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RESEARCH ARTICLE

Conjugacy in finite state wreath powers of finite permutation groups

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Abstract. It is proved that conjugated periodic elements of the infinite wreath power of a finite permutation group are conjugated in the finite state wreath power of this group. Counter-examples for non-periodic elements are given.

1. Introduction

The conjugacy classes in the full automorphism group of a regular rooted tree are described in [1]. But for its subgroup of finite state automorphisms the corresponding description is a challenging task. In particular, there exist finite state level-transitive automorphisms (and therefore, conjugated in the full automorphism group) which are not conjugates in the finite state subgroup [2]. Deep results about conjugation of some special finite state automorphisms were obtained in [3] and [4].

The most natural way to introduce finite state automorphisms uses automata theory. But regarding our purposes we choose a language of infinite wreath products. The full automorphism group of m-regular rooted tree is the infinite wreath power of the symmetric group of degree m. If we restrict ourselves to some subgroup (or even subsemigroup) G of this symmetric group we naturally obtain the infinite and finite state wreath powers of G (cf. [5, 6]).

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The work is organized as follows. In Section 2 we recall the finite state wreath power of a finite permutation group. In Section 3 we prove the main result of the paper. Namely, if periodic finite state elements of the infinite wreath power of a finite permutation group are conjugated in this wreath power then they are conjugated in the finite state wreath power of a given permutation group. In Section 4 we show how to construct non-periodic finite state elements conjugated in the wreath power but not conjugated in the finite state wreath power of a given permutation group.

For all other definitions used in the paper one can refer to [2].

2. Finite state wreath power

Let A be a finite set of cardinality $m \geqslant 2$. Consider a finite group G acting faithfully on the set A. In other words, the permutation group (G, A) is a subgroup of the symmetric group $Sym(A)$. In the sequel we assume that the groups act on the sets from the right and denote by a^g the result of the action of a group element g on a point a.

Denote by $W^\infty(G, A)$ the infinitely iterated wreath product of (G, A). The group $W^\infty(G, A)$ consists of permutations of the infinite cartesian product X^∞ given by infinite sequences of the form

$$\mathsf{g} = [\,g_1, g_2(x_1), \ldots, g_n(x_1, \ldots, x_{n-1}), \ldots], \tag{1}$$

where $g_1 \in G$, $g_2(x_1) : A \to G$, $\ldots$, $g_n(x_1, \ldots, x_{n-1}) : A^{n-1} \to G$, $\ldots$ For each $n \geqslant 1$ we call g_n the n-th term of g and denote it by $[\mathsf{g}]_n$. An element g acts on a point

$$\bar{a} = (a_1, a_2, \ldots, a_n, \ldots) \in A^\infty$$

by the rule

$$\bar{a}^{\mathsf{g}} = (a_1^{g_1}, a_2^{g_2(a_1)}, \ldots, a_n^{g_n(a_1,\ldots,a_{n-1})}, \ldots).$$

Let $\mathsf{g} = [g_1, g_2(x_1), g_3(x_1, x_2), \ldots] \in W^\infty(G, A)$ and $\bar{a} = (a_1, \ldots, a_n) \in A^n$ for some $n \geqslant 1$. Define an element $\mathsf{rest}(\mathsf{g}, \bar{a}) \in W^\infty(G, A)$ as

$$\mathsf{rest}(\mathsf{g}, \bar{a}) = [h_1, h_2(x_1), h_3(x_1, x_2), \ldots],$$

where

$$\begin{aligned} h_1 &= g_{n+1}(a_1, \ldots, a_n), \\ h_2(x_1) &= g_{n+2}(a_1, \ldots, a_n, x_1), \\ h_3(x_1, x_2) &= g_{n+3}(a_1, \ldots, a_n, x_1, x_2), \ldots \end{aligned}$$

The tuple $\mathsf{rest}(\mathsf{g},\bar{a})$ has the form (1) as well and therefore belongs to the group $W^\infty(G,A)$. The element $\mathsf{rest}(\mathsf{g},\bar{a})$ is called the state of g at $\bar{a}$. Also we consider g as a state of itself.

Define the set

$$\mathcal{Q}(\mathsf{g}) = \{\mathsf{rest}(\mathsf{g},\bar{a}) : \bar{a} \in A^n, n \geqslant 1\} \cup \{\mathsf{g}\}$$

of all states of g. In particular, for the identity element $\mathsf{e} \in W^\infty(G,A)$ we get the equality $\mathcal{Q}(\mathsf{e}) = \{\mathsf{e}\}$. The following lemma is directly verified.

Lemma 1. *Let* $\mathsf{g},\mathsf{h} \in W^\infty(G,A)$. *Then*

$$\mathcal{Q}(\mathsf{gh}) \subseteq \mathcal{Q}(\mathsf{g}) \cdot \mathcal{Q}(\mathsf{h}), \quad \mathcal{Q}(\mathsf{g}^{-1}) = \mathcal{Q}(\mathsf{g})^{-1}.$$

If $\mathsf{h} \in \mathcal{Q}(\mathsf{g})$ *then* $\mathcal{Q}(\mathsf{h}) \subseteq \mathcal{Q}(\mathsf{g})$.

Let

$$FW^\infty(G,A) = \{\mathsf{g} \in W^\infty(G,A) : |\mathcal{Q}(\mathsf{g})| < \infty\}.$$

Lemma 1 implies that this set form a subgroup of $W^\infty(G,A)$.

Definition 1. The group $FW^\infty(G,A)$ is called the finite state wreath power of the permutation group (G,A).

The group $FW^\infty(G,A)$ is countable while $W^\infty(G,A)$ is not.

Another useful remark is that both permutation groups $(W^\infty(G,A),A^\infty)$ and $(FW^\infty(G,A),A^\infty)$ split into the wreath product of (G,A) and itself, i.e.

$$(W^\infty(G,A),A^\infty) = (G,A) \wr (W^\infty(G,A),A^\infty) \tag{2}$$

and

$$(FW^\infty(G,A),A^\infty) = (G,A) \wr (FW^\infty(G,A),A^\infty). \tag{3}$$

This allows us to present an element $\mathsf{g} = [g_1, g_2(x_1), g_3(x_1,x_2), \ldots] \in W^\infty(G,A)$ in the form

$$\mathsf{g} = [g_1; \mathsf{rest}(\mathsf{g},a), a \in A]. \tag{4}$$

3. Conjugation of periodic elements

It is convenient for us to identify the set A with the set $\{1,\ldots,m\}$. We need some additional notation here. Let elements $g \in G$ and $a \in A$ be fixed. Consider the cyclic decomposition of g as a permutation on A. The length of the cycle containing a will be denoted by $l(g,a)$. Then $a^{g^{l(g,a)}} = a$

and $l(g,a)$ is the smallest integer satisfying this equality. The minimal element of this cycle we denote by $s(g,a)$. Note, that $l(g, s(g,a)) = l(g,a)$. The smallest integer $d \geqslant 0$ such that $a^{g^d} = s(g,a)$ will be denoted by $d(g,a)$. Then $0 \leqslant d(g,a) \leqslant l(g,a) - 1$.

We have the following

Lemma 2. *Let u, v and h be elements of G such that $u = h^{-1}vh$. Then for arbitrary $a \in A$ the equality $l(u, a^h) = l(v, a)$ holds.*

Proof. From the definition of $l(v,a)$ we have $a^{v^{l(v,a)}} = a$. Thus

$$(a^h)^{u^{l(v,a)}} = (a^h)^{(h^{-1}vh)^{l(v,a)}} = (a^h)^{h^{-1}v^{l(v,a)}h} = (a^{v^{l(v,a)}})^h = a^h.$$

Hence, $l(u, a^h) \leqslant l(v,a)$. The inequality $l(u, a^h) \geqslant l(v,a)$ is proved analogously. □

The rules of multiplication and taking inverses in wreath products imply that for arbitrary $\mathsf{u} = [u_1, \ldots], \mathsf{v} = [v_1, \ldots] \in W^\infty(G, A)$ and $a \in A$ one have the equalities:

$$\mathsf{rest}(\mathsf{uv}, a) = \mathsf{rest}(\mathsf{u}, a)\mathsf{rest}(\mathsf{v}, a^{u_1}) \quad \text{and} \quad \mathsf{rest}(\mathsf{u}^{-1}, a) = \mathsf{rest}(\mathsf{u}, a^{u_1^{-1}})^{-1}.$$

Then we can prove

Lemma 3. *Let $\mathsf{u} = [u_1, \ldots]$, $\mathsf{v} = [v_1, \ldots]$ and $\mathsf{h} = [h_1, \ldots]$ be elements of the group $W^\infty(G, A)$ such that $\mathsf{u} = \mathsf{h}^{-1}\mathsf{vh}$. Then for arbitrary $a \in A$ the equality*

$$\mathsf{rest}(\mathsf{u}^{l(u_1, a^{h_1})}, a^{h_1}) = (\mathsf{rest}(\mathsf{h}, a))^{-1}\mathsf{rest}(\mathsf{v}^{l(v_1,a)}, a)\mathsf{rest}(\mathsf{h}, a)$$

holds.

Proof. Note, that $u_1 = h_1^{-1}v_1h_1$. By Lemma 2 we have equalities

$$\mathsf{u}^{l(u_1, a^{h_1})} = \mathsf{u}^{l(v_1,a)} = \mathsf{h}^{-1}\mathsf{v}^{l(v_1,a)}\mathsf{h}.$$

Hence,

$$\begin{aligned}
\mathsf{rest}(\mathsf{u}^{l(u_1, a^{h_1})}, a^{h_1}) &= \mathsf{rest}(\mathsf{h}^{-1}\mathsf{v}^{l(v_1,a)}\mathsf{h}, a^{h_1}) \\
&= \mathsf{rest}(\mathsf{h}^{-1}, a^{h_1})\mathsf{rest}(\mathsf{v}^{l(v_1,a)}, a)\mathsf{rest}(\mathsf{h}, a^{v_1^{l(v_1,a)}}) \\
&= (\mathsf{rest}(\mathsf{h}, a))^{-1}\mathsf{rest}(\mathsf{v}^{l(v_1,a)}, a)\mathsf{rest}(\mathsf{h}, a).
\end{aligned}$$

□

Theorem 1. *Arbitrary elements of finite order of the group $FW^\infty(G,A)$, conjugated in the group $W^\infty(G,A)$, are conjugated in the group $FW^\infty(G,A)$ as well.*

Proof. Denote by M the set of all pairs $(\mathsf{u},\mathsf{v}) \in FW^\infty(G,A)\times FW^\infty(G,A)$ such that elements u and v have finite order and are conjugated in the group $W^\infty(G,A)$. For each pair $\theta = (\mathsf{u},\mathsf{v}) \in M$ let us fix an element $\Psi(\theta) \in W^\infty(G,A)$ such that the equality $\mathsf{u} = (\Psi(\theta))^{-1}\mathsf{v}\Psi(\theta)$ holds. The correspondence $\theta \mapsto \Psi(\theta)$ may be regarded as a mapping $\Psi : M \to W^\infty(G,A)$.

Now we proceed as follows. Using Ψ we construct new mappings

$$\Psi_* : M \to W^\infty(G,A) \quad \text{and} \quad \Phi : M \times A \to W^\infty(G,A).$$

Then we show that for each pair $\theta = (\mathsf{u},\mathsf{v}) \in M$ the equality

$$\mathsf{u} = (\Psi_*(\theta))^{-1}\mathsf{v}\Psi_*(\theta)$$

holds. Finally, we prove that indeed $\Psi_*(M) \subset FW^\infty(G,A)$. Then the statement of the theorem follows.

For a pair $\theta = (\mathsf{u},\mathsf{v}) \in M$, where $\mathsf{u} = [u_1, u_2(x_1), \ldots]$ and $\mathsf{u} = [v_1, v_2(x_1), \ldots]$, we will use the notation $\Psi(\theta) = [h_1, h_2(x_1), \ldots]$.

Step 1. Let us define mappings Ψ_* and Φ.

It is sufficient to check that recursive equalities

$$\Psi_*(\theta) = [h_1; \mathsf{rest}(\mathsf{v}^{d(v_1,a)}, a)\Phi(\theta,a)\big(\mathsf{rest}(\mathsf{u}^{d(v_1,a)}, a^{h_1})\big)^{-1}, a \in A], \tag{5}$$

$$\Phi(\theta,a) = \Psi_*\Big(\mathsf{rest}\big(\mathsf{u}^{l(u_1,a^{h_1})}, (s(v_1,a))^{h_1}\big), \mathsf{rest}\big(\mathsf{v}^{l(v_1,a)}, s(v_1,a)\big)\Big), \ a \in A \tag{6}$$

correctly define required mappings Ψ_* and Φ. First of all, from (5) we have $[\Psi_*(\theta)]_1 = [\Psi(\theta)]_1$ and hence the term $[\Psi_*(\theta)]_1$ is well-defined. To define other terms we need $\Phi(\theta,a)$, $a \in A$. Then, by Lemma 3, for each $a \in A$ the pair

$$\Big(\mathsf{rest}\big(\mathsf{u}^{l(u_1,a^{h_1})}, (s(v_1,a))^{h_1}\big), \mathsf{rest}\big(\mathsf{v}^{l(v_1,a)}, s(v_1,a)\big)\Big)$$

belongs to M. Hence, equality (6) defines the first term of $\Phi(\theta,a)$, $a \in A$. Again looking at (5), we obtain the second term of $\Psi_*(\theta)$ and so on. Inductively, for arbitrary $k \geqslant 1$, having defined the kth term of $\Psi_*(\theta)$ by (5), we define the kth term of $\Phi(\theta)$ by (6) and this gives us a possibility to define the $(k+1)$th term of $\Psi_*(\theta)$ by (5).

Note that for every $a \in A$ the equality $\Phi(\theta, a^{v_1}) = \Phi(\theta, a)$ holds.

Step 2. Let us prove the equality $\mathsf{u} = (\Psi_*(\theta))^{-1}\mathsf{v}\Psi_*(\theta)$, where $\theta = (\mathsf{u}, \mathsf{v}) \in M$.

We will prove by induction on k the equality

$$[\mathsf{u}]_k = [(\Psi_*(\theta))^{-1}\mathsf{v}\Psi_*(\theta)]_k.$$

Since $[\Psi_*(\theta)]_1 = [\Psi(\theta)]_1$ and

$$[\mathsf{u}]_1 = [(\Psi(\theta))^{-1}\mathsf{v}\Psi(\theta)]_1$$

we obtain the required statement for $k = 1$.

Assume that for the $(k-1)$th terms the equality is proved. Proceed with the kth ones. Fix an element $a \in A$. Denote by g the state of $(\Psi_*(\theta))^{-1}\mathsf{v}\Psi_*(\theta)$ at a^{h_1}. It is sufficient to check the equality $[\mathsf{rest}(\mathsf{u}, a^{h_1})]_k = [\mathsf{g}]_k$. For g we have the equalities:

$$\begin{aligned} g &= \mathsf{rest}\big((\Psi_*(\theta))^{-1}\mathsf{v}\Psi_*(\theta), a^{h_1}\big) \\ &= \mathsf{rest}\big((\Psi_*(\theta))^{-1}, a^{h_1}\big)\mathsf{rest}(\mathsf{v}, a)\mathsf{rest}\big(\Psi_*(\theta), a^{v_1}\big) \\ &= (\mathsf{rest}(\Psi_*(\theta), a))^{-1}\mathsf{rest}(v, a)\mathsf{rest}(\Psi_*(\theta), a^{v_1}) \\ &= \mathsf{rest}(\mathsf{u}^{d(v_1,a)}, a^{h_1})(\Phi(\theta, a))^{-1}(\mathsf{rest}(\mathsf{v}^{d(v_1,a)}, a))^{-1}\mathsf{rest}(\mathsf{v}, a) \\ &\qquad \cdot \mathsf{rest}(\mathsf{v}^{d(v_1,a^{v_1})}, a^{v_1})\Phi(\theta, a^{v_1})(\mathsf{rest}(\mathsf{u}^{d(v_1,a^{v_1})}, a^{v_1h_1}))^{-1} \\ &= \mathsf{rest}(\mathsf{u}^{d(v_1,a)}, a^{h_1})(\Phi(\theta, a))^{-1}(\mathsf{rest}(\mathsf{v}^{d(v_1,a)}, a))^{-1} \\ &\qquad \cdot \mathsf{rest}(\mathsf{v}^{d(v_1,a^{v_1})+1}, a)\Phi(\theta, a^{v_1})(\mathsf{rest}(\mathsf{u}^{d(v_1,a^{v_1})}, a^{v_1h_1}))^{-1}. \end{aligned}$$

There are two possibilities: $s(v_1, a) = a$ or $s(v_1, a) \neq a$. Consider these cases.

1) Let $s(v_1, a) = a$. Then $d(v_1, a) = 0$ and $d(v_1, a^{v_1}) = l(v_1, a) - 1$. This implies $\mathsf{rest}(\mathsf{u}^{d(v_1,a)}, a^{h_1}) = \mathsf{e}$ and $\mathsf{rest}(\mathsf{v}^{d(v_1,a)}, a) = \mathsf{e}$. Lemma 2 and equality (6) then implies

$$\Phi(\theta, a) = \Psi_*(\mathsf{rest}(\mathsf{u}^{l(v_1,a)}, a^{h_1}), \mathsf{rest}(\mathsf{v}^{l(v_1,a)}, a)).$$

Then, in view of the inductive hypothesis, the equalities follow:

$$\begin{aligned} [g]_k &= [(\Phi(\theta, a))^{-1}\mathsf{rest}(\mathsf{v}^{l(v_1,a)}, a)\Phi(\theta, a^{v_1})(\mathsf{rest}(\mathsf{u}^{l(v_1,a)-1}, a^{v_1h_1}))^{-1}]_k \\ &= [\mathsf{rest}(\mathsf{u}^{l(v_1,a)}, a^{h_1})(\mathsf{rest}(\mathsf{u}^{l(v_1,a)-1}, a^{h_1u_1}))^{-1}]_k \\ &= [\mathsf{rest}(\mathsf{u}, a^{h_1})\mathsf{rest}(\mathsf{u}^{l(v_1,a)-1}, a^{h_1u_1})(\mathsf{rest}(\mathsf{u}^{l(v_1,a)-1}, a^{h_1u_1}))^{-1}]_k \\ &= [\mathsf{rest}(\mathsf{u}, a^{h_1})]_k. \end{aligned}$$

2) Let $s(v_1, a) \neq a$. Then $d(v_1, a^{v_1}) = d(v_1, a) - 1$. For g now we have:

$$\begin{aligned} g &= \mathsf{rest}(\mathsf{u}^{d(v_1,a)}, a^{h_1})(\Phi(\theta, a))^{-1}(\mathsf{rest}(\mathsf{v}^{d(v_1,a)}, a))^{-1} \\ &\qquad \cdot \mathsf{rest}(\mathsf{v}^{d(v_1,a^{v_1})+1}, a)\Phi(\theta, a^{v_1})(\mathsf{rest}(\mathsf{u}^{d(v_1,a^{v_1})}, a^{v_1 h_1}))^{-1} \\ &= \mathsf{rest}(\mathsf{u}^{d(v_1,a)}, a^{h_1})(\mathsf{rest}(\mathsf{u}^{d(v_1,a)-1}, a^{h_1 u_1}))^{-1} \\ &= \mathsf{rest}(\mathsf{u}, a^{h_1})\mathsf{rest}(\mathsf{u}^{d(v_1,a)-1}, a^{h_1 u_1})(\mathsf{rest}(\mathsf{u}^{d(v_1,a)-1}, a^{h_1 u_1}))^{-1} \\ &= \mathsf{rest}(\mathsf{u}, a^{h_1}). \end{aligned}$$

In both cases we obtained the equality $[\mathsf{rest}(\mathsf{u}, a^{h_1})]_k = [\mathsf{g}]_k$. Hence, our statement is true for the kth terms.

Step 3. Let us check the inclusion $\Psi_*(M) \subset FW^\infty(G, A)$.

Denote by M_k the subset of all pairs $(u, v) \in M$ such that the orders of u and v equal k. These subsets are pairwise disjoint and

$$M = \bigcup_{k=1}^{\infty} M_k.$$

Let us prove by induction on k that $\Psi_*(M_k) \subset FW^\infty(G, A)$.

In case $k = 1$ we have $M_1 = \{(\mathsf{e}, \mathsf{e})\}$. Since

$$\mathsf{rest}(\mathsf{e}, a) = \mathsf{e}, \quad a \in A,$$

equalities (5) and (6) imply

$$\mathsf{rest}(\Psi_*(\mathsf{e}, \mathsf{e}), a) = \Psi_*(\mathsf{e}, \mathsf{e}), \quad a \in A.$$

Hence, $\mathsf{g} \in FW^\infty(G, A)$.

Suppose that $\Psi_*(M_i) \subset FW^\infty(G, A)$ for all $i < k$. We are going to prove the inclusion $\Psi_*(M_k) \subset FW^\infty(G, A)$. If the set M_k is empty then the statement is true. Let $\theta = (\mathsf{u}, \mathsf{v})$ be a pair belonging to the set M_k. We have to show that $|\mathcal{Q}(\Psi_*(\theta))| < \infty$.

For an element $\mathsf{g} \in W^\infty(G, A)$ its set of stable states is defined as

$$\mathcal{SQ}(\mathsf{g}) = \{\mathsf{rest}(\mathsf{g}, \bar{a}) : \mathsf{rest}(\mathsf{g}^2, \bar{a}) = (\mathsf{rest}(\mathsf{g}, \bar{a}))^2, \bar{a} \in A^n, n \geqslant 1\} \cup \{\mathsf{g}\}.$$

Since $\mathsf{u}, \mathsf{v} \in FW^\infty(G, A)$ the set

$$Q_1 = \Psi_*\Big(\big(\mathcal{SQ}(\mathsf{u}) \times \mathcal{SQ}(\mathsf{v})\big) \cap M_k\Big)$$

is finite. In particular, $\Psi_*\big((\mathsf{u}, \mathsf{v})\big) \in Q_1$. We are going to show that $Q_1 \subset FW^\infty(G, A)$.

Fix arbitrary $\mathsf{g} \in Q_1$. Then

$$\mathsf{g} = \Psi_*\big((\tilde{\mathsf{u}}, \tilde{\mathsf{v}})\big)$$

for some $\tilde{\mathsf{u}} \in \mathcal{SQ}(\mathsf{u})$, $\tilde{\mathsf{v}} \in \mathcal{SQ}(\mathsf{v})$ such that $(\tilde{\mathsf{u}}, \tilde{\mathsf{v}}) \in M_k$.

Let $a \in A$. Denote $l([\tilde{\mathsf{v}}]_1, a)$ by ℓ. Two possible cases arise.

Case 1: $\ell = 1$. Then $s([\tilde{\mathsf{v}}]_1, a) = a$ and $d([\tilde{\mathsf{v}}]_1, a) = 0$. By equalities (5) and (6) we now obtain

$$\begin{aligned}\mathsf{rest}(\mathsf{g}, a) &= \mathsf{rest}\big(\Psi_*((\tilde{\mathsf{u}}, \tilde{\mathsf{v}})), a\big) = \Phi\big((\tilde{\mathsf{u}}, \tilde{\mathsf{v}}), a\big) \\ &= \Psi_*\Big(\mathsf{rest}\big(\tilde{\mathsf{u}}, a^{[\Psi((\tilde{\mathsf{u}}, \tilde{\mathsf{v}}))]_1}\big), \mathsf{rest}(\tilde{\mathsf{v}}, a)\Big).\end{aligned}$$

The latter element belongs to Q_1. Hence, in this case

$$\mathsf{rest}(\mathsf{g}, a) \in Q_1.$$

Case 2: $\ell > 1$. By Lemma 2 the equality $l([\tilde{\mathsf{u}}]_1, a^{[\Psi((\tilde{\mathsf{u}}, \tilde{\mathsf{v}}))]_1}) = l([\tilde{\mathsf{v}}]_1, a) = \ell$ holds. By (6) we have

$$\Phi\big((\tilde{\mathsf{u}}, \tilde{\mathsf{v}}), a\big) = \Psi_*\Big(\mathsf{rest}\big(\tilde{\mathsf{u}}^\ell, (s([\tilde{\mathsf{v}}]_1, a))^{[\Psi((\tilde{\mathsf{u}}, \tilde{\mathsf{v}}))]_1}\big), \mathsf{rest}\big(\tilde{\mathsf{v}}^\ell, s([\tilde{\mathsf{v}}]_1, a)\big)\Big). \quad (7)$$

Since cyclic decompositions of both $[\tilde{\mathsf{u}}]_1$ and $[\tilde{\mathsf{v}}]_1$ contain a cycle of length ℓ, the number ℓ divides the orders of both $\tilde{\mathsf{u}}$ and $\tilde{\mathsf{v}}$. It implies that the orders of arguments of Ψ_* in (7) are strictly less then k. Indeed, they are states of the ℓth powers of elements $\tilde{\mathsf{u}}$ and $\tilde{\mathsf{v}}$ correspondingly at elements belonging to cycles of length ℓ. Applying the inductive hypothesis we get $\Phi((\tilde{\mathsf{u}}, \tilde{\mathsf{v}}), a) \in FW^\infty(G, A)$. Now from the definition of Ψ_* we obtain that the state

$$\mathsf{rest}(\mathsf{g}, a) = \mathsf{rest}\big(\Psi_*((\tilde{\mathsf{u}}, \tilde{\mathsf{v}})), a\big)$$

belongs to $FW^\infty(G, A)$ as the product of elements from $FW^\infty(G, A)$.

Thus, the state $\mathsf{rest}(\mathsf{g}, a)$ belongs to the finite set Q_1 or the set $\mathcal{Q}(\mathsf{rest}(\mathsf{g}, a))$ is finite.

Let

$$Q_2 = \{\mathsf{rest}(\mathsf{g}, a) : \mathsf{g} \in Q_1, a \in A\} \cap FW^\infty(G, A)$$

and

$$Q_3 = \bigcup_{\mathsf{h} \in Q_2} \mathcal{Q}(\mathsf{h}).$$

Since sets Q_1 and A are finite, the set Q_2 is finite. Being a union of finite number of finite sets, the set Q_3 is finite as well. Then, using the definition of the state we obtain

$$\mathcal{Q}(\mathsf{g}) \subset Q_1 \cup Q_3.$$

Therefore, $\mathsf{g} \in FW^\infty(G, A)$. The proof is complete. □

Observe that rewriting mapping Ψ_* constructed in the proof of theorem 1 may be defined on the set of all pairs of conjugated elements of $W^\infty(G, A)$. Additional conditions on such elements were used only to prove that the image of Ψ_* belongs to the finite state wreath power of (G, A). It would be interesting to examine this image in general case.

4. Non-conjugated elements of infinite order

Let us show how to construct two elements of the group $FW^\infty(G, A)$ which are conjugated in the group $W^\infty(G, A)$ but are not conjugated in the group $FW^\infty(G, A)$.

Let g be a non-identity element of the group G. If n is the order of the element g and p is a prime divisor of n then the element $g_* = g^{n/p}$ has order p and as a permutation on A is a product of independent cycles of length p. Without loss of generality we assume that g_* has no fixed points. We will identify the set A with the set $\{0, \ldots, m-1\}$ in such a way that for some $k \geqslant 1$ the permutation g_* will be expressed in the form

$$g_* = (0, \ldots, p-1)(p, \ldots, 2p-1) \cdots ((k-1)p, \ldots, kp-1).$$

Let us consider the set $A_p = \{0, \ldots, p-1\}$, the cyclic group $G_p = \langle\sigma\rangle$ generated by the permutation $\sigma = (0, \ldots, p-1)$ and the mapping $c : G_p \to G$ that maps an element $h \in G_p$ to the permutation acting on the set $\{0, \ldots, kp-1\}$ by the rule $x \mapsto (x \bmod p)^h + [x/p] \cdot p$ and trivially on other elements of the set A. In other words the mapping c duplicates action on the set A_p onto the sets $\{p, \ldots, 2p-1\}$, ..., $\{(k-1)p, \ldots, kp-1\}$.

Using mapping c one can transform any element $\mathsf{u} \in W^\infty(G_p, A_p)$ into an element $\mathsf{u}^{(k)} \in W^\infty(G, A)$ by the equality

$$[\mathsf{u}^{(k)}]_n(x_1, \ldots, x_{n-1}) = \begin{cases} c\big([\mathsf{u}]_n(x_1 \bmod p, \ldots, x_{n-1} \bmod p)\big), \\ \qquad\qquad\qquad\qquad 0 \leqslant x_1, \ldots, x_{n-1} < kp, \\ e, \qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

Denote by f the function that for any $\mathsf{u} \in W^\infty(G_p, A_p)$ computes $\mathsf{u}^{(k)} \in W^\infty(G, A)$. The function f is well-defined.

Lemma 4. *If $\mathsf{u} \in FW^\infty(G_p, A_p)$ then $\mathsf{u}^{(k)} \in FW^\infty(G, A)$.*

Proof. If $\mathsf{u} \in W^\infty(G_p, A_p)$ then the value of $[\mathsf{u}]_n$ equals to some power of σ. By definition of the transformation the value of $[\mathsf{u}^{(k)}]_n$ equals to the same power of g_* or e depending on the arguments. Thus $\mathsf{u}^{(k)} \in W^\infty(G, A)$.

Denote by A_{kp} the set $\{0,\dots,kp-1\}$ and denote by $\bar a \bmod p$ the element

$$(a_1 \bmod p, a_2 \bmod p, \dots, a_n \bmod p) \in A_p^n$$

for $\bar a = (a_1, a_2, \dots, a_n) \in A^n$. We are going to prove for $\mathsf{u} \in W^\infty(G_p, A_p)$ that

$$\mathsf{rest}(f(\mathsf{u}), \bar a) = \begin{cases} f(\mathsf{rest}(\mathsf{u}, \bar a \bmod p)), & \bar a \in A_{kp}^n, \quad n \geqslant 1, \\ \mathsf{e}, & \text{otherwise.} \end{cases} \tag{8}$$

For $\bar a = (a_1, \dots, a_n) \in A^n$ and $n \geqslant 1$ we have

$$[\mathsf{rest}(f(\mathsf{u}), \bar a)]_m(x_1, \dots, x_{m-1}) = [f(\mathsf{u})]_{n+m}(a_1, \dots, a_n, x_1, \dots, x_{m-1}).$$

If $\bar a \notin A_{kp}^n$ then $[\mathsf{rest}(f(\mathsf{u}), \bar a)]_m(x_1, \dots, x_{m-1}) = e$ for all $x_1, \dots, x_{m-1}$ that implies $\mathsf{rest}(f(\mathsf{u}), \bar a) = \mathsf{e}$. In case $\bar a \in A_{kp}^n$ the equality

$$[\mathsf{rest}(f(\mathsf{u}), \bar a)]_m(x_1, \dots, x_{m-1}) =$$

$$= \begin{cases} c\big([\mathsf{u}]_{n+m}(a_1 \bmod p, \dots, x_{m-1} \bmod p)\big), & 0 \leqslant x_1, \dots, x_{m-1} < kp, \\ e, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} c\big([\mathsf{rest}(\mathsf{u}, \bar a \bmod p)]_m(x_1 \bmod p, \dots, x_{m-1} \bmod p)\big), & \\ & 0 \leqslant x_1, \dots, x_{m-1} < kp, \\ e, & \text{otherwise,} \end{cases}$$

$$= [f(\mathsf{rest}(\mathsf{u}, \bar a \bmod p))]_m(x_1, \dots, x_{m-1})$$

holds. Thus $\mathsf{rest}(f(\mathsf{u}), \bar a) = f(\mathsf{rest}(\mathsf{u}, \bar a \bmod p))$.

From equality (8) for $\mathsf{u} \in W^\infty(G_p, A_p)$ we get

$$\begin{aligned} \mathcal{Q}(\mathsf{u}^{(k)}) &= \mathcal{Q}(f(\mathsf{u})) = \{\mathsf{rest}(f(\mathsf{u}), \bar a) : \bar a \in A^n, n \geqslant 1\} \cup \{f(\mathsf{u})\} \subset \\ &\subset \{f(\mathsf{rest}(\mathsf{u}, \bar a \bmod p)) : \bar a \in A_{kp}^n, n \geqslant 1\} \cup \{\mathsf{e}\} \cup \{f(\mathsf{u})\} \subset \\ &\subset f(\mathcal{Q}(\mathsf{u})) \cup \{\mathsf{e}\} \cup \{f(\mathsf{u})\} = f(\mathcal{Q}(\mathsf{u})) \cup \{\mathsf{e}\}. \end{aligned}$$

This implies that if $\mathsf{u} \in FW^\infty(G_p, A_p)$ then $\mathsf{u}^{(k)} \in FW^\infty(G, A)$. □

Suppose that we have two elements $\mathsf{u}, \mathsf{v} \in FW^\infty(G_p, A_p)$ that satisfy conditions: 1) u and v are conjugated in the group $W^\infty(G_p, A_p)$; 2) growth of u is logarithmic; 3) growth of v is exponential. Using elements u and v we construct elements $\mathsf{u}^{(k)}$ and $\mathsf{v}^{(k)}$. By lemma 4 these new elements belongs to the group $FW^\infty(G, A)$. Since

$$f(\mathcal{Q}(\mathsf{u})) \subset \mathcal{Q}(f(\mathsf{u})) \subset f(\mathcal{Q}(\mathsf{u})) \cup \{\mathsf{e}\}.$$

u and $f(\mathsf{u})$ have equivalent growth. If $\mathsf{gu} = \mathsf{vg}$ then $f(\mathsf{g})f(\mathsf{u}) = f(\mathsf{v})f(\mathsf{g})$. Therefore u and $f(\mathsf{u})$ satisfy the following conditions: 1') $\mathsf{u}^{(k)}$ and $\mathsf{v}^{(k)}$ are conjugated in the group $W^\infty(G, A)$; 2') growth of $\mathsf{u}^{(k)}$ is logarithmic; 3') growth of $\mathsf{v}^{(k)}$ is exponential. Since the growth of an element is invariant under conjugation in $FW^\infty(G, A)$ (see [2, subsection 4.3]) This implies that elements $\mathsf{u}^{(k)}$ and $\mathsf{v}^{(k)}$ are non-conjugated in the group $FW^\infty(G, A)$.

Let us consider the following elements of the group $FW^\infty(G_p, A_p)$

$$\mathsf{e} = [e; \mathsf{e}, \ldots, \mathsf{e}], \qquad \mathsf{a}_i = [\sigma^i; \mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_{p-1}], \quad 0 \leqslant i < p,$$
$$\mathsf{s} = [\sigma; \mathsf{e}, \ldots, \mathsf{e}, \mathsf{s}], \qquad \mathsf{b}_i = [\sigma^i; \mathsf{a}_i, \mathsf{a}_i, \ldots, \mathsf{a}_i, \mathsf{b}_{i+1}], \quad 0 \leqslant i < p.$$

To simplify notations we will identify b_p and b_0. Let us show that the elements s and b_1 satisfy conditions 1)–3).

Lemma 5. *An element $\mathsf{g} \in W^\infty(G_p, A_p)$ is level transitive (acts transitively on the sets A_p^k, $k \geqslant 1$) if and only if $\mathsf{g}_k^* = \prod_{v \in A_p^{k-1}} [\mathsf{g}]_k(v) \neq e$ for all $k \geqslant 1$.*

Proof. The proof is similar to the proof of lemma 4.4 in [2] and we use two additional facts that the group G_p is abelian and every non-unity element generates a transitive subgroup. □

Lemma 6. *Let p be a odd prime number. Then the element b_1 satisfies equalities $(\mathsf{b}_1)_k^* = \sigma$ for all $k \geqslant 1$ which implies that b_1 is level transitive.*

Proof. Equalities $(\mathsf{a}_i)_1^* = (\mathsf{b}_i)_1^* = \sigma^i$ are obvious and the recurrent formulas

$$(\mathsf{a}_i)_{k+1}^* = (\mathsf{a}_0)_k^* (\mathsf{a}_1)_k^* \cdots (\mathsf{a}_{p-1})_k^*,$$
$$(\mathsf{b}_i)_{k+1}^* = ((\mathsf{a}_i)_k^*)^{p-1} (\mathsf{b}_{i+1})_k^*$$

follow from definitions. The first of the recurrent formulas implies

$$(\mathsf{a}_i)_2^* = (\mathsf{a}_0)_1^* (\mathsf{a}_1)_1^* \cdots (\mathsf{a}_{p-1})_1^* = \sigma^{0+1+\ldots+(p-1)} = \sigma^{\frac{p(p-1)}{2}} = e$$

and by induction we get $(\mathsf{a}_i)_k^* = 1$ for all $k \geqslant 2$. The second of the recurrent formulas implies

$$(\mathsf{b}_i)_2^* = ((\mathsf{a}_i)_1^*)^{p-1} (\mathsf{b}_{i+1})_1^* = \sigma^{-i}\sigma^{i+1} = \sigma,$$
$$(\mathsf{b}_i)_k^* = ((\mathsf{a}_i)_{k-1}^*)^{p-1} (\mathsf{b}_{i+1})_{k-1}^* = (\mathsf{b}_{i+1})_{k-1}^* = \sigma, \quad k \geqslant 3.$$ □

Lemma 7. *The elements s and b_1 are conjugated in the group $W^\infty(G_p, A_p)$.*

Proof. The adding machine s is level transitive. The element b_1 is level transitive by the lemma 6. Thus the elements s and b_1 are conjugated in the group $W^\infty(S_p, A_p)$.

Suppose that equality $\mathsf{b}_1 = \mathsf{g}^{-1}\mathsf{s}\mathsf{g}$ holds for some $\mathsf{g} \in W^\infty(S_p, A_p)$. Let us prove that $\mathsf{g} \in W^\infty(G_p, A_p)$. The element g for every $k \geqslant 0$ satisfies equality $\mathsf{b}_1^{p^k} = \mathsf{g}^{-1}\mathsf{s}^{p^k}\mathsf{g}$ which implies $[\mathsf{g}]_k(\bar{a})(\mathsf{b}_1)^*_k = (\mathsf{s})^*_k[\mathsf{g}]_k(\bar{a})$ for $\bar{a} \in A_p^k$. From the last equality it follows by lemma 6 that $[\mathsf{g}]_k(\bar{a})\sigma = \sigma[\mathsf{g}]_k(\bar{a})$ and finally we get $[\mathsf{g}]_k(\bar{a}) \in G_p$. □

Lemma 8. *The element* b_1 *has exponential growth.*

Proof. The proof is analogous to the proof of the proposition 4.2 in [2]. □

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