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RESEARCH ARTICLE

On invariants of polynomial functions, II

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Abstract. Let P be a finite partially ordered set. In our previous paper, we defined the sectional geometric genus $g_i(P)$ of P and studied $g_i(P)$. In this paper, by using this sectional geometric genus of P, we will give a criterion about the case in which P has no order.

Introduction

In our previous paper [1], we studied polynomial functions. In particular, we generalized the notion of the ith sectional geometric genus of polarized varieties, which is an important invariant of polarized varieties (see [2, Definition 3.2]), to the case of polynomial functions. Here we note that if $i = 1$, then the first sectional geometric genus of any polynomial function in two variables associated with a polynomial function h is equal to the sectional genus of h which was defined by Ooishi [5, Definition 1.3] (see Remark 2). By using this invariant, we proved the following theorem about partially ordered sets (see [1, Theorem 4.1]).

Theorem 1. *Let P be a finite partially ordered set, and let i be an integer with $1 \leqslant i \leqslant d(P)$. Then $g_i(P) = 0$ if and only if $l(P) \geqslant d(P) - i$, where $d(P) = \sharp(P)$ and $l(P)$ denotes the length of P (see Definition 6 (3)).*

As a corollary, we can get the following result which gives a neccesary and sufficient condition for being totally ordered.

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Corollary 1. *Let P be a finite partially ordered set. Then $g_1(P) = 0$ if and only if P is a totally ordered set.*

In this paper, by using the sectional geometric genus of a finite partially ordered set P, we will give two criterions about the case in which P has no order (Theorems 5 and 6).

1. Preliminaries

Notation 1. For a real number m and a non-negative integer n, let

$$[m]_n = \begin{cases} m(m-1)\cdots(m-n+1) & \text{if } n \geqslant 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then for n fixed, $[t]_n$ is a polynomial in t whose degree is n.
For any non-negative integer n,

$$n! := \begin{cases} [n]_n & \text{if } n \geqslant 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Assume that m and n are non-negative integers. Then we put

$$\binom{m}{n} := \begin{cases} \frac{[m]_n}{n!} & \text{if } m \geqslant n, \\ 0 & \text{if } m < n. \end{cases}$$

We note that

$$\binom{m}{0} = 1 \quad \text{if } m \geqslant 0.$$

Definition 1 (see [4, §1])**.** (1) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function. Then f is called a *polynomial function* if f satisfies the following.
- (A) There exist an integer N_1 and a polynomial $P(n) \in \mathbb{C}[n]$ such that $f(n) = P(n)$ for every integer n with $n > N_1$.
- (B) There exists an integer N_2 such that $f(m) = 0$ for every integer m with $m < N_2$.

In this case we put $P_f(t) := P(t)$ because $P(t)$ depends on the function f. We call this polynomial $P_f(t)$ the *polynomial associated with f*.

(2) Let $\phi(t) \in \mathbb{C}[t, t^{-1}]$ and we put $\phi(t) = \sum_i a_i t^i$. Then we put

$$d(\phi) := \max\{k \mid a_k \neq 0\}.$$

(3) Let f be a polynomial function such that $P_f(t) \neq 0_{\mathbb{C}[t]}$. Then we put $d(f) := d(P_f)$.

Remark 1. Let f be a polynomial function. Then $P_f(t) \in \mathbb{Q}[t]$ (see [1, Remark 2.1 (1)]).

Notation 2. (1) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function. We put

$$\nabla f(n) := \sum_{i \leqslant n} f(i) \quad \text{and} \quad F_f(t) := \sum_{n \in \mathbb{Z}} f(n) t^n.$$

(2) $\mathcal{PF}$ denotes the set of polynomial functions, and

$$\mathcal{PF}^{\geqslant 0} := \{ g(t) \in \mathcal{PF} \mid g(t) = 0 \text{ for } \forall t < 0 \}.$$

Notation 3. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a function in two variables. We put $f_1(x) := f(x, 0)$ and $f_2(y) := f(0, y)$.

Definition 2. Let $h(x)$ be a polynomial function in x. Then a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is called a *polynomial function in two variables associated with* $h(x)$ if the following hold.

(1) $f_1(x) = h(x)$.

(2) There exists an integer N such that $f_2(y) = 0$ for every integer y with $y \leqslant N$.

The following lemma is well-known:

Lemma 1. *Let $p(t) \in \mathbb{C}[t]$ be a polynomial in t such that $p(t) \neq 0_{\mathbb{C}[t]}$ and $p(n) \in \mathbb{Z}$ for any integer n, and let k be the degree of $p(t)$. Then for every integer d with $0 \leqslant k \leqslant d$ there exists a unique sequence of integers $(b_0, b_1, \ldots, b_d)$ such that the following holds.*

$$p(t) = \sum_{j=0}^{d} (-1)^j b_j \binom{t + d - j}{d - j}.$$

Notation 4. Let $h(t) \in \mathbb{C}[t]$ be a polynomial in t such that $h(t) \neq 0_{\mathbb{C}[t]}$ and $h(n) \in \mathbb{Z}$ for every integer n. We put $k = \deg h(t)$, and let d be an integer with $k \leqslant d$. Then by Lemma 1, there exists a unique sequence of integers $(b_0, b_1, \ldots, b_d)$ such that the following holds.

$$h(t) = b_0 \binom{t + d}{d} - b_1 \binom{t + d - 1}{d - 1} + \cdots + (-1)^d b_d.$$

Here we put $e_i^{k,d}(h(t)) := b_i$, and if $k = d$, then we put $e_i(h(t)) := e_i^{d,d}(h(t))$. Here we note that if $k < d$, then $e_i^{k,d}(h(t)) = 0$ for every integer i with $0 \leqslant i \leqslant d - k - 1$.

Definition 3. (1) Let $h(t) \in \mathbb{C}[t]$ be a polynomial in t such that $h(t) \neq 0_{\mathbb{C}[t]}$ and $h(n) \in \mathbb{Z}$ for every integer n. We use Notation 4. Then for every integer i with $0 \leqslant i \leqslant d(h)$, we define the *ith sectional H-arithmetic genus $\chi_i^H(h)$ of $h(t)$* as follows.

$$\chi_i^H(h) := \sum_{j=0}^{i} (-1)^j e_j(h(t)).$$

(2) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a polynomial function, and let $P_f(t) \in \mathbb{Q}[t]$ be the polynomial associated with f such that $P_f(t) \neq 0_{\mathbb{Q}[t]}$. Then, for every integer i with $0 \leqslant i \leqslant d(f)$, we define the *ith sectional H-arithmetic genus $\chi_i^H(f)$ of f* as follows.

$$\chi_i^H(f) := \chi_i^H(P_f).$$

Definition 4. Let $h(x)$ be a polynomial function in x such that $h(x) \neq 0_{\mathbb{Q}[x]}$, and let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a two variable polynomial function associated with $h(x)$. We use Notations 2 and 3. For every integer i with $0 \leqslant i \leqslant d(h)$, the *ith sectional geometric genus $g_i(f)$ of f* is defined by the following:

$$g_i(f) := (-1)^i \left(\chi_i^H(h) - \nabla f_2(0) + \nabla f_2(-i) \right).$$

Remark 2. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a polynomial function in two variables associated with a polynomial function $h(x)$. If $i = 1$ and $d(h) \geqslant 1$, then $g_1(f) = g_s(h)$, that is, $g_1(f)$ is the sectional genus of h (see [1, Remark 3.4 (1)]).

Definition 5. From now on, a partially ordered set is called a *poset* for short in this paper.

(1) Let P be a finite poset. We put

$$d(P) := \sharp(P)$$

and for every $n \in \mathbb{N}$

$$\Omega(P, n) := \sharp\{\sigma : P \to \{1, \dots, n\} \mid \sigma(x_i) \leqslant \sigma(x_j) \text{ if } x_i \leqslant x_j.\}.$$

Then $\Omega(P, n)$ is a polynomial in n whose degree is $d(P)$ (see [6, 3.12]). This is called the *order polynomial of P*.

(2) Let P be a finite poset and let $\Omega(P, n)$ be the order polynomial of P. We put

$$h_P(x) := \begin{cases} \Omega(P, x) & \text{if } x \in \mathbb{Z} \text{ with } 1 \leqslant x, \\ 0 & \text{otherwise.} \end{cases}$$

Then h_P is a polynomial function, and we define a polynomial function in two variables $f_P(x,y)$ associated with $h_P(x)$ as follows.

$$f_P(x,y) := h_P(x)\cdot\rho(y),$$

where

$$\rho(y) := \begin{cases} 1 & \text{if } y=0, \\ 0 & \text{otherwise.} \end{cases}$$

We call $f_P(x,y)$ the *generalized polynomial function associated with* P. We note that $d(P)=d(h_P)$.

(3) For every integer i with $0\leqslant i\leqslant d(P)$ we put $\chi_i^H(P):=\chi_i^H(h_P)$ and $g_i(P):=g_i(f_P)$, which are called the ith sectional H-arithmetic genus and the ith sectional geometric genus of P respectively. We note that $g_1(P)=g_s(h_P)$ (see Remark 2).

Definition 6. (1) Let P be a finite poset, and let C be a subset of P. Then C is called a *chain of* P if any two elements of C are comparable.

(2) Let P be a finite poset, and let C be a chain of P. Then we put $l(C) := \sharp(C)-1$.

(3) Let P be a finite poset. Then we put

$$l(P) := \max\{l(C)\mid C \text{ is a chain of } P\},$$

which is called the *length of* P.

(4) Let n be a natural number and let Σ_n be the set of all permutations of $\{1,\dots,n\}$. For $\sigma\in\Sigma_n$ with

$$\sigma=\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix},$$

we put

$$[\![a_1,\dots,a_n]\!] := \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}.$$

(5) Let P be a finite poset. We put $P=\{x_1,\dots,x_{d(P)}\}$ and

$$A(P) := \{\mu : P\to\{1,\dots,d(P)\}\mid \mu \text{ is a bijection such that } \mu(x_i)<\mu(x_j) \text{ if } x_i<x_j\}.$$

(5.1) We fix an element $\mu\in A(P)$. Then we put

$$\mathcal{L}(P;\mu) := \{[\![\mu\circ\sigma^{-1}(1),\dots,\mu\circ\sigma^{-1}(d(P))]\!]\mid\sigma\in A(P)\}.$$

By definition, we get that $\mathcal{L}(P;\mu) \subset \Sigma_{d(P)}$. We call the set $\mathcal{L}(P;\mu)$ the *Jordan-Hölder set of P with respect to μ* (see [6, 3.13]).

(5.2) For $\pi \in \Sigma_n$ with $\pi = (a_1, \ldots, a_n)$, we put

$$D(\pi) := \{a_i \mid a_i > a_{i+1}\} \quad \text{and} \quad \delta(\pi) := \sharp D(\pi).$$

For a natural number n and a subset $S \subset \{1, \ldots, n-1\}$, we put

$$D_n(S) := \{\pi \in \Sigma_n \mid S = D(\pi)\}.$$

Let $\mu, \mu' \in A(P)$ with $\mu \neq \mu'$. Then by [6, 3.13.1 Theorem], we obtain the following. For any subset $S \subset \{1, \ldots, d(P)-1\}$, $\sharp(D_{d(P)}(S) \cap \mathcal{L}(P;\mu)) = \sharp(D_{d(P)}(S) \cap \mathcal{L}(P;\mu'))$. In particular, we get that for every integer i with $1 \leqslant i \leqslant d(P)-1$

$$\sharp\{\pi \in \mathcal{L}(P;\mu) \mid \delta(\pi) = i\} = \sharp\{\pi \in \mathcal{L}(P;\mu') \mid \delta(\pi) = i\}.$$

So when we use results concerned with $\delta(\pi)$ (for example, Proposition 1 below), we describe the Jordan-Hölder set as $\mathcal{L}(P)$ instead of $\mathcal{L}(P;\mu)$ for $\mu \in A(P)$.

Proposition 1. *Let P be a finite poset and let $h_P(x)$ be the following:*

$$h_P(x) := \begin{cases} \Omega(P, x) & \text{if } x \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$F_{h_P}(t) = \left(\sum_{\pi \in \mathcal{L}(P)} t^{1+\delta(\pi)} \right) (1-t)^{-d(P)-1}.$$

Proof. See [6, 3.15.8 Theorem]. □

Notation 5. Let P be a finite poset, and let $f_P(x, y)$ be the generalized polynomial function associated with P. By [1, Theorem 2.1 and Remark 2.4] there exists a polynomial $\phi(t) \in \mathbb{Z}[t]$ such that

$$F_{h_P}(t) = \frac{\phi(t)}{(1-t)^d},$$

where $d = d(h_P) + 1$. Here we put $F_{f_P}(t) := F_{h_P}(t)$, $\phi_{f_P}(t) := \phi(t)$, and

$$\phi_{f_P}(t) = \sum_{j=0}^{d(\phi_{f_P})} a_j t^j,$$

where $a_j \in \mathbb{Z}$ for every j. Let $a_j(P) := a_j$.

Remark 3. We note that $h_P(t) \in \mathcal{PF}^{\geqslant 0}$, $P_{h_P}(t) \neq 0_{\mathbb{Q}[t]}$, and $f_P(0,m) = f_P(1,m) = 0$ for every integer m with $m \neq 0$. Hence by [1, Theorem 3.2] we get the following:

(1) For every integer i with $1 \leqslant i \leqslant d(P)$

$$g_i(P) = \begin{cases} \sum_{k=i+1}^{d(\phi_{f_P})} \binom{k-1}{i} a_k(P) & \text{if } i+1 \leqslant d(\phi_{f_P}), \\ 0 & \text{if } d(\phi_{f_P}) \leqslant i. \end{cases}$$

(2) Since $f_P(x,y)$ satisfies the assumptions in [1, Corollary 3.2], we get $g_i(P) \geqslant 0$ for every integer i with $1 \leqslant i \leqslant d(P)$. Moreover if $i = d(P)$, then $g_{d(P)}(P) = 0$ by Proposition 1 and (1) above.

2. Main results

In this section, by using the sectional geometric genus of a poset P, we are going to give two criterions about the case in which P has no order.

Theorem 2. *Let d and k be nonnegative integers and $S(a,b)$ the Stirling number of the second kind, where a and b are nonnegative integers. Then for every integer j with $0 \leqslant j \leqslant d$ we have*

$$e_j^{k,d}(t^k) = (-1)^{d-k}(d-j)!S(k+1, d+1-j).$$

Proof. By Lemma 1 and Notation 4 we have

$$t^k = \sum_{j=0}^{d} (-1)^j e_j^{k,d}(t^k) \binom{t+d-j}{d-j}. \tag{1}$$

Using this equation (1), we have

$$\begin{aligned} (-t)^k &= \sum_{j=0}^{d} (-1)^j e_j^{k,d}(t^k) \binom{-t+d-j}{d-j} \\ &= \sum_{j=0}^{d} (-1)^j e_j^{k,d}(t^k) \frac{(-t+d-j)(-t+d-j-1)\cdots(-t+1)}{(d-j)!} \\ &= \sum_{j=0}^{d} (-1)^j (-1)^{d-j} e_j^{k,d}(t^k) \frac{(t-d+j)(t-d+j+1)\cdots(t-1)}{(d-j)!} \\ &= \sum_{j=0}^{d} (-1)^d e_j^{k,d}(t^k) \binom{t-1}{d-j}. \end{aligned}$$

Hence we have

$$(-1)^{d-k}t^k = \sum_{j=0}^{d} e_j^{k,d}(t^k)\binom{t-1}{d-j}. \tag{2}$$

Moreover by (2) we have

$$\begin{aligned}
(-1)^{d-k}t^{k+1} &= \sum_{j=0}^{d} e_j^{k,d}(t^k)t\binom{t-1}{d-j} \\
&= \sum_{j=0}^{d} e_j^{k,d}(t^k)\frac{t(t-1)\cdots(t-d+j)}{(d-j)!} \\
&= \sum_{j=0}^{d} e_j^{k,d}(t^k)\frac{t(t-1)\cdots(t-d+j)}{(d-j+1)!}(d-j+1) \\
&= \sum_{j=0}^{d}(d-j+1)e_j^{k,d}(t^k)\binom{t}{d-j+1} \\
&= \sum_{j=d-k}^{d}(d-j+1)e_j^{k,d}(t^k)\binom{t}{d-j+1} \\
&= \sum_{j=0}^{k}(k-j+1)e_{d-k+j}^{k,d}(t^k)\binom{t}{k-j+1}.
\end{aligned}$$

On the other hand, by [6, (1.94d)] we get

$$\begin{aligned}
t^{k+1} &= \sum_{j=0}^{k+1}(k-j+1)!S(k+1,k+1-j)\binom{t}{k-j+1} \\
&= \sum_{j=0}^{k}(k-j+1)!S(k+1,k+1-j)\binom{t}{k-j+1}
\end{aligned}$$

because $S(k+1,0)=0$.

Therefore we get $e_{d-k+j}^{k,d}(t^k) = (-1)^{d-k}(k-j)!S(k+1,k+1-j)$ for every integer j with $0 \leqslant j \leqslant k$. Hence we get $e_j^{k,d}(t^k) = (-1)^{d-k}(d-j)!S(k+1,d+1-j)$ for every integer j with $d-k \leqslant j \leqslant d$. On the other hand, we have $e_j^{k,d}(t^k) = 0 = (-1)^{d-k}(d-j)!S(k+1,d+1-j)$ for every integer j with $0 \leqslant j \leqslant d-k-1$. So we get the assertion. □

Corollary 2. *Let d be a nonnegative integer and $S(a,b)$ the Stirling number of the second kind. Then for every integer j with $0 \leqslant j \leqslant d$ we have*

$$e_j(t^d) = (d-j)!S(d+1, d+1-j).$$

Theorem 3. *Let d be a nonnegative integer and $h(t) \in \mathbb{C}[t]$ be a polynomial $h(t) = \sum_{k=0}^{d} c_k t^k$ in t such that $h(n) \in \mathbb{Z}$ for every integer n, where $c_d \neq 0$. Then for every integer i with $0 \leqslant i \leqslant d$ the following equality holds.*

$$\chi_i^H(h) = \sum_{l=0}^{d}(-1)^l c_{d-l} \left\{ \sum_{j=0}^{i}(-1)^j (d-j)!S(d+1-l, d+1-j) \right\}.$$

Proof. First we note that $t^k = \sum_{j=0}^{d}(-1)^j e_j^{k,d}(t^k)\binom{t+d-j}{d-j}$. Then

$$\begin{aligned} h(t) = \sum_{k=0}^{d} c_k t^k &= \sum_{k=0}^{d} c_k \left\{ \sum_{j=0}^{d}(-1)^j e_j^{k,d}(t^k) \binom{t+d-j}{d-j} \right\} \\ &= \sum_{j=0}^{d}(-1)^j \left\{ \sum_{k=0}^{d} c_k e_j^{k,d}(t^k) \right\} \binom{t+d-j}{d-j} \\ &= \sum_{j=0}^{d}(-1)^j \left\{ \sum_{l=0}^{d} c_{d-l} e_j^{d-l,d}(t^{d-l}) \right\} \binom{t+d-j}{d-j}. \end{aligned}$$

So we get $e_j(h(t)) = \sum_{l=0}^{d} c_{d-l} e_j^{d-l,d}(t^{d-l})$ (see Notation 4). Hence

$$\chi_i^H(h) = \sum_{j=0}^{i}(-1)^j \left\{ \sum_{l=0}^{d} c_{d-l} e_j^{d-l,d}(t^{d-l}) \right\}.$$

By using Theorem 2 we have

$$\begin{aligned} \chi_i^H(h) &= \sum_{j=0}^{i}(-1)^j \left\{ \sum_{l=0}^{d} c_{d-l}(-1)^l (d-j)!S(d-l+1, d+1-j) \right\} \\ &= \sum_{l=0}^{d}(-1)^l c_{d-l} \left\{ \sum_{j=0}^{i}(-1)^j (d-j)!S(d+1-l, d+1-j) \right\}. \end{aligned}$$

Therefore we get the assertion. □

By Theorem 3 we see that if $h(t) = t^d$, then

$$\chi_i^H(h) = \sum_{j=0}^{i} (-1)^j (d-j)! S(d+1, d+1-j).$$

In this case, we can also prove the following theorem.

Theorem 4. *Let d be a nonnegative integer and $S(a, b)$ the Stirling number of the second kind, where a and b are nonnegative integers. Let $h(t) = t^d$. Then for every integer i with $0 \leqslant i \leqslant d$ we have*

$$\chi_i^H(h) = (-1)^i (d-i)! S(d, d-i).$$

Proof. We prove this by induction on i.
(i) If $i = 0$, then $\chi_0^H(h) = e_0(t^d) = d! S(d+1, d+1) = d! S(d, d)$ and this shows that it is true for the case of $i = 0$.
(ii) Assume that it is true for the case of $i = k-1$. So we have

$$\sum_{j=0}^{k-1} (-1)^j e_j(t^d) = \chi_{k-1}^H(h) = (-1)^{k-1} (d-k+1)! S(d, d-k+1). \quad (3)$$

Next we consider the case of $i = k$. Then by Corollary 2, [6, (1.93)] and (3)

$$\begin{aligned}
\chi_k^H(h) &= \sum_{j=0}^{k} (-1)^j e_j(t^d) = (-1)^k e_k(t^d) + \sum_{j=0}^{k-1} (-1)^j e_j(t^d) \\
&= (-1)^k (d-k)! S(d+1, d+1-k) \\
&\quad + (-1)^{k-1} (d-k+1)! S(d, d+1-k) \\
&= (-1)^{k-1} (d-k)! \{-S(d+1, d+1-k) \\
&\quad + (d-k+1) S(d, d+1-k)\} \\
&= (-1)^{k-1} (d-k)! \{-(d+1-k) S(d, d+1-k) \\
&\quad - S(d, d-k) + (d-k+1) S(d, d+1-k)\} \\
&= (-1)^k (d-k)! S(d, d-k).
\end{aligned}$$

So we get the assertion. □

The following corollary of Theorem 4 has been proved in [3] by a different method.

Corollary 3 (Kaki). *Let P be a poset such that P has no order. Then for every integer i with $0 \leqslant i \leqslant d(P)$ we have*

$$g_i(P) = (d(P) - i)!S(d(P), d(P) - i).$$

Proof. Here we note that $h_P(t) = t^{d(P)}$ in this case. By Theorem 4 we have $\chi_i^H(P) = \chi_i^H(h_P) = (-1)^i(d(P) - i)!S(d(P), d(P) - i)$. On the other hand,

$$\nabla f_2(0) = \sum_{y \leqslant 0} f_2(y) = \sum_{y \leqslant 0} h_P(0)\rho(y) = h_P(0)\rho(0) = h_P(0) = 0$$

and

$$\begin{aligned} \nabla f_2(-i) &= \sum_{y \leqslant -i} f_2(y) = \sum_{y \leqslant -i} h_P(0)\rho(y) \\ &= \begin{cases} h_P(0)\rho(0), & \text{if } i = 0, \\ 0, & \text{if } i \geqslant 1 \end{cases} \\ &= 0. \end{aligned}$$

Therefore by Definition 4 we get the assertion. □

Theorem 5. *Let P be a finite poset. Then the following are equivalent each other.*

(i) *P has no order.*
(ii) *$g_{d(P)-1}(P) \neq 0$.*
(iii) *$g_{d(P)-1}(P) = 1$.*

Proof. By Theorem 1 we see that $g_{d(P)-1}(P) \neq 0$ if and only if $l(P) < d(P) - (d(P) - 1) = 1$. Hence $g_{d(P)-1}(P) \neq 0$ if and only if $l(P) = 0$. So we get the equivalence (i) and (ii) because $l(P) = 0$ means that P has no order.

On the other hand, by Corollary 3 we see that

$$g_{d(P)-1}(P) = S(d(P), 1) = 1$$

if P has no order. Therefore (i) implies (iii). Since (iii) implies (ii), we get the assertion. □

Theorem 6. *Let P be a finite poset. Then*

$$g_i(P) \leqslant (d(P) - i)!S(d(P), d(P) - i)$$

holds for every integer i with $0 \leqslant i \leqslant d(P) - 1$. Moreover this equality holds for some i with $0 \leqslant i \leqslant d(P) - 1$ if and only if P has no order.

Proof. Here we note the following

Claim 1. Let P be a finite poset. Then P has an order if and only if $\sigma \notin \mathcal{L}(P)$, where

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & d(P)-1 & d(P) \\ d(P) & d(P)-1 & \cdots & 2 & 1 \end{pmatrix}$$

Proof. First we note that $S_{d(P)} = \mathcal{L}(P)$ if P has no order. Hence $\sigma \in \mathcal{L}(P)$ if P has no order. So it suffices to show that $\sigma \notin \mathcal{L}(P)$ if P has an order. Assume that P has an order. Then there exist elements $x_i, x_j \in P$ such that $x_i < x_j$. For any $\delta \in \mathcal{L}(P)$, there exist $\mu, \pi \in A(P)$ such that $\delta = \pi \circ \mu^{-1}$. On the other hand, we have $\mu(x_i) = a_1 < b_1 = \mu(x_j)$ and $\pi(x_i) = a_2 < b_2 = \pi(x_j)$. Then we have $\delta(a_1) = a_2 < b_2 = \delta(b_1)$. Since σ does not satisfiy this property, we see that $\sigma \notin \mathcal{L}(P)$ and we get the assertion of this claim. □

Using this claim, we can prove the following.

Claim 2. Let Q and R be finite posets with $d(Q) = d(R)$. Assume that Q has no order and R has an order. Then $g_i(Q) > g_i(R)$ for every integer i with $0 \leqslant i \leqslant d(Q) - 1 = d(R) - 1$.

Proof. We use Notation 5. First of all, we note that $\mathcal{L}(Q) = S_{d(Q)}$. We see from Proposition 1 that $d(\phi_{f_Q}) = d(Q)$. So by Remark 3 (1) we have

$$g_i(Q) = \sum_{k=i+1}^{d(Q)} \binom{k-1}{i} a_k(Q)$$

for every integer i with $0 \leqslant i \leqslant d(Q) - 1$. We note that $a_{d(Q)}(Q) > 0$ by Proposition 1 and Claim 1.

On the other hand, $\mathcal{L}(R) \subset S_{d(R)} = S_{d(Q)} = \mathcal{L}(Q)$ holds by assumption. We see from Proposition 1 and Claim 1 that $d(\phi_{f_R}) < d(R) = d(Q)$. We note that by Proposition 1 we obtain $a_k(Q) \geqslant a_k(R)$ for every integer k with $0 \leqslant k \leqslant d(Q) = d(R)$. Hence we see from Proposition 1 that

$$\begin{aligned} g_i(Q) - g_i(R) &= \sum_{k=i+1}^{d(Q)} \binom{k-1}{i} a_k(Q) - \sum_{k=i+1}^{d(\phi_{f_R})} \binom{k-1}{i} a_k(R) \\ &\geqslant a_{d(Q)}(Q) > 0. \end{aligned}$$

Therefore we get the assertion of Claim 2. □

By Claim 2 and Corollary 3, we get the assertion of Theorem 6. □

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