Algebra and Discrete Mathematics
Volume **21** (2016). Number 1, pp. 153–162 RESEARCH ARTICLE


# On nilpotent Lie algebras of derivations with large center

**Kateryna Sysak**

Communicated by A. P. Petravchuk

Abstract. Let $\mathbb{K}$ be a field of characteristic zero and $A$ an associative commutative $\mathbb{K}$-algebra that is an integral domain. Denote by $R$ the quotient field of $A$ and by $W(A) = R\,\mathrm{Der}\,A$ the Lie algebra of derivations on $R$ that are products of elements of $R$ and derivations on $A$. Nilpotent Lie subalgebras of the Lie algebra $W(A)$ of rank $n$ over $R$ with the center of rank $n-1$ are studied. It is proved that such a Lie algebra $L$ is isomorphic to a subalgebra of the Lie algebra $u_n(F)$ of triangular polynomial derivations where $F$ is the field of constants for $L$.

## Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and $A$ be an associative commutative algebra over $\mathbb{K}$ with identity, without zero divisors. A $\mathbb{K}$-linear mapping $D : A \longrightarrow A$ is called $\mathbb{K}$-*derivation* of $A$ if $D$ satisfies the Leibniz's rule: $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. The set $\mathrm{Der}\,A$ of all $\mathbb{K}$-derivations on $A$ forms a Lie algebra over $\mathbb{K}$ with respect to the operation $[D_1, D_2] = D_1D_2 - D_2D_1,\ D_1, D_2 \in \mathrm{Der}\,A$. Denote by $R$ the quotient field of the integral domain $A$. Each derivation $D$ of $A$ is uniquely extended to a derivation of $R$ by the rule: $D(a/b) = (D(a)b - aD(b))/b^2$. Denote by $\mathrm{Der}\,R$ the Lie algebra (over $\mathbb{K}$) of all $\mathbb{K}$-derivations on $R$.

**2010 MSC:** Primary 17B66; Secondary 17B30, 13N15.

**Key words and phrases:** derivation, Lie algebra, nilpotent Lie subalgebra, triangular derivation, polynomial algebra.

Define the mapping $rD : R \longrightarrow R$ by $(rD)(s) = r \cdot D(s)$ for all $r, s \in R$. It is easy to see that $rD$ is a derivation of $R$. The $R$-linear hull of the set $\{rD | r \in R, D \in \operatorname{Der} A\}$ forms the vector space $R \operatorname{Der} A$ over $R$, which is a Lie subalgebra of $\operatorname{Der} R$. Observe that $R \operatorname{Der} A$ is a Lie algebra over $\mathbb{K}$ but not always over $R$, and $\operatorname{Der} A$ is embedded in a natural way into $R \operatorname{Der} A$. Many authors study the Lie algebra of derivations $\operatorname{Der} A$ and its subalgebras, see [2–7].

This paper deals with nilpotent Lie subalgebras of the Lie algebra $R \operatorname{Der} A$. Let $L$ be a Lie subalgebra of $R \operatorname{Der} A$. The subfield $F = F(L)$ of $R$ consisting of all $a \in R$ such that $D(a) = 0$ for all $D \in L$ is called *the field of constants for* $L$. Let us denote by $RL$ the $R$-linear hull of $L$ and, analogously, by $FL$ the linear hull of $L$ over its field of constants $F = F(L)$. The *rank of* $L$ *over* $R$ is defined as the dimension of the vector space $RL$ over $R$, i.e. $\operatorname{rank}_R L = \dim_R RL$.

The main results of the paper:

- (Theorem 1) If $L$ is a nilpotent Lie subalgebra of the Lie algebra $R \operatorname{Der} A$ of rank $n$ over $R$ such that its center $Z(L)$ is of rank $n-1$ over $R$ and $F$ is the field of constants for $L$, then the Lie algebra $FL$ is contained in the Lie subalgebra of $R \operatorname{Der} A$ of the form

$$F \left\langle D_1, aD_1, \frac{a^2}{2!} D_1, \ldots, \frac{a^s}{s!} D_1, D_2, aD_2, \ldots, \frac{a^s}{s!} D_2, \ldots, \right. \\ \left. D_{n-1}, \ldots, \frac{a^s}{s!} D_{n-1}, D_n \right\rangle,$$

  where $D_1, D_2, \ldots, D_n \in FL$ are such that $[D_i, D_j] = 0$, $i, j = 1, \ldots, n$, and $a \in R$ is such that $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$ and $D_n(a) = 1$.
- (Theorem 2) The Lie algebra $FL$ is isomorphic to some subalgebra of the Lie algebra $u_n(F)$ of triangular polynomial derivations.

Recall that the Lie algebra $u_n(\mathbb{K})$ of triangular polynomial derivations consists of all derivations of the form

$$D = f_1(x_2, \ldots, x_n) \frac{\partial}{\partial x_1} + f_2(x_3, \ldots, x_n) \frac{\partial}{\partial x_2} + \cdots \\ + f_{n-1}(x_n) \frac{\partial}{\partial x_{n-1}} + f_n \frac{\partial}{\partial x_n},$$

where $f_i \in \mathbb{K}[x_{i+1}, \ldots, x_n], i = 1, \ldots, n-1$, and $f_n \in \mathbb{K}$. It is a Lie subalgebra of the Lie algebra $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations on the polynomial algebra $\mathbb{K}[x_1, \ldots, x_n]$. Such subalgebras are studied in [2, 3]. As Lie algebras, they are locally nilpotent but not nilpotent.

We use the standard notations. The Lie algebra $R\operatorname{Der} A$ is denoted by $W(A)$, as in [7]. The linear hull of elements $D_1, D_2, \dots, D_n$ over the field $\mathbb{K}$ is denoted by $\mathbb{K}\langle D_1, D_2, \dots, D_n\rangle$. If $L$ is a Lie subalgebra of a Lie algebra $M$, then the normalizer of $L$ in $M$ is the set $N_M(L) = \{x \in M \mid [x, L] \subseteq L\}$. Obviously, $N_M(L)$ is the largest subalgebra of $M$ in which $L$ is an ideal.

## 1. Nilpotent Lie subalgebras of $R\operatorname{Der} A$ with the center of large rank

We use Lemmas 1-5 proved in [7].

**Lemma 1** ([7, Lemma 1])**.** *Let $D_1, D_2 \in W(A)$ and $a, b \in R$. Then*
(a) $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$.
(b) *If $a, b \in \operatorname{Ker} D_1 \cap \operatorname{Ker} D_2$, then $[aD_1, bD_2] = ab[D_1, D_2]$.*

**Lemma 2** ([7, Lemma 2])**.** *Let $L$ be a nonzero Lie subalgebra of the Lie algebra $W(A)$, and $F$ be the field of constants for $L$. Then $FL$ is a Lie algebra over $F$, and if $L$ is abelian, nilpotent or solvable, then the Lie algebra $FL$ has the same property.*

**Lemma 3** ([7, Theorem 1])**.** *Let $L$ be a nilpotent Lie subalgebra of finite rank over $R$ of the Lie algebra $W(A)$, and $F$ be the field of constants for $L$. Then $FL$ is finite dimensional over $F$.*

**Lemma 4** ([7, Lemma 4])**.** *Let $L$ be a Lie subalgebra of the Lie algebra $W(A)$, and $I$ be an ideal of $L$. Then the vector space $RI \cap L$ over $\mathbb{K}$ is an ideal of $L$.*

**Lemma 5** ([7, Lemma 5])**.** *Let $L$ be a nilpotent Lie subalgebra of rank $n > 0$ over $R$ of the Lie algebra $W(A)$. Then:*
(a) *$L$ contains a series of ideals*

$$0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = L$$

*such that* $\operatorname{rank}_R I_k = k$ *for each* $k = 1, \dots, n$.
(b) *$L$ possesses a basis $D_1, \dots, D_n$ over $R$ such that $I_k = (RD_1 + \cdots + RD_k) \cap L$, $k = 1, \dots, n$, and $[L, D_k] \subset I_{k-1}$.*
(c) $\dim_F FL/FI_{n-1} = 1$.

**Lemma 6.** *Let $L$ be a nilpotent Lie subalgebra of the Lie algebra $W(A)$, and $F$ be the field of constants for $L$. If $L$ is of rank $n > 0$ over $R$ and its center $Z(L)$ is of rank $n - 1$ over $R$, then $L$ contains an abelian ideal $I$ such that* $\dim_F FL/FI = 1$.

*Proof.* Since the center $Z(L)$ is of rank $n-1$ over $R$, we can take linearly independent over $R$ elements $D_1, D_2, \dots, D_{n-1} \in Z(L)$. Let us consider

$$I = RZ(L) \cap L = (RD_1 + \cdots + RD_{n-1}) \cap L.$$

In view of Lemma 4, $I$ is an ideal of the Lie algebra $L$. Let us show that $I$ is an abelian ideal.

We first show that for an arbitrary element $D = r_1 D_1 + r_2 D_2 + \cdots + r_{n-1} D_{n-1} \in I$, its coefficients $r_1, r_2, \dots, r_{n-1} \in \bigcap_{i=1}^{n-1} \operatorname{Ker} D_i$. For each $D_i \in Z(L)$, $i = 1, \dots, n-1$, let us consider

$$[D_i, D] = [D_i, r_1 D_1 + r_2 D_2 + \cdots + r_{n-1} D_{n-1}] = \sum_{j=1}^{n-1} [D_i, r_j D_j].$$

By Lemma 1, $[D_i, r_j D_j] = r_j [D_i, D_j] + D_i(r_j) D_j = D_i(r_j) D_j$. Since $D_i \in Z(L)$, we get

$$[D_i, D] = \sum_{j=1}^{n-1} D_i(r_j) D_j = 0.$$

This implies that $D_i(r_1) = D_i(r_2) = \cdots = D_i(r_{n-1}) = 0$ because $D_1, D_2, \dots, D_{n-1} \in L$ are linearly independent over $R$. Therefore, $r_j \in \bigcap_{i=1}^{n-1} \operatorname{Ker} D_i$ for $j = 1, \dots, n-1$.

Now we take arbitrary $D, D' \in I$ and show that $[D, D'] = 0$. Let $D = a_1 D_1 + a_2 D_2 + \cdots + a_{n-1} D_{n-1}$ and $D' = b_1 D_1 + b_2 D_2 + \cdots + b_{n-1} D_{n-1}$. Then

$$[D, D'] = \sum_{i,j=1}^{n-1} (a_i b_j [D_i, D_j] + a_i D_i(b_j) D_j - b_j D_j(a_i) D_i) = 0$$

since $a_i, b_j \in \bigcap_{i=1}^{n-1} \operatorname{Ker} D_i$ for all $i, j = 1, \dots, n-1$, and $I$ is an abelian ideal.

It is easy to see that $FI$ is an abelian ideal of the Lie algebra $FL$ over $F$ and $\dim_F FL/FI = 1$ in view of Lemma 5(c). □

**Remark 1.** It follows from the proof of Lemma 6 that for an arbitrary $D = a_1 D_1 + a_2 D_2 + \cdots + a_{n-1} D_{n-1} \in FI$, the inclusions $a_i \in \bigcap_{k=1}^{n-1} \operatorname{Ker} D_k$ hold for $i = 1, \dots, n-1$.

**Lemma 7.** *Let $L$ be a Lie subalgebra of rank $n$ over $R$ of the Lie algebra $W(A)$, $\{D_1, D_2, \ldots, D_n\}$ be a basis of $L$ over $R$, and $F$ be the field of constants for $L$. Let there exists $a \in R$ such that $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$ and $D_n(a) = 1$. Then if $b \in R$ satisfies the conditions $D_1(b) = D_2(b) = \cdots = D_{n-1}(b) = 0$ and $D_n(b) \in F\langle 1, a, \ldots, \frac{a^s}{s!}\rangle$ for some integer $s \geqslant 0$, then $b \in F\langle 1, a, \ldots, \frac{a^s}{s!}, \frac{a^{s+1}}{(s+1)!}\rangle$.*

*Proof.* Since $D_n(b) \in F\langle 1, a, \ldots, \frac{a^s}{s!}\rangle$, the equality $D_n(b) = \sum\limits_{i=0}^{s} \beta_i \frac{a^i}{i!}$ holds for some $\beta_i \in F$, $i = 0, \ldots, s$. Take an element $c = \sum\limits_{i=0}^{s} \beta_i \frac{a^{i+1}}{(i+1)!}$ from $R$. It is easy to check that $D_1(c) = D_2(c) = \cdots = D_{n-1}(c) = 0$, because $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$ by the conditions of the lemma. Since $D_n(a) = 1$, the equality $D_n(c) = \sum\limits_{i=0}^{s} \beta_i \frac{a^i}{i!} = D_n(b)$ holds, and so $D_k(b - c) = 0$ for all $k = 1, \ldots, n$. This implies that $b - c \in F$, hence for some $\gamma \in F$, we obtain

$$b = \gamma + c = \gamma + \sum_{i=0}^{s} \beta_i \frac{a^{i+1}}{(i+1)!}.$$

Thus,

$$b \in F\left\langle 1, a, \ldots, \frac{a^s}{s!}, \frac{a^{s+1}}{(s+1)!}\right\rangle,$$

and the proof is complete. □

**Theorem 1.** *Let $L$ be a nilpotent Lie subalgebra of the Lie algebra $W(A)$, and let $F = F(L)$ be the field of constants for $L$. If $L$ is of rank $n$ and its center $Z(L)$ is of rank $n-1$ over $R$, then there exist $D_1, D_2, \ldots, D_n \in FL$, $a \in R$, and an integer $s \geqslant 0$ such that $FL$ is contained in the Lie subalgebra of $W(A)$ of the form*

$$F\left\langle D_1, aD_1, \frac{a^2}{2!}D_1, \ldots, \frac{a^s}{s!}D_1, D_2, aD_2, \ldots, \frac{a^s}{s!}D_2, \ldots, D_{n-1}, \ldots, \frac{a^s}{s!}D_{n-1}, D_n\right\rangle,$$

*where $[D_i, D_j] = 0$ for $i, j = 1, \ldots, n$, $D_n(a) = 1$, and $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$.*

*Proof.* It is easy to see that the vector space over $F$ of the form

$$F\left\langle D_1, aD_1, \frac{a^2}{2!}D_1, \dots, \frac{a^s}{s!}D_1, D_2, aD_2, \dots, \frac{a^s}{s!}D_2, \dots, \right.$$
$$\left. D_{n-1}, \dots, \frac{a^s}{s!}D_{n-1}, D_n\right\rangle$$

is a Lie algebra over $F$. We denote it by $\widetilde{L}$.

By Lemma 6, the Lie algebra $L$ contains an abelian ideal $I$ such that $FI$ is of codimension 1 in $FL$ over $F$. The ideal $I$ contains an $R$-basis $\{D_1, D_2, \dots, D_{n-1}\}$ of the center $Z(L)$. Let us take an arbitrary element $D_n \in L$ that is not in $Z(L)$. Then $\{D_1, D_2, \dots, D_{n-1}, D_n\}$ is an $R$-basis of $L$ and $FL = FI + FD_n$, where $FI$ is an abelian ideal of $FL$.

Let us consider the action of the inner derivation $\operatorname{ad} D_n$ on the vector space $FI$. It is easy to see that $\dim_F \operatorname{Ker}(\operatorname{ad} D_n) = n-1$. Indeed, let

$$D = r_1D_1 + r_2D_2 + \dots + r_{n-1}D_{n-1} \in \operatorname{Ker}(\operatorname{ad} D_n).$$

Then

$$[D_n, D] = \sum_{i=1}^{n-1}[D_n, r_iD_i] = \sum_{i=1}^{n-1} D_n(r_i)D_i = 0$$

whence $D_n(r_i) = 0$ for all $i = 1, \dots, n-1$.

By Remark 1, $r_1, r_2, \dots, r_{n-1} \in F$. Thus, $\operatorname{Ker}(\operatorname{ad} D_n) = F\langle D_1, D_2, \dots, D_{n-1}\rangle$ and $\dim_F \operatorname{Ker}(\operatorname{ad} D_n) = n-1$.

The Jordan matrix of the nilpotent operator $\operatorname{ad} D_n$ over $F$ has $n-1$ Jordan blocks. Denote by $J_1, J_2, \dots, J_{n-1}$ the corresponding Jordan chains. Without loss of generality, we may take $D_1 \in J_1$, $D_2 \in J_2$, $\dots$, $D_{n-1} \in J_{n-1}$ to be the first elements in the corresponding Jordan bases.

If $\dim_F F\langle J_1\rangle = \dim_F F\langle J_2\rangle = \dots = \dim_F F\langle J_{n-1}\rangle = 1$, then $FL = F\langle D_1, D_2, \dots, D_n\rangle$ and $FL$ is an abelian Lie algebra. It is the algebra from the conditions of the theorem if $s = 0$.

Let

$$\dim_F F\langle J_1\rangle \geqslant \dim_F F\langle J_2\rangle \geqslant \dots \geqslant \dim_F F\langle J_{n-1}\rangle$$

and $\dim_F F\langle J_1\rangle = s+1$, $s \geqslant 1$. Write the elements of the basis $J_1$ as follows:

$$J_1 = \Big\{D_1, \sum_{i=1}^{n-1} a_{1i}D_i, \sum_{i=1}^{n-1} a_{2i}D_i, \dots, \sum_{i=1}^{n-1} a_{si}D_i\Big\}.$$

By the definition of a Jordan basis,

$$D_1 = [D_n, \sum_{i=1}^{n-1} a_{1i}D_i] = \sum_{i=1}^{n-1} D_n(a_{1i})D_i$$

whence $D_n(a_{11}) = 1$ and $D_n(a_{1i}) = 0$ for all $i \neq 1$.

By Remark 1, $\sum\limits_{i=1}^{n-1} a_{1i}D_i \in FI$ implies $a_{1i} \in \bigcap\limits_{k=1}^{n-1} \operatorname{Ker} D_k,\ i = 1, \ldots, n-1$, and thus $a_{12}, a_{13}, \ldots, a_{1,n-1} \in F,$ and $a_{11} \notin F$. Let us write $a_{11} = a$. Then $a_{11}, a_{12}, \ldots, a_{1,n-1} \in F\langle 1, a\rangle$.

We shall show that $a_{21}, a_{22}, \ldots, a_{2,n-1} \in F\langle 1, a, \frac{a^2}{2!}\rangle$. By the definition of a Jordan basis,

$$[D_n, \sum_{i=1}^{n-1} a_{2i}D_i] = \sum_{i=1}^{n-1} D_n(a_{2i})D_i = \sum_{i=1}^{n-1} a_{1i}D_i$$

whence $D_n(a_{2i}) = a_{1i} \in F\langle 1, a\rangle$ for $i = 1, \ldots, n-1$. Then, by Lemma 7, $a_{2i} \in \bigcap\limits_{k=1}^{n-1} \operatorname{Ker} D_k$ implies $a_{2i} \in F\langle 1, a, \frac{a^2}{2!}\rangle$, $i = 1, \ldots, n-1$. Assume that $a_{mi} \in F\langle 1, a, \ldots, \frac{a^m}{m!}\rangle$ for all $m = 1, \ldots, s-1$ and $i = 1, \ldots, n-1$. Then

$$[D_n, \sum_{i=1}^{n-1} a_{m+1,i}D_i] = \sum_{i=1}^{n-1} a_{mi}D_i$$

whence $D_n(a_{m+1,i}) = a_{mi}$ for $i = 1, \ldots, n-1$. The coefficients $a_{m+1,i}$ satisfy the conditions of Lemma 7, so that $a_{m+1,i} \in F\langle 1, a, \ldots, \frac{a^{m+1}}{(m+1)!}\rangle$. Reasoning by induction, we get that all coefficients $a_{ji}$, $i = 1, \ldots, n-1$, $j = 1, \ldots, s$, of the elements from the basis $J_1$ belong to $F\langle 1, a, \ldots, \frac{a^s}{s!}\rangle$, and thus $F\langle J_1\rangle \subseteq \widetilde{L}$.

Consider the basis

$$J_2 = \Big\{D_2, \sum_{i=1}^{n-1} b_{1i}D_i, \sum_{i=1}^{n-1} b_{2i}D_i, \ldots, \sum_{i=1}^{n-1} b_{ti}D_i\Big\},$$

where $1 \leqslant t+1 \leqslant s$ and $\dim_F F\langle J_2\rangle = t+1$. By the definition of a Jordan basis, $[D_n, \sum\limits_{i=1}^{n-1} b_{1i}D_i] = \sum\limits_{i=1}^{n-1} D_n(b_{1i})D_i = D_2$, and thus $D_n(b_{12}) = 1$ and $D_n(b_{1i}) = 0$ for all $i \neq 2$. Set $b_{12} = b \notin F$ and consider $D_n(b-a) = 0$. It follows from Remark 1 that $a, b \in \bigcap\limits_{i=1}^{n-1} \operatorname{Ker} D_i$, so $b - a = \delta \in F$. The latter means that $b \in F\langle 1, a\rangle$. Moreover, $b_{1i} \in F$ for $i \neq 2$ in view of Remark 1. Thus, $b_{11}, b_{12}, \ldots, b_{1,n-1} \in F\langle 1, a\rangle$. Reasoning as for $J_1$ and using Lemma 7, one can show that $b_{2i} \in F\langle 1, a, \frac{a^2}{2!}\rangle$ and prove by induction that $b_{ji} \in F\langle 1, a, \ldots, \frac{a^t}{t!}\rangle$ for all $j = 1, \ldots, t$ and $i = 1, \ldots, n-1$. Since $t \leqslant s$, we have $F\langle J_2\rangle \subseteq \widetilde{L}$.

In the same way, one can show that the subspaces $F\langle J_3\rangle,\ F\langle J_4\rangle, \ldots, F\langle J_{n-1}\rangle$ lie in $\widetilde{L}$. Therefore, the Lie algebra $FL$ is contained in the Lie subalgebra $\widetilde{L}$ of $W(A)$. □

**Theorem 2.** *Let $L$ be a nilpotent Lie subalgebra of the Lie algebra $W(A)$, and let $F = F(L)$ be its field of constants. If $L$ is of rank $n \geqslant 3$ and its center $Z(L)$ is of rank $n-1$ over $R$, then the Lie algebra $FL$ over $F$ is isomorphic to a finite dimensional subalgebra of the Lie algebra $u_n(F)$ of triangular polynomial derivations.*

*Proof.* By Theorem 1, the Lie algebra $FL$ is contained in the Lie subalgebra $\widetilde{L}$ of $W(A)$, which is of the form $F\langle D_1, aD_1, \frac{a^2}{2!}D_1, \ldots, \frac{a^s}{s!}D_1, D_2, aD_2, \ldots, \frac{a^s}{s!}D_2, \ldots, D_{n-1}, \ldots, \frac{a^s}{s!}D_{n-1}, D_n\rangle$, where $[D_i, D_j] = 0$ for $i, j = 1, \ldots, n$, $D_n(a) = 1$ and $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$. The Lie algebra $\widetilde{L}$ is isomorphic (as a Lie algebra over $F$) to the subalgebra

$$F\left\langle \frac{\partial}{\partial x_1}, x_n\frac{\partial}{\partial x_1}, \frac{x_n^2}{2!}\frac{\partial}{\partial x_1}, \ldots, \frac{x_n^s}{s!}\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}, x_n\frac{\partial}{\partial x_{n-1}}, \ldots, \frac{x_n^s}{s!}\frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_n}\right\rangle$$

of the Lie algebra $u_n(F)$ of triangular polynomial derivations over $F$. □

## 2. Example of a maximal nilpotent Lie subalgebra of the Lie algebra $\widetilde{W}_n(\mathbb{K})$

**Lemma 8** ([8, Lemma 4])**.** *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. For a rational function $\phi \in \mathbb{K}(t)$, write $\phi' = \frac{d\phi}{dt}$. If $\phi \in \mathbb{K}(t) \setminus \mathbb{K}$, then does not exist a function $\psi \in \mathbb{K}(t)$ such that $\psi' = \frac{\phi'}{\phi}$.*

Let us denote by $\mathbb{K}[X] = \mathbb{K}[x_1, x_2, \ldots, x_n]$ the polynomial algebra, by $\mathbb{K}(X) = \mathbb{K}(x_1, x_2, \ldots, x_n)$ the field of rational functions in $n$ variables over $\mathbb{K}$, and by $\widetilde{W}_n(\mathbb{K})$ the Lie algebra of derivations on the field $\mathbb{K}(X)$. We think that the first part of the following statement is known.

**Proposition 1.** *The subalgebra $L = \mathbb{K}\langle x_1\frac{\partial}{\partial x_1}, x_2\frac{\partial}{\partial x_2}, \ldots, x_n\frac{\partial}{\partial x_n}\rangle$ of the Lie subalgebra of $\widetilde{W}_n(\mathbb{K})$ is isomorphic to a Lie subalgebra of the Lie algebra $u_n(\mathbb{K})$ of triangular polynomial derivations, but is not conjugated with any Lie subalgebra of this Lie algebra by an automorphism of the Lie algebra $\widetilde{W}_n(\mathbb{K})$.*

*Proof.* Let us show that $L$ is a maximal nilpotent Lie subalgebra of $\widetilde{W}_n(\mathbb{K})$. Obviously, $L$ is abelian, and so it is nilpotent. Let us show that $L$ coincides with its normalizer in $\widetilde{W}_n(\mathbb{K})$, which will imply that $L$ is maximal nilpotent (in view of the well-known fact from the theory of Lie algebras that a proper Lie subalgebra of a nilpotent Lie algebra does not coincide with its normalizer, see [1, p.58]).

Let $D$ be an arbitrary element of the normalizer $N = N_{\widetilde{W}_n(\mathbb{K})}(L)$. Then $[D, x_i\frac{\partial}{\partial x_i}] \in L$ for each $i = 1, \dots, n$. $D$ can be uniquely written as $D = \sum\limits_{j=1}^{n} f_j \frac{\partial}{\partial x_j}$, where $f_1, \dots, f_n \in \mathbb{K}(X)$. Using the following equations

$$\Big[x_i\frac{\partial}{\partial x_i}, \sum_{j=1}^{n} f_j\frac{\partial}{\partial x_j}\Big] = \sum_{j=1}^{n}\Big[x_i\frac{\partial}{\partial x_i}, f_j\frac{\partial}{\partial x_j}\Big] =$$

$$= \sum_{\substack{j=1\\ j\neq i}}^{n} x_i\frac{\partial f_j}{\partial x_i}\frac{\partial}{\partial x_j} + \Big(x_i\frac{\partial f_i}{\partial x_i} - f_i\Big)\frac{\partial}{\partial x_i},$$

we obtain that

$$x_i\frac{\partial f_j}{\partial x_i} = \alpha_j x_j, i \neq j, \text{ and } x_i\frac{\partial f_i}{\partial x_i} - f_i = \alpha_i x_i \tag{1}$$

for $\alpha_i, \alpha_j \in \mathbb{K}, i, j = 1, \dots, n$. We rewrite the first equation in (1) in the form $\frac{\partial f_j}{\partial x_i} = \frac{\alpha_j x_j}{x_i}$ and consider $f_j$ as a rational function in $x_i$ over the field $\mathbb{K}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. By Lemma 8 with $\phi = \phi(x_i) = x_i$, we have $\alpha_j = 0$. Thus, $\frac{\partial f_j}{\partial x_i} = 0$ for all $i \neq j$. This means that $f_j \in \mathbb{K}(x_j)$ for each $j = 1, \dots, n$.

Write $f_i = \frac{u_i}{v_i}$, where $u_i, v_i \in \mathbb{K}[x_i]$ are relatively prime and $v_i \neq 0$. Then the second equation in (1) is rewritten as

$$x_i\frac{u_i'v_i - u_iv_i' - \alpha_i v_i^2}{v_i^2} = \frac{u_i}{v_i},$$

where $'$ denotes the derivative with respect to the variable $x_i$. But then $x_i(u_i'v_i - u_iv_i' - \alpha_i v_i^2)v_i = u_iv_i^2$, whence we have that the polynomial $v_i$ must divide $v_i'$. It is possible only if $v_i \in \mathbb{K}^*$, i.e. $f_i$ is a polynomial in $x_i$ with coefficients in $\mathbb{K}$. Since $x_i(f_i' - \alpha_i) = f_i$, we have that $f_i$ is a polynomial of degree 1. It is easy to see that $f_i = \gamma_i x_i$ with $\gamma_i \in \mathbb{K}$ for all $i = 1, \dots, n$. Thus $D \in L$, that is, $L = N$ and $L$ is a maximal nilpotent Lie subalgebra of $\widetilde{W}_n(\mathbb{K})$.

If $L$ is conjugated by an automorphism of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ with some Lie subalgebra of $u_n(\mathbb{K})$, then $L$ is contained in a nilpotent Lie subalgebra of $u_n(\mathbb{K})$. Therefore, $L$ is not coincide with its normalizer in $\widetilde{W}_n(\mathbb{K})$, which contradicts the fact proved above. However, the subalgebra $\mathbb{K}\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\rangle$ of the Lie algebra $u_n(\mathbb{K})$ is obviously isomorphic to $L$. $\square$

## References

[1] Yu. A. Bahturin, *Identical Relations in Lie Algebras* (in Russian), Nauka, Moscow (1985).

[2] V. V. Bavula, *Lie algebras of triangular polynomial derivations and an isomorphism criterion for their Lie factor algebras* Izv. RAN. Ser. Mat., **77** (2013), Issue 6, 3-44.

[3] V. V. Bavula, *The groups of automorphisms of the Lie algebras of triangular polynomial derivations*, J. Pure Appl. Algebra, **218** (2014), Issue 5, 829-851.

[4] J. Draisma, *Transitive Lie algebras of vector fields: an overview*, Qual. Theory Dyn. Syst., **11** (2012), no. 1, 39-60.

[5] A. González-López, N. Kamran and P. J. Olver, *Lie algebras of differential operators in two complex variables*, Amer. J. Math., **114** (1992), 1163-1185.

[6] S. Lie, *Theorie der Transformationsgruppen*, Vol. 3, Leipzig (1893).

[7] Ie. O. Makedonskyi and A. P. Petravchuk, *On nilpotent and solvable Lie algebras of derivations*, Journal of Algebra, **401** (2014), 245-257.

[8] Ie. O. Makedonskyi and A. P. Petravchuk, *On finite dimensional Lie algebras of planar vector fields with rational coefficients*, Methods Func. Analysis Topology, **19** (2013), no. 4, 376-388.

### Contact information

**K. Sysak** Department of Algebra and Mathematical Logic, Faculty of Mechanics and Mathematics, Kyiv National Taras Shevchenko University, 64, Volodymyrska street, 01033 Kyiv, Ukraine
*E-Mail(s)*: sysakkya@gmail.com

Received by the editors: 24.12.2015
and in final form 10.02.2016.