda^{s_{n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{2n'}} \\ \dots & \dots & \dots & \dots \\ \lambda^{s_{n-n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n-n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_n} \end{pmatrix},$$

where $s_i \geqslant 0$, $i = 1, \dots, n$, such that $M(\lambda, k, n)S = SM(\lambda, k', n')$ if and only if $\frac{n}{n'} = \frac{k}{k'}$.

Proof. Let $l_1 = \dots = l_k = 0$, $l_{k+1} = \dots = l_n = 1$, $r_1 = \dots = r_{k'} = 0$ and $r_{k'+1} = \dots = r_{n'} = 1$ be such that

$$M(\lambda, k, n) = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{l_k} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{l_{k+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix}$$

and

$$M(\lambda, k', n') = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{r_{n'}} \\ \lambda^{r_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{r_{k'}} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{r_{k'+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{r_{n'-1}} & 0 \end{pmatrix}.$$

We denote by (i, j) the scalar equality

$$(M(\lambda, k, n)S)_{ij} = (SM(\lambda, k', n'))_{ij}.$$

Obviously, in each of matrices $M(\lambda, k, n)S$ and $SM(\lambda, k', n')$ there are exactly n non-zero element, which are in the i, j positions (i-th row, j-th column), where $i \equiv j + 1 \pmod{n'}$. Let $\delta_{n'}(i) = (i - 1) \mod n' + 1$ or, equivalently, $\delta_{n'}(i) \equiv i \pmod{n'}$, $1 \leqslant \delta_{n'}(i) \leqslant n'$. Thus $M(\lambda, k, n)S = SM(\lambda, k', n')$ if and only if scalar equalities

$$\begin{cases} (i+1, \delta_{n'}(i)) : & \lambda^{l_i}\lambda^{s_i} = \lambda^{s_{i+1}}\lambda^{r_{\delta_{n'}(i)}} \quad (i = 1, \dots, n-1), \\ (1, n') : & \lambda^{l_n}\lambda^{s_n} = \lambda^{s_1}\lambda^{r_{n'}} \end{cases}$$

hold. Obviously, these equalities are equivalent to the equalities

$$\begin{cases} (i+1, \delta_{n'}(i)) : & l_i + s_i = s_{i+1} + r_{\delta_{n'}(i)} \quad (i = 1, \dots, n-1), \\ (1, n') : & l_n + s_n = s_1 + r_{n'}. \end{cases} \tag{1}$$

Assume that for some $s_i \geqslant 0$, $i = 1, \dots, n$ $M(\lambda, k, n)S = SM(\lambda, k', n')$. Then (1) holds. Summing the equations (1), we obtain

$$\sum_{i=1}^{n-1} l_i + \sum_{i=1}^{n-1} s_i + l_n + s_n = \sum_{i=1}^{n-1} s_{i+1} + \sum_{i=1}^{n-1} r_{\delta_{n'}(i)} + s_1 + r_{n'}.$$

But since $\delta_{n'}(n) = n'$ we have that

$$\sum_{i=1}^{n} l_i + \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} s_i + \sum_{i=1}^{n} r_{\delta_{n'}(i)},$$

or $\sum_{i=1}^{n} l_i = \sum_{i=1}^{n} r_{\delta_{n'}(i)}$. This is equivalent to $\sum_{i=1}^{n} l_i = \frac{n}{n'} \sum_{i=1}^{n'} r_i$ or $k = \frac{n}{n'} k'$ and $\frac{n}{n'} = \frac{k}{k'}$.

Now, assume that $\frac{n}{n'} = \frac{k}{k'}$ and we want to prove that for some $s_i \geqslant 0$, $i = 1, \dots, n$,

$$M(\lambda, k, n)S = SM(\lambda, k', n').$$

It remains to prove that the equations in (1) hold for non negative integers s_i. We will prove it for arbitrary integers s_i since that addition of any number to s_i will also be a solution. Let $s_1 = 0$, $s_{i+1} = l_i + s_i - r_{\delta_{n'}(i)}$ ($i = 1, \dots, n-1$). It follows immediately that all, accept last equation in (1) hold and $s_n = \sum_{i=1}^{n-1} l_i - \sum_{i=1}^{n-1} r_{\delta_{n'}(i)}$. If we replace s_n by

$\sum_{i=1}^{n-1} l_i - \sum_{i=1}^{n-1} r_{\delta_{n'}(i)}$ and s_1 by 0 in the last equation in (1), we obtain the following equation:

$$l_n + \sum_{i=1}^{n-1} l_i - \sum_{i=1}^{n-1} r_{\delta_{n'}(i)} = r_{n'} \quad \text{or} \quad l_n + \sum_{i=1}^{n-1} l_i = \sum_{i=1}^{n-1} r_{\delta_{n'}(i)} + r_{n'}.$$

This equation is equivalent to

$$\sum_{i=1}^{n} l_i = \sum_{i=1}^{n} r_{\delta_{n'}(i)}$$

and $\sum_{i=1}^{n} l_i = \frac{n}{n'} \sum_{i=1}^{n'} r_i$ which in turn is equivalent to $k = \frac{n}{n'} k'$ or $\frac{n}{n'} = \frac{k}{k'}$. The proof is complete. □

Using a similar argument applied in the previous proof, we can state the following result:

Proposition 2. *Let $n'|n$, $l_i \geqslant 0$ $(i = 1, \ldots, n)$, $\sum_{i=1}^{n} l_i = k$,*

$$M = \begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \lambda^{l_k} & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \lambda^{l_{k+1}} & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & \lambda^{l_{n-1}} & 0 \end{pmatrix}.$$

Then there exists an $n \times n'$-matrix

$$S = \begin{pmatrix} \lambda^{s_1} & 0 & \ldots & 0 \\ 0 & \lambda^{s_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda^{s_{n'}} \\ \lambda^{s_{n'+1}} & 0 & \ldots & 0 \\ 0 & \lambda^{s_{n'+2}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda^{s_{2n'}} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda^{s_{n-n'+1}} & 0 & \ldots & 0 \\ 0 & \lambda^{s_{n-n'+2}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda^{s_n} \end{pmatrix},$$

where $s_i \geqslant 0$, $i = 1, \dots, n$, *such that* $MS = SM(\lambda, k', n')$ *if and only if* $\frac{n}{n'} = \frac{k}{k'}$.

Next, we provide a result regarding the similarity of $M(\lambda, k, n)$ and a certain matrix.

Proposition 3. *Let* $n'|n$, $\frac{n}{n'} = \frac{k}{k'}$. *Than* $k' < k$ *and* $M(\lambda, k, n)$ *is similar (over* $\mathbb{Z}[\lambda]$*) to a matrix of the form*

$$M = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{l_{k'}} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{l_{k'+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix},$$

where $l_1 = \dots = l_{k'} = 0$, $l_{k'+1} = \dots = l_{k'+n-k} = 1$, $l_{k'+n-k+1} = \dots = l_n = 0$.

Proof. Clearly $k' < k$ as $\frac{k}{k'} = \frac{n}{n'} > 1$. Now, rearrange the rows and columns of the matrix $M(\lambda, k, n)$ in the order $k-k'+1, k-k'+2, \dots n, 1, 2, \dots, k-k'$ and denote the new matrix by M:

$$M = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{l_{k'}} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{l_{k'+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix},$$

where $l_1 = \dots = l_{k'} = 0$, $l_{k'+1} = \dots = l_{k'+n-k} = 1$, $l_{k'+n-k+1} = \dots = l_n = 0$. $\square$

The next result connects the previous two results.

Proposition 4. *Let* $n'|n$, $k'<k$.

$$M=\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n}\\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & \lambda^{l_{k'}} & 0 & \dots & 0 & 0\\ 0 & \dots & 0 & \lambda^{l_{k'+1}} & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix},$$

where $l_1=\cdots=l_{k'}=0$, $l_{k'+1}=\cdots=l_{k'+n-k}=1$, $l_{k'+n-k+1}=\cdots=l_n=0$. *Then there exists an* $n\times n'$*-matrix*

$$S=\begin{pmatrix} 1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\\ 1 & 0 & \dots & 0\\ 0 & \lambda^{s_{n'+2}} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda^{s_{2n'}}\\ \dots & \dots & \dots & \dots\\ \lambda^{s_{n-n'+1}} & 0 & \dots & 0\\ 0 & \lambda^{s_{n-n'+2}} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda^{s_n} \end{pmatrix},$$

where $s_i\geqslant 0$, $(i=n'+2,\dots,n)$, *such that* $MS=SM(\lambda,k',n')$ *if and only if* $\frac{n}{n'}=\frac{k}{k'}$.

Proof. Clearly if $MS=SM(\lambda,k',n')$, then $\frac{n}{n'}=\frac{k}{k'}$ by Proposition 2. Assume that $\frac{n}{n'}=\frac{k}{k'}$. Let $r_1=\cdots=r_{k'}=0$, $r_{k'+1}=\cdots=r_{n'}=1$ such that

$$M(\lambda,k',n')=\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{r_{n'}}\\ \lambda^{r_1} & \dots & 0 & 0 & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & \lambda^{r_{k'}} & 0 & \dots & 0 & 0\\ 0 & \dots & 0 & \lambda^{r_{k'+1}} & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & 0 & \dots & \lambda^{r_{n'-1}} & 0 \end{pmatrix}.$$

We will prove that $M(\lambda, k, n)S = SM(\lambda, k', n')$ for some $s_i \geqslant 0$ $(i = n'+2, \ldots, n)$.

It remains to prove that (1) holds, where $s_1 = \cdots = s_{k'} = s_{k'+1} = 0$. Let

$$s_1 = 0, \quad s_{i+1} = l_i + s_i - r_{\delta_{n'}(i)} \quad (i = 1, \ldots, n-1). \tag{2}$$

Using a similar argument to the one used in (1) all equations hold. Furthermore, if $\frac{n-k}{n'-k'} = \frac{n}{n'} = \frac{k}{k'} > 1$ it follows $n' - k' < n - k$,

$$l_1 = \cdots = l_{k'} = 0 = r_1 = \cdots = r_{k'}$$

and

$$l_{k'+1} = \cdots = l_{n'} = 1 = r_{k'+1} = \cdots = r_{n'}.$$

Therefore $l_i = r_i$ $(i = 1, \ldots, n')$ and $s_{i+1} = s_i$ $(i = 1, \ldots, n')$ by (2). We can also see that $s_1 = \cdots = s_{n'} = s_{n'+1} = 0$.

It remains to prove that $s_i \geqslant 0$ $(i = n'+2, \ldots, n)$. Let $s_{n+1} = s_1$. It follows from (2) end last equation from (1), that $s_{n+1} = l_n + s_n - r_{\delta_{n'}(n)}$, which is equivalent to $s_1 = s_{n+1} = 0$ and

$$s_{i+1} = \sum_{j=1}^{i} l_j - \sum_{j=1}^{i} r_{\delta_{n'}(j)} \quad (i = 1, \ldots, n). \tag{3}$$

Let us consider $s(i) = s_i$, as a function of an integer i $(1 \leqslant i \leqslant n+1)$. Then $s(i) = 0$ if $1 \leqslant i \leqslant n'+1$. Thus, $s(i)$ is a constant for $1 \leqslant i \leqslant n'+1$. If $n'+1 \leqslant i < i+1 \leqslant k'+n-k+1$, then it follows from (3) that $s(i+1) - s(i) = l_i - r_{\delta_{n'}(i)} = 1 - r_{\delta_{n'}(i)} \geqslant 0$. Therefore $s(i)$ either increases or remains constant for each step and $s(i) \geqslant s(n'+1) = 0$. If $k'+n-k+1 \leqslant i < i+1 \leqslant n+1$, then it follows from (3) that $s(i+1) - s(i) = l_i - r_{\delta_{n'}(i)} = 0 - r_{\delta_{n'}(i)} \leqslant 0$. Consequently $s(i)$ either decreases or remains constant for each step and $s(i) \geqslant s(n+1) = 0$. Therefore, $s_i = s(i) \geqslant 0$ $(i = n'+2, \ldots, n)$. □

Finally, we are in a position to prove our main result.

Proof of Theorem 2. Recall that $0 < k < n$, $\frac{k}{k'} = \frac{n}{n'} = d > 1$ and $0 < n' < n$, $0 < k' < k$. By Propositin 3, $M(\lambda, k, n)$ is similar (over $\mathbb{Z}[\lambda]$)

to a matrix of the form

$$M = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{l_{k'}} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{l_{k'+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix},$$

where $l_1 = \cdots = l_{k'} = 0$, $l_{k'+1} = \cdots = l_{k'+n-k} = 1$, $l_{k'+n-k+1} = \cdots = l_n = 0$.

By Proposition 4 there exists an $n \times n'$-matrix

$$S = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & \lambda^{s_{n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{2n'}} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ \lambda^{s_{n-n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n-n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_n} \end{pmatrix} = \begin{pmatrix} I_{n'} \\ S' \end{pmatrix},$$

where

$$S' = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \lambda^{s_{n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{2n'}} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ \lambda^{s_{n-n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n-n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_n} \end{pmatrix},$$

$I_{n'}$ is the identity $n' \times n'$ matrix, $s_i \geqslant 0$ $(i = n'+2, \dots, n)$ such that $MS = SM(\lambda, k', n')$. Now,

$$\begin{pmatrix} I_{n'} & 0 \\ S' & I_{n-n'} \end{pmatrix}^{-1} M \begin{pmatrix} I_{n'} & 0 \\ S' & I_{n-n'} \end{pmatrix} = \begin{pmatrix} I_{n'} & 0 \\ -S' & I_{n-n'} \end{pmatrix} M \begin{pmatrix} I_{n'} & 0 \\ S' & I_{n-n'} \end{pmatrix}.$$

If we omit the last $n - n'$ columns of the last matrix, we obtain

$$\begin{pmatrix} I_{n'} & 0 \\ -S' & I_{n-n'} \end{pmatrix} M \begin{pmatrix} I_{n'} \\ S' \end{pmatrix} = \begin{pmatrix} I_{n'} & 0 \\ -S' & I_{n-n'} \end{pmatrix} \begin{pmatrix} I_{n'} \\ S' \end{pmatrix} M(\lambda, k', n')$$
$$= \begin{pmatrix} I_{n'} \\ 0 \end{pmatrix} M(\lambda, k', n') = \begin{pmatrix} M(\lambda, k', n') \\ 0 \end{pmatrix}.$$

In conclusion, we note that matrix M and the matrix $M(\lambda, k, n)$ are similar (over $\mathbb{Z}[\lambda]$) to a matrix of the form

$$\begin{pmatrix} M(\lambda, k', n') & B \\ 0 & A \end{pmatrix} \in M(n, \mathbb{Z}[\lambda]),$$

as claimed. □

Note that Theorem 1 folows from the last theorem and the existence of the homomorphism of rings $f\colon \mathbb{Z}[\lambda] \to R$ where $f(1) = 1$ and $f(\lambda) = t$.

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