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Some remarks about minimal prime ideals of skew Poincaré-Birkhoff-Witt extensions

A. Niño and A. Reyes*

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Dedicated to professor Oswaldo Lezama for his brilliant academic career at the Universidad Nacional de Colombia, Sede Bogotá.

Abstract. In this paper, we characterize the minimal prime ideals of skew PBW extensions over several classes of rings. We unify different results established in the literature for Ore extensions, and extend all of them to a several families of noncommutative rings of polynomial type which cannot be expressed as these extensions.

Introduction

For a ring B, the *set of prime ideals* of B is denoted by $\mathrm{Spec}(B)$, and the *set of minimal prime ideals* of B is denoted by $\mathrm{MinSpec}(B)$. The *lower nil radical* or the *prime radical* and the *set of nilpotent elements* of B are denoted by $\mathrm{Nil}_*(B)$ and $\mathrm{Nil}(B)$, respectively. We recall that any prime ideal U of B contains a minimal prime ideal (Goodearl and Warfield [16], Proposition 3.3). As a matter of fact, if B is a right or left Noetherian ring, then there exist only finitely many minimal prime ideals, and there is a finite product of minimal prime ideals (repetitions allowed) that equals zero ([16], Theorem 3.4).

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Recall that a ring B is said to be *2-primal* if and only if $\mathrm{Nil}_*(B) = \mathrm{Nil}(B)$, i.e., if the prime radical is completely semiprime (an ideal I of B is completely semiprime, if $a^2 \in I$ implies $a \in I$). The importance of 2-primal rings is that they can be considered as a generalization of commutative rings and reduced rings (a ring B is *reduced*, if B has no nonzero nilpotent elements). Commutative and reduced rings are strictly contained in 2-primal rings (see Marks [40] for a beautiful and detailed exposition about the relations between these rings). Several results about 2-primal rings can be found in the literature. For instance, Shin [60], Proposition 1.11, showed that a ring B is 2-primal if and only if every minimal prime ideal P of B is completely prime. He also proved that the minimal prime spectrum of a 2-primal ring is a Hausdorff space with a basis of clopen sets ([60], Proposition 4.7). Birkenmeier et al., [9] proved that the 2-primal condition is inherited by ordinary polynomial extensions.

With respect to the well-known *Ore extensions* (also called *skew polynomial rings*) defined by Ore [44], objects of interest for us in this paper, Ferrero and Kishimoto [11], Example 2.1, showed that if B is 2-primal, the differential polynomial ring $B[x;\delta]$ need not to be 2-primal. For a 2-primal ring B, Marks [38] investigated conditions on ideals of B that ensure that a skew polynomial ring $B[x;\sigma]$ or a differential polynomial ring $B[x;\delta]$ be 2-primal. On the other hand, Marks [39] considered the 2-primal property of the Ore extension $B[x;\sigma,\delta]$ where B is a local ring and σ is an automorphism of B. Marks showed that for a local ring with a nilpotent maximal ideal, the Ore extension $B[x;\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of B. If $B[x;\sigma,\delta]$ is 2-primal, it will satisfy an even stronger condition; if $B[x;\sigma,\delta]$ is not 2-primal, it will fail to satisfy an even weaker condition. In particular, Marks [39], Example 2.2, showed that Ore extension of automorphism type $B[x;\sigma]$ (i.e., σ is an automorphism of B) need not be 2-primal. With respect to the minimal prime ideals of 2-primal rings, Kim and Kwak [26] is one of the most important works. Several treatments about minimal prime ideals of Ore extensions can be found in Gabriel [12], Goodearl and Letzter [15], Goodearl and Warfield [16], Chapter 3, or McConnell and Robson [41].

Besides of Ore extensions, another classes of rings are of interest for us in this paper. These are the following: the *σ-rigid rings* defined by Krempa [27], the *$\sigma(*)$-rings* introduced by Kwak [28] and the *weak σ-rigid rings* defined by Ouyang [45]. We will say a few words about each one of these algebraic structures. Following Krempa [27], an endomorphism σ of a ring B is called *rigid*, if $a\sigma(a) = 0$ implies $a = 0$, for $a \in B$. B is called

σ-*rigid*, if there exists a rigid endomorphism σ of B. It is easy to see that any rigid endomorphism of a ring is injective and that σ-rigid rings are reduced (see [22] for details and [47] for a complete list of works about these rings). Several properties of σ-rigid rings have been established in the literature (e.g., [22], [27] and [47]). Now, following Kwak [28], if B is a ring and σ is an endomorphism of B, then B is said to be a $\sigma(*)$-*ring*, if $a\sigma(a) \in \mathrm{Nil}_*(B)$ implies $a \in \mathrm{Nil}_*(B)$, for $a \in B$. Kwak established a relation between 2-primal rings and $\sigma(*)$-rings. For instance, he proved that if B is a 2-primal ring and σ is an automorphism of B, then B is a $\sigma(*)$-ring if and only if $\sigma(P) = P$, for all $P \in \mathrm{MinSpec}(B)$, and that if B is a $\sigma(*)$-ring with $\sigma(\mathrm{Nil}_*(B)) = \mathrm{Nil}_*(B)$, then $B[x;\sigma]$ is 2-primal if and only if $\mathrm{Nil}_*(B)[x;\sigma] = \mathrm{Nil}_*(B[x;\sigma])$ ([28], Theorems 5 and 12). Finally, following Ouyang [45], if σ is an endomorphism of a ring B, B is said to be *weak* σ-*rigid*, when $a\sigma(a) \in \mathrm{Nil}(B)$ if and only if $a \in \mathrm{Nil}(B)$. Ouyang showed that B is σ-rigid if and only if B is weak σ-rigid and reduced ([45], Proposition 2.2). In this way, weak σ-rigid rings are a generalization of σ-rigid rings deleting the condition of being reduced.

Relations between the three families of rings described above and the notion of 2-primal ring have been established by Bhat [5], [6], [7] and [8]. Our objective in this paper is to generalize all results obtained by Bhat in these four papers in the context of Ore extensions of automorphism type (i.e., when σ is an automorphism of B), to the setting of skew Poincaré-Birkhoff-Witt extensions (PBW, for short) of bijective type introduced by Gallego and Lezama [13], which are strictly more general than Ore extensions of automorphism type (see [34], Section 3.2; in [58], Example 1, there are remarkable noncommutative rings of skew PBW extensions which can not be expressed as Ore extensions). Skew PBW extensions were introduced with the aim of generalizing the PBW extensions defined by Bell and Goodearl [4]. Some words about the generality of skew PBW extensions with respect to another families of noncommutative rings are said in Section 1. Homological and ring-theoretical properties for these extensions have been investigated by several people (e.g., Artamonov [2], Hashemi et al., [17], [18], [19], and [20], Lezama et al., [1], [24], [29], [31], [33], [35], Louzari et al., [37], Tumwesigye et al., [61], Zambrano [62], and the authors, [42], [43], [48], [51], [52] and [55]). As a matter of fact, a book containing research results about these extensions has recently been published, see [10].

It is important to say that this paper continues the study of ideals of these extensions initiated in [21], [30], [37], [53], [56] and [58]. We remark that in [37] and [56] the second author considered the question about the property of being 2-primal and the minimal prime ideals for skew PBW extensions by using a different approach to the presented here.

Next, we describe the structure of the article. In Section 1 we establish some useful results about skew PBW extensions for the rest of the paper. Section 2 contains the generalization of all the results obtained by Bhat in [5], [6] and [7], from Ore extensions of automorphism type to bijective skew PBW extensions. The Σ-rigid rings and the weak Σ-rigid rings defined by the second author in [47] and [54], respectively, are key in this generalization (they are the corresponding generalization of σ-rigid rings and weak σ-rigid rings, respectively). Also, in this section, we define the $\Sigma(*)$-rings as a natural generalization of $\sigma(*)$-rings. Finally, in Section 3 we generalize the assertions presented by Bhat [8], again, from Ore extensions of automorphism type to bijective skew PBW extensions. In Sections 2 and 3 we explicitly write the results we are extending. In [34], [55], and [58] there are several examples of these extensiones where the theory developed in both sections can be illustrated. In the last section, Section 4, we say a few words about the topic of interest in this paper and a possible future work. The results presented here are new for skew PBW extensions and can be considered as a contribution to the study of minimal prime ideals of noncommutative rings of polynomial type.

Throughout the paper, the word ring means an associative ring (not necessarily commutative) with unity. The symbol $\Bbbk$ will denote a field. The set of positive integers is denoted by $\mathbb{N}$.

1. Preliminaries on skew PBW extensions

Skew PBW extensions (also known as σ-PBW extensions) were defined by Gallego and Lezama with the aim of generalizing the PBW extensions introduced by Bell and Goodearl [4]. As time went by, we began to realize that skew PBW extensions generalize important families of noncommutative rings (not only PBW extensions) appearing in representation theory, Hopf algebras, quantum groups, noncommutative algebraic geometry and another algebras of interest in the context of mathematical physics. Some of these families are the following: Ore extensions of injective type, almost normalizing extensions defined by McConnell and Robson in [41], solvable polynomial rings introduced by Kandri-Rody and Weispfenning in [25], 3-dimensional skew polynomial algebras considered by Rosenberg [59], and diffusion algebras defined by Isaev, Pyatov, and Rittenberg [23]. The advantage of skew PBW extensions is that they do not require the coefficients to commute with the variables and, moreover, the coefficients need not come from a field (see Definition 1). In fact, the skew PBW extensions contain well-known groups of algebras such as some types of G-algebras in the sense of Apel [3], Auslander-Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul algebras, quantum polynomials, and some quantum universal enveloping algebras, see [10] for more details. Of course, from the definition of skew PBW extensions it is very clear their relation with quadratic algebras having PBW bases, see [46]. Therefore, as we can see, skew PBW extensions cover a wide spectrum of noncommutative rings formulated in the literature.

Next, we recall some results about skew PBW extensions which are important for the rest of the paper.

Definition 1 ([13], Definition 1)**.** Let R and A be rings. We say that A is a skew PBW extension of R, which is denoted by $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$, if the following conditions hold:

(i) R is a subring of A sharing the same multiplicative identity element.

(ii) There exist elements $x_1, \ldots, x_n \in A$ such that A is a left free R-module, with basis

$$\operatorname{Mon}(A) := \{x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\},$$

and $x_1^0 \cdots x_n^0 := 1 \in \operatorname{Mon}(A)$.

(iii) For each $1 \leqslant i \leqslant n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any elements $1 \leqslant i, j \leqslant n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$ (i.e., there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \ldots, r_n^{(i,j)}$ of R with $x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \sum_{l=1}^{n} r_l^{(i,j)} x_l$).

Since $\operatorname{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $d_{i,j}$ are unique, ([13], Remark 2).

Proposition 1 ([13], Proposition 3)**.** *Let A be a skew PBW extension of R. For each $1 \leqslant i \leqslant n$, there exist an injective endomorphism $\sigma_i : R \to R$ and an σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$. From now on, we will write $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$, and $\Delta := \{\delta_1, \ldots, \delta_n\}$.*

Definition 2 ([13], Definition 4; [30], Definition 2.3)**.** Let A be a skew PBW extension of R.

(a) A is called quasi-commutative, if the conditions (iii) and (iv) in Definition 1 are replaced by the following: (iii') for each $1 \leqslant i \leqslant n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$;

(iv') for any $1 \leqslant i, j \leqslant n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = d_{i,j} x_i x_j$.

(b) A is called bijective, if σ_i is bijective for each $1 \leqslant i \leqslant n$, and $d_{i,j}$ is invertible, for any $1 \leqslant i, j \leqslant n$.

(c) A is called of endomorphism type, if $\delta_i = 0$, for every i. In addition, if every σ_i is bijective, A is said to be a skew PBW extension of automorphism type.

Remark 1 ([13], Section 3)**.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension.

(i) Consider the families Σ and Δ in Proposition 1. Throughout the paper, for any element $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we will write $\sigma^{\alpha} := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^{\alpha} = \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes composition, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) Given the importance of monomial orders in the proofs of some results presented in the paper, next we recall some key facts about these for skew PBW extensions.

Let $\succeq$ be a total order defined on $\mathrm{Mon}(A)$. If $x^{\alpha} \succeq x^{\beta}$ but $x^{\alpha} \neq x^{\beta}$, we will write $x^{\alpha} \succ x^{\beta}$. If f is a nonzero element of A, then f can be expressed uniquely as $f = a_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R$, and $X_m \succ \cdots \succ X_1$ (eventually, we will use expressions as $f = a_0 + a_1 Y_1 + \cdots + a_m Y_m$, with $a_i \in R$, and $Y_m \succ \cdots \succ Y_1$). With this notation, we define $\mathrm{lm}(f) := X_m$, the *leading monomial* of f; $\mathrm{lc}(f) := a_m$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_m X_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{m}$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. Thus, we extend $\succeq$ to $\mathrm{Mon}(A) \cup \{0\}$.

Following [13], Definition 11, if $\succeq$ is a total order on $\mathrm{Mon}(A)$, we say that $\succeq$ is a *monomial order* on $\mathrm{Mon}(A)$, if the following conditions hold:

- For every $x^{\beta}, x^{\alpha}, x^{\gamma}, x^{\lambda} \in \mathrm{Mon}(A)$, $x^{\beta} \succeq x^{\alpha} \Rightarrow \mathrm{lm}(x^{\gamma} x^{\beta} x^{\lambda}) \succeq \mathrm{lm}(x^{\gamma} x^{\alpha} x^{\lambda})$ (the total order is compatible with multiplication).
- $x^{\alpha} \succeq 1$, for every $x^{\alpha} \in \mathrm{Mon}(A)$.
- $\succeq$ is degree compatible, i.e., $|\beta| \succeq |\alpha| \Rightarrow x^{\beta} \succeq x^{\alpha}$.

Monomial orders are also called *admissible orders*. The third condition of the previous definition is needed in the proof of the fact that every monomial order on $\mathrm{Mon}(A)$ is a well order, that is, there are not infinite decreasing chains in $\mathrm{Mon}(A)$ (see [13], Proposition 12). The importance of considering monomial orders on $\mathrm{Mon}(A)$ can be appreciated in [13] or [24].

Proposition 2 ([13], Theorem 7). *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

(1) *For each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.*

(2) *For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, there exist unique elements $d_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = d_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $d_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.*

Remark 2 ([47], Proposition 2.9 and Remark 2.10 (iv)). Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension. Then the following assertions hold:

(a) If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and r is an element of R, then

$$\begin{aligned}
x^\alpha r &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} r \\
&= x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n - j} \delta_n(\sigma_n^{j-1}(r)) x_n^{j-1} \right) \\
&\quad + x_1^{\alpha_1} \cdots x_{n-2}^{\alpha_{n-2}} \left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1} - j} \delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r))) x_{n-1}^{j-1} \right) x_n^{\alpha_n} \\
&\quad + x_1^{\alpha_1} \cdots x_{n-3}^{\alpha_{n-3}} \left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2} - j} \delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(r)))) x_{n-2}^{j-1} \right) \\
&\qquad \times x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&\quad + \cdots + x_1^{\alpha_1} \left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2 - j} \delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))))) x_2^{j-1} \right) \\
&\qquad \times x_3^{\alpha_3} x_4^{\alpha_4} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&\quad + \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))) x_1^{\alpha_1} \cdots x_n^{\alpha_n},
\end{aligned}$$

where $\sigma_j^0 := \mathrm{id}_R$ for $1 \leqslant j \leqslant n$.

(b) If $X_i := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $Y_j := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$, and a_i, b_j are elements of R, when we compute every summand of $a_i X_i b_j Y_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i. This assertion follows from the expression:

$$a_i X_i b_j Y_j = a_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j} + a_i p_{\alpha_{i1}, \sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} x^{\beta_j}$$

$$+ a_i x_1^{\alpha_{i1}} p_{\alpha_{i2},\sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_3^{\alpha_{i3}} \cdots x_n^{\alpha_{in}} x^{\beta_j}$$
$$+ a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_4^{\alpha_{i4}} \cdots x_n^{\alpha_{in}} x^{\beta_j}$$
$$+ \cdots + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_{i(n-2)}^{\alpha_{i(n-2)}} p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b_j)} x_n^{\alpha_{in}} x^{\beta_j}$$
$$+ a_i x_1^{\alpha_{i1}} \cdots x_{i(n-1)}^{\alpha_{i(n-1)}} p_{\alpha_{in},b_j} x^{\beta_j}.$$

2. Skew PBW extensions over weak Σ-rigid rings and $\Sigma(*)$-rings

In this section, we generalize the results presented by Bhat [5], [6] and [7], from Ore extensions of automorphism type to skew PBW extensions where every element $\sigma_i \in \Sigma$ is an automorphism.

For a ring B with a ring endomorphism $\sigma : B \to B$, Krempa [27] defined σ as a *rigid endomorphism*, if $b\sigma(b) = 0$ implies $b = 0$, for $b \in B$. Krempa called B a *σ-rigid*, if there exists a rigid endomorphism σ of B. In [27], Theorem 3.3, Krempa proved that if σ is a monomorphism, then the Ore extension $B[x;\sigma,\delta]$ is reduced if and only if B is reduced and σ is rigid. In this case, any minimal prime ideal (annihilator) of $B[x;\sigma,\delta]$ is of the form $PB[x;\sigma,\delta]$, where P is a minimal prime ideal (annihilator) in B. Now, since Ore extensions of injective type are particular examples of skew PBW extensions (see [34], Section 3.2), the second author introduced the following definition with the purpose of studying the notion of *rigidness* in this more general setting. Consider the notation in Remark 1 (i).

Definition 3 ([47], Definition 3.2)**.** Let B be a ring and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ a finite family of endomorphisms of B. Σ is called a rigid endomorphisms family, if $r\sigma^{\alpha}(r) = 0$ implies $r = 0$, for every $r \in B$ and each $\alpha \in \mathbb{N}^n$. A ring B is said to be Σ-rigid, if there exists a rigid endomorphisms family Σ of B.

Note that if Σ is a finite rigid endomorphisms family, then every element $\sigma_i \in \Sigma$ is a monomorphism. In fact, Σ-rigid rings are reduced rings: if B is a Σ-rigid ring and $r^2 = 0$ for $r \in B$, then $0 = r\sigma^{\alpha}(r^2)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r\sigma^{\alpha}(r))$, i.e., $r\sigma^{\alpha}(r) = 0$ and so $r = 0$, for every $\alpha \in \mathbb{N}^n$, that is, B is reduced (note that there exists an endomorphism of a reduced ring which is not a rigid endomorphism, see [22], Example 9). With this in mind, we consider the family of injective endomorphisms Σ and the family Δ of Σ-derivations in a skew PBW extension A over a ring R established in Proposition 1. Examples and some ring theoretical properties of Σ-rigid rings have been established by the authors, see [43], [50], [55], [56] and [58].

On the other hand, Kwak [28] introduced the notion of $\sigma(*)$-ring in the following way: Let B be a ring and σ an endomorphism of B. B is said to be a $\sigma(*)$-*ring*, if $a\sigma(a) \in \mathrm{Nil}_*(B)$ implies $a \in \mathrm{Nil}_*(B)$, for $a \in B$. Motivated by this definition and with the aim of extending in the natural way to the more general setting of skew PBW extensions, we present the following definition having in mind the family Σ of endomorphisms established in Proposition 1.

Definition 4. Let B be a ring with a family $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ of endomorphisms of B. B is said to be a $\Sigma(*)$-ring, if $a\sigma_i(a) \in \mathrm{Nil}_*(B)$ implies that $a \in \mathrm{Nil}_*(B)$, for all $a \in B$ and every i.

Remark 3. • Kwak [28], Example 2, presented the following noncommutative ring which is a $\sigma(*)$-ring but not σ-rigid. Let $B = \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix}$. Then $\mathrm{Nil}_*(B) = \begin{pmatrix} 0 & \Bbbk \\ 0 & 0 \end{pmatrix}$. If $\sigma : B \to B$ is defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, then it can be seen that σ is an endomorphism of B and that B is a $\sigma(*)$-ring. Nevertheless, B is not σ-rigid, since for any nonzero element $a \in \Bbbk$, we have that

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

• Kwak [28], Example 4, showed an example of a 2-primal ring but not a $\sigma(*)$-ring. Let us recall it. Let $B = \Bbbk[x]$ be the polynomial ring over $\Bbbk$. Then B is a domain and hence 2-primal with $\mathrm{Nil}_*(B) = \{0\}$. Consider the endomorphism $\sigma : B \to B$ defined by $\sigma(f(x)) = f(0)$. It is easy to see that B is not a $\sigma(*)$-ring. For example, take $f(x) = ax$ with a a nonzero element of $\Bbbk$, and note that $f(x)\sigma(f(x)) = 0 \in \mathrm{Nil}_*(B)$, but $f(x) \notin \mathrm{Nil}_*(B)$.

• Kwak [28], Theorem 5, proved that if B is a 2-primal ring and σ is an automorphism of B, then B is a $\sigma(*)$-ring if and only if $\sigma(P) = P$, for all $P \in \mathrm{MinSpec}(B)$. Theorem 12 of Kwak's paper assert that if B is a $\sigma(*)$-ring with $\sigma(\mathrm{Nil}_*(B)) = \mathrm{Nil}_*(B)$, then $B[x;\sigma]$ is 2-primal if and only if $\mathrm{Nil}_*(B)[x;\sigma] = \mathrm{Nil}_*(B[x;\sigma])$.

All results mentioned in Remark 3 are very important for the assertions we want to obtain for skew PBW extensions. In Remark 4, we present some key facts for our purposes.

Remark 4. The following results are direct consequences of Definition 4 and its relations with the assertions established in [5], Propositions 1, 2, and Theorem 3; [6], Theorem 1; [7], Proposition 1 and Theorem 1; [8], Propositions 1 and 2. As a matter of fact, in [5], Proposition 3 or [7], Proposition 1, there is a little mistake (the condition $\sigma\delta = \delta\sigma$ was omitted) which was solved in [8], Proposition 2 and Remark 2. This same mistake appear in [5], Proposition 4, and it was corrected in [8], Proposition 4.

(1) Let R be a ring with a family $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ of automorphisms of R. If R is a $\Sigma(*)$-ring, then $\text{Nil}_*(R)$ is completely semiprime (the converse of this implication is false, as Kwak showed in [28]: Take $R = \Bbbk \times \Bbbk$ and consider the automorphism σ of R defined by $\sigma((a, b)) = (b, a)$, for any elements $a, b \in \Bbbk$. Then R is reduced and so $\text{Nil}_*(R) = \{0\}$ is a completely semiprime ideal. However, the ring R is not a $\sigma(*)$-ring, since $(1, 0)\sigma((1, 0)) = (0, 0)$, but $(1, 0) \notin \text{Nil}_*(R)$).

(2) Let R be a Noetherian ring with a family $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ of automorphisms of R. If R is a $\Sigma(*)$-ring, then R is a 2-primal ring.

(3) If R is a Noetherian ring with a family $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ of automorphisms of R, then R is a $\Sigma(*)$-ring if and only if for every minimal prime ideal U of R, $\sigma_i(U) = U$, for all i, and U is a completely prime ideal of R.

(4) Let R be a Noetherian ring which is an algebra over $\mathbb{Q}$. Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a family of automorphisms of R such that R is a $\Sigma(*)$-ring and $\Delta = \{\delta_1, \dots, \delta_n\}$ a family of Σ-derivations of R. If $\sigma_i\delta_j = \delta_j\sigma_i$, for all $1 \leqslant i, j \leqslant n$, then $\delta_i(U) \subseteq U$, for each i and every $U \in \text{MinSpec}(R)$.

From now on, for a subset S of R, if A is a skew PBW extension over R, SA will denote the set $\{a_0 + a_1X_1 + \cdots + a_mX_m \in A \mid a_i \in S, \text{ for all } i\}$.

We start with the following result which generalizes Bhat's Theorem 4 in [5].

Theorem 1. *If A is a skew PBW extension over a ring R where every element of Σ is an automorphism, then:*

(1) *For every completely prime ideal P of R with $\delta_i(P) \subseteq P$ and $\sigma_i(P) = P$, for each i, PA is a completely prime ideal of A.*

(2) *For every completely prime ideal U of A, $U \cap R$ is a completely prime ideal of R.*

Proof. (1) Fix a monomial order on $\text{Mon}(A)$. Consider P a completely prime ideal of R. Let $f = \sum_{i=0}^{m} a_iX_i$ and $g = \sum_{j=0}^{t} b_jY_j$ be elements of A with $fg \in PA$. Let $\alpha_l := \exp(X_l)$. Suppose that $f \notin PA$. The idea is to show that $g \in PA$. We will use induction on m and t. If $m = t = 1$, the assertion is clear. Let us see the case $m = 2, t = 1$. We have $f = a_0 + a_1X_1 + a_2X_2$ and $g = b_0 + b_1Y_1$. Proposition 2 establishes that

$$
\begin{aligned}
fg &= (a_0 + a_1X_1 + a_2X_2)(b_0 + b_1Y_1)\\
&= a_0b_0 + a_0b_1Y_1 + a_1X_1b_0 + a_1X_1b_1Y_1 + a_2X_2b_0 + a_2X_2b_1Y_1\\
&= a_0b_0 + a_0b_1Y_1 + a_1(\sigma^{\alpha_1}(b_0)X_1 + p_{\alpha_1,b_0}) + a_1(\sigma^{\alpha_1}(b_1)X_1 + p_{\alpha_1,b_1})Y_1\\
&+ a_2(\sigma^{\alpha_2}(b_0)X_2 + p_{\alpha_2,b_0}) + a_2(\sigma^{\alpha_2}(b_1)X_2 + p_{\alpha_2,b_1})Y_1\\
&= a_0b_0 + a_0b_1Y_1 + a_1\sigma^{\alpha_1}(b_0)X_1 + a_1p_{\alpha_1,b_0} + a_1\sigma^{\alpha_1}(b_1)X_1Y_1\\
&+ a_1p_{\alpha_1,b_1}Y_1 + a_2\sigma^{\alpha_2}(b_0)X_2 + a_2p_{\alpha_2,b_0} + a_2\sigma^{\alpha_2}(b_1)X_2Y_1 + a_2p_{\alpha_2,b_1}Y_1.
\end{aligned}
$$

Since $fg \in PA$ and $f \notin PA$, we have the following possibilities: (a) $a_2 \notin P$ (b) $a_1 \notin P$ (c) $a_0 \notin P$ (d) any two of a_2, a_1, a_0 do not belong to P (e) all the elements a_2, a_1, a_0 do not belong to P. As an illustration, let us see (a) and (b). (a) $a_2 \notin P$: Since $fg \in PA$, we have $a_2\sigma^{\alpha_2}(b_0), a_2\sigma^{\alpha_2}(b_1) \in P$ whence $\sigma^{\alpha_2}(b_0), \sigma^{\alpha_2}(b_1) \in P$, and using that $\sigma_i(P) = P$, then $b_0, b_1 \in P$, i.e., $g \in P$. (b) $a_1 \notin P$: Note that $a_1\sigma^{\alpha_1}(b_0), a_1\sigma^{\alpha_1}(b_1) \in P$, and so we have that $\sigma^{\alpha_1}(b_0), \sigma^{\alpha_1}(b_1) \in P$, that is, $b_0, b_1 \in P$, and so $g \in P$. The remaining cases can be treated in a similar way.

Suppose that the assertion is true for k, where $m = k > 2$ and $t = 1$. Let us prove the case $m = k+1$. Consider $f = a_0 + a_1X_1 + \cdots + a_kX_k + a_{k+1}X_{k+1}$ and $g = b_0 + b_1Y_1$ with $fg \in PA$, but $f \notin PA$. As before, let us see that $g \in PA$. Since $f \notin PA$, then $a_{k+1} \notin P$, but note that $a_{k+1}\sigma^{\alpha_{k+1}}(b_0), a_{k+1}\sigma^{\alpha_{k+1}}(b_1)$, and hence $b_0, b_1 \in P$, i.e., $g \in PA$.

Now, if $a_j \notin P$, $0 \leqslant j \leqslant k$, then by induction hypothesis, it follows that $g \in PA$, so the statement is true for every m. By a similar argument, one can see that the statement is also true for all t.

(2) Let U be a completely prime ideal of A. Let a, b be elements of R with $ab \in U \cap R$ but $a \notin U \cap R$. Then $a \notin U$, and so $ab \in U \cap R \subseteq U$, with $a \notin U$, and hence $b \in U$, i.e., $b \in U \cap R$. □

The following result extends Bhat [5], Proposition 4.

Proposition 3. *If A is a skew PBW extension over a Noetherian $\Sigma(*)$-ring R which is also an algebra over $\mathbb{Q}$ such that $\sigma_i\delta_j = \delta_j\sigma_i$, for all $1 \leqslant i, j \leqslant n$, where every σ_i is bijective, for all i, then $U \in \mathrm{MinSpec}(R)$ implies that UA is a completely prime ideal of A.*

Proof. From Remark 4 part (1) we know that $\mathrm{Nil}_*(R)$ is a completely semiprime ideal of R. Let $U \in \mathrm{MinSpec}(R)$. Remark 4 parts (3) and (4) show that $\sigma_i(U) = U$, U is completely prime and $\delta_i(U) \subseteq U$, for every i. Finally, Theorem 1 implies that UA is a completely prime ideal of A. $\square$

With the aim of extending the σ-rigid rings defined by Krempa [27], Ouyang [45] introduced the weak σ-rigid rings. For the skew PBW extensions, this weak notion of rigidness was considered by the second author in [54] with the purpose of generalizing the Σ-rigid rings. Let us recall it. Consider the notation in Remark 1 (i).

Definition 5 ([54], Definition 3.2)**.** Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ and $\Delta = \{\delta_1, \dots, \delta_n\}$ be a family of endomorphisms and Σ-derivations of a ring R, respectively. R is called a weak Σ-rigid ring, when $a\sigma^{\theta}(a) \in \mathrm{Nil}(R)$ if and only if $a \in \mathrm{Nil}(R)$, for each element $a \in R$ and every $\theta \in \mathbb{N}^n$.

Remark 5. It is clear that Σ-rigid rings are weak Σ-rigid. However, the converse is false as we can appreciated in the following example taken from [45], Example 2.1. Let σ be an endomorphism of a ring R which is an σ-rigid ring. Consider the ring

$$R_3 := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in R \right\}.$$

If we extend the endomorphism σ of R to the endomorphism $\overline{\sigma} : R_3 \to R_3$ defined by $\overline{\sigma}(a_{ij}) = (\sigma(a_{ij}))$, then R_3 is a weak $\overline{\sigma}$-rigid ring but R_3 is not $\overline{\sigma}$-rigid. Therefore, weak Σ-rigid rings are a generalization of Σ-rigid rings to the case where the ring of coefficients is not assumed to be reduced (note that the ring R_3 is not reduced). Nevertheless, from [54], Theorem 3.4, we know that if $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ and $\Delta = \{\delta_1, \dots, \delta_n\}$ are families of endomorphisms and Σ-derivations of R, respectively, then R is Σ-rigid if and only if R is weak Σ-rigid and reduced (this result extends [45], Proposition 2.2).

The next assertion establishes a relation between $\Sigma(*)$-rings and weak Σ-rigid rings. This result generalizes [5], Theorem 5; [6], Theorem 2; [8], Proposition 3 (in the formulation of [6], Theorem 2, there is a little mistake: it says endomorphism but actually is automorphism).

Theorem 2. *If R is a Noetherian ring and Σ is a family of automorphisms of R such that R is a $\Sigma(*)$-ring, then R is a weak Σ-rigid ring. Conversely, every 2-primal weak Σ-rigid ring R is a $\Sigma(*)$-ring.*

Proof. Suppose that Σ is a family of automorphisms of R such that R is a $\Sigma(*)$-ring. From Remark 4 part (2) we know that R is 2-primal, that is, $\mathrm{Nil}_*(R) = \mathrm{Nil}(R)$, whence $a\sigma^\theta(a) \in \mathrm{Nil}(R) = \mathrm{Nil}_*(R)$, for every $\theta \in \mathbb{N}^n$, implies that $a \in \mathrm{Nil}_*(R) = \mathrm{Nil}(R)$, for each $\theta \in \mathbb{N}^n$. This proves that R is a weak Σ-rigid ring.

Now, suppose that R is a 2-primal weak Σ-rigid ring. It is clear that $\mathrm{Nil}(R) = \mathrm{Nil}_*(R)$ and that $a\sigma^\theta(a) \in \mathrm{Nil}(R)$ implies $a \in \mathrm{Nil}(R)$, for every $\theta \in \mathbb{N}^n$. Hence, $a\sigma^\theta(a) \in \mathrm{Nil}_*(R)$ implies $a \in \mathrm{Nil}_*(R)$, for all $\theta \in \mathbb{N}^n$, that is, R is a $\Sigma(*)$-ring. □

The following theorem extends [5], Theorem 6 and [6], Proposition 3.

Theorem 3. *If R is a Noetherian ring with a family of automorphisms Σ such that R is weak Σ-rigid, then* $\mathrm{Nil}(R)$ *is completely semiprime.*

Proof. It is easy to see that $\sigma_i(\mathrm{Nil}(R)) = \mathrm{Nil}(R)$, for every i. Let R be a weak Σ-rigid ring and consider $a \in R$ with $a^2 \in \mathrm{Nil}(R)$. Since $a\sigma^\theta(a)\sigma^\theta(a\sigma^\theta(a)) = a\sigma^\theta(a)\sigma^\theta(a)\sigma^\theta(\sigma^\theta(a)) \in \sigma^\theta(\mathrm{Nil}(R)) = \mathrm{Nil}(R)$, for all $\theta \in \mathbb{N}^n$, then $a\sigma^\theta(a) \in \mathrm{Nil}(R)$ implies that $a \in \mathrm{Nil}(R)$, for every $\theta \in \mathbb{N}^n$, which means that $\mathrm{Nil}(R)$ is completely semiprime. □

Remark 4 part (3) and Theorem 3 imply the following result.

Corollary 1. *If R is a Noetherian ring with Σ a family of automorphisms of R, then R is a 2-primal weak Σ-rigid ring if and only if for every minimal prime ideal U of R, $\sigma_i(U) = U$, for every i, and U is a completely prime ideal of R.*

The following proposition generalizes [5], Proposition 5.

Proposition 4. *If A is a skew PBW extension over a commutative Noetherian $\Sigma(*)$-ring R, where Σ is a family of automorphisms of R, then* $\mathrm{Nil}(R)A = \mathrm{Nil}(A)$.

Proof. Remark 4 part (2) shows that R is 2-primal. It is easy to see that $\mathrm{Nil}(R)A \subseteq \mathrm{Nil}(A)$, so we will only prove that $\mathrm{Nil}(A) \subseteq \mathrm{Nil}(R)A$. Fix a monomial order on $\mathrm{Mon}(A)$. Let $f = \sum_{i=0}^{m} a_iX_i$ be an element of $\mathrm{Nil}(A)$ (with $X_1 \prec X_2 \prec \cdots \prec X_m$), and let $X_m := x^{\alpha_m} = x_1^{\alpha_{m1}} \cdots x_n^{\alpha_{mn}}$. Note that

$$
\begin{aligned}
f^2 &= (a_mX_m + \cdots + a_1x_1 + a_0)(a_mX_m + \cdots + a_1x_1 + a_0)\\
&= a_mX_ma_mX_m + \text{ other terms less than } \exp(X_m)\\
&= a_m[\sigma^{\alpha_m}(a_m)X_m + p_{\alpha_m,a_m}]X_m + \text{ other terms less than } \exp(X_m)\\
&= a_m\sigma^{\alpha_m}(a_m)X_mX_m + a_mp_{\alpha_m,a_m}X_m + \text{other terms less than } \exp(X_m)\\
&= a_m\sigma^{\alpha_m}(a_m)[d_{\alpha_m,\alpha_m}x^{2\alpha_m} + p_{\alpha_m,\alpha_m}] + a_mp_{\alpha_m,a_m}X_m\\
&\quad + \text{other terms less than } \exp(X_m)\\
&= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}x^{2\alpha_m} + \text{ other terms less than } \exp(x^{2\alpha_m}),
\end{aligned}
$$

and hence,

$$
\begin{aligned}
f^3 &= (a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}x^{2\alpha_m} + \text{ other terms less than } \exp(x^{2\alpha_m}))\\
&\qquad \times (a_mX_m + \cdots + a_1x_1 + a_0)\\
&= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}x^{2\alpha_m}a_mX_m + \text{ other terms less than } \exp(x^{3\alpha_m})\\
&= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}[\sigma^{2\alpha_m}(a_m)x^{2\alpha_m} + p_{2\alpha_m,a_m}]X_m\\
&\qquad + \text{ other terms less than } \exp(x^{3\alpha_m})\\
&= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}\sigma^{2\alpha_m}(a_m)x^{2\alpha_m}X_m\\
&\qquad + \text{ other terms less than } \exp(x^{3\alpha_m})\\
&= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}\sigma^{2\alpha_m}(a_m)[d_{2\alpha_m,\alpha_m}x^{3\alpha_m} + p_{2\alpha_m,\alpha_m}]\\
&= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}\sigma^{2\alpha_m}(a_m)d_{2\alpha_m,\alpha_m}x^{3\alpha_m}\\
&\qquad + \text{ other terms less than } \exp(x^{3\alpha_m}).
\end{aligned}
$$

Continuing in this way, one can show that for f^k,

$$
f^k = a_m\prod_{l=1}^{k-1}\sigma^{l\alpha_m}(a_m)d_{l\alpha_m,\alpha_m}x^{k\alpha_m} + \text{ other terms less than } \exp(x^{k\alpha_m}),
$$

whence $0 = \mathrm{lc}(f^k) = a_m\prod_{l=1}^{k-1}\sigma^{l\alpha_m}(a_m)d_{l\alpha_m,\alpha_m}$, and since the elements d's are central in R and left invertible (Proposition 2), we have $0 = \mathrm{lc}(f^k) = a_m\prod_{l=1}^{k-1}\sigma^{l\alpha_m}(a_m)$. Since $0 \subseteq P$, for all $P \in \mathrm{MinSpec}(R)$, then for some $1 \leqslant j \leqslant k-1$, $a_m\sigma^{j\alpha_m}(a_m) \in P$, for all $P \in \mathrm{MinSpec}(R)$. Using that R is a $\Sigma(*)$-ring, one can see that $a_m \in P$, for all $P \in \mathrm{MinSpec}(R)$, and so $a_m \in \mathrm{Nil}_*(R)$. Now, having in mind that R is 2-primal, we obtain that $a_m \in \mathrm{Nil}(R)$, which means that $a_mX_m \in \mathrm{Nil}(R)A \subseteq \mathrm{Nil}(A)$, i.e., $\sum_{i=0}^{m-1} a_iX_i \in \mathrm{Nil}(A)$. Repeating this process one can see that $a_i \in \mathrm{Nil}_*(R) = \mathrm{Nil}(R)$, for every $0 \leqslant i \leqslant m-1$, which implies that $f \in \mathrm{Nil}(R)A$. Therefore, $\mathrm{Nil}(A) \subseteq \mathrm{Nil}(R)A$. □

With the purpose of establishing Theorem 4 (which extends [5], Theorem 7), we need the following preliminary result which says us how to extend the families of functions Σ and Δ from R to a families $\overline{\Sigma}$ and $\overline{\Delta}$ of a skew PBW extension A over R.

Proposition 5 ([52], Theorem 5.1)**.** *Let A be a skew PBW extension of a ring R. Suppose that $\sigma_i\delta_j = \delta_j\sigma_i$, $\delta_i\delta_j = \delta_j\delta_i$, and $\delta_k(d_{i,j}) = \delta_k(r_l^{(i,j)}) = 0$, for $1 \leqslant i, j, k, l \leqslant n$, where $d_{i,j}$ and $r_l^{(i,j)}$ are the elements established in Definition 1. If $\overline{\sigma_k} : A \to A$ and $\overline{\delta_k} : A \to A$ are the functions given by $\overline{\sigma_k}(f) := \sigma_k(a_0) + \sigma_k(a_1)X_1 + \cdots + \sigma_k(a_m)X_m$ and $\overline{\delta_k}(f) := \delta_k(a_0) + \delta_k(a_1)X_1 + \cdots + \delta_k(a_m)X_m$, for every $f = a_0 + a_1X_1 + \cdots + a_mX_m \in A$, respectively, and $\overline{\sigma_k}(r) := \sigma_i(k)$, for every $1 \leqslant i \leqslant n$, then $\overline{\sigma_k}$ is an injective endomorphism of A and $\overline{\delta_k}$ is a $\overline{\sigma_k}$-derivation of A, for all k.*

Theorem 4. *Let A be a skew PBW extension over a 2-primal commutative Noetherian ring R. Suppose that we have the conditions established in Proposition 5. If R is a weak Σ-rigid ring, then A is a weak $\overline{\Sigma}$-rigid ring.*

Proof. Suppose that R is a weak Σ-rigid ring. From Theorem 2 we know that R is a $\Sigma(*)$-ring, and by Proposition 4, $\mathrm{Nil}(R)A = \mathrm{Nil}(A)$. Again, fix a monomial order on $\mathrm{Mon}(A)$. Consider $f \in A$ given by $f = \sum_{i=0}^{m} a_iX_i$ such that $f\overline{\sigma^{\theta}}(f) \in \mathrm{Nil}(A)$. We will use induction on m with the aim of proving the assertion. If $m = 1$, then $f = a_0 + a_1X_1$. But $f\overline{\sigma^{\theta}}(f) \in \mathrm{Nil}(A)$ implies that $(a_0 + a_1X_1)(\sigma^{\theta}(a_0) + \sigma^{\theta}(a_1)X_1) \in \mathrm{Nil}(A) = \mathrm{Nil}(R)A$, that is, $a_0\sigma^{\theta}(a_0) + a_0\sigma^{\theta}(a_1)X_1 + a_1X_1\sigma^{\theta}(a_0) + a_1X_1\sigma^{\theta}(a_1)X_1$, or what is the same, $a_0\sigma^{\theta}(a_0) + a_0\sigma^{\theta}(a_1)X_1 + a_1(\sigma^{\alpha_1}(\sigma^{\theta}(a_0))X_1 + p_{\alpha_1,\sigma^{\theta}(a_0)}) + a_1(\sigma^{\alpha_1}(\sigma^{\theta}(a_1))X_1 + p_{\alpha_1,\sigma^{\theta}(a_0)})X_1$ belongs to $\mathrm{Nil}(A)$, from which we obtain that $a_0\sigma^{\theta}(a_0), a_1\sigma^{\alpha_1}(\sigma^{\theta}(a_1)) \in \mathrm{Nil}(A)$, and using that $\sigma_i(\mathrm{Nil}(R)) = \mathrm{Nil}(R)$, for each i (Corollary 1), we obtain that $a_0, a_1 \in \mathrm{Nil}(R)$, and so $f \in \mathrm{Nil}(R)A = \mathrm{Nil}(A)$. An argument by induction gives us the result. □

The following theorem generalizes [7], Theorem 2.

Theorem 5. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over a Noetherian ring R which is an algebra over $\mathbb{Q}$. Suppose that every element of $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is an automorphism and that the conditions established in Proposition 5 hold. If R is a $\Sigma(*)$-ring, then A is a Noetherian $\overline{\Sigma}(*)$-ring.*

Proof. From [34], Corollary 2.4, we know that A is Noetherian. Let us prove that A is a $\overline{\Sigma}(*)$-ring. With this aim, we will show that every minimal prime ideal P of A is completely prime and $\overline{\sigma_i}(P) = P$, for every i. It is easy to see that $P \cap R \in \mathrm{MinSpec}(R)$. Since R is a $\Sigma(*)$-ring, then $\sigma_i(P \cap R) = P \cap R$, and by Remark 4 part (4), $P \cap R$ is a completely prime ideal of R. Now, Remark 4 part (3) guarantees that $\delta_i(P \cap R) \subseteq P \cap R$, for every i. We can check that $(P \cap R)A$ is a completely prime ideal of A. Hence, $(P \cap R)A \subseteq P$ implies that $(P \cap R) = P$ because P is minimal. Therefore, using that $\sigma_i(P \cap R) = P \cap R$, for each i, it follows that $\overline{\sigma_i}(P) = P$, for all i. The assertion follows from Remark 4, (4). □

The following result extends [7], Theorem 3.

Theorem 6. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a right Noetherian ring and $\Sigma(*)$-ring R which is an algebra over $\mathbb{Q}$. Suppose that we have the conditions established in Proposition 5. We have the following assertions:*

1. *If U is a minimal prime ideal of R, then UA is a minimal prime ideal of A and $UA \cap R = U$.*
2. *If P is a minimal prime ideal of A, then $P \cap R$ is a minimal prime ideal of R.*

Proof. 1. From Remark 4 parts (3) and (4), we know that $\sigma_i(U) = U$ and $\delta_i(U) \subseteq U$, for every $i = 1, \dots, n$. It is clear that $UA \in \mathrm{Spec}(A)$ and $UA \cap R = U$.

2. By Theorem 5, A is a Noetherian $\overline{\Sigma}(*)$-ring. Remark 4 parts (3) and (4) show that $\overline{\sigma_i}(P) = P$ and $\overline{\delta_i}(P) = P$, for every i. In this way, $\sigma_i(P \cap R) = P \cap R$ and $\delta_i(P \cap R) \subseteq P \cap R$, for all i. It is easy to see that $P \cap R \in \mathrm{Spec}(R)$, whence $(P \cap R)A \in \mathrm{Spec}(A)$. Finally, since $(P \cap R)A \subseteq P$, it follows that $(P \cap R)A = P$. □

3. 2-primal skew PBW extensions over Noetherian weak Σ-rigid rings

In this section, we extend the results presented in [8], from Ore extensions of automorphism type to skew PBW extensions where every element σ_i of Σ is an automorphism.

We start with the following theorem which is the analogue to the result formulated in [8], Theorem 1.

Theorem 7. *If R is a commutative Noetherian ring and Σ is the family of automorphisms $\{\sigma_1, \dots, \sigma_n\}$ of R, then R is a weak Σ-rigid ring if and only if $\mathrm{Nil}(R)$ is a completely semiprime ideal of R.*

Proof. We follow the ideas presented in [8]. First of all, note that $\mathrm{Nil}(R)$ is an ideal of R due to the commutativity of R. Second of all, let us show that $\sigma_i(\mathrm{Nil}(R)) = \mathrm{Nil}(R)$, for every i. We fix i. Since $\sigma_i(\mathrm{Nil}(R))$ is a nilpotent ideal of R, then $\sigma_i(\mathrm{Nil}(R)) \subseteq \mathrm{Nil}(R)$, and having in mind that every σ_i is an automorphism, for every element $r \in \mathrm{Nil}(R)$, there exists an unique element $s \in R$ such that $\sigma_i(s) = r$. Consider the set $J := \sigma_i^{-1}(\mathrm{Nil}(R)) = \{s \in R \mid \sigma_i(s) = r\}$. It is clear that J is an ideal of R, and as a matter of fact, J is nilpotent, whence $J \subseteq \mathrm{Nil}(R)$ and so $\mathrm{Nil}(R) \subseteq \sigma_i(\mathrm{Nil}(R))$. Hence $\sigma_i(\mathrm{Nil}(R)) = \mathrm{Nil}(R)$, for all i.

Suppose that R is a weak Σ-rigid ring. Let $a \in R$ with $a^2 \in \mathrm{Nil}(R)$. Having in mind that $a\sigma^\theta(a)\sigma^\theta(a\sigma^\theta(a)) = a\sigma^\theta(a)\sigma^\theta(a)\sigma^\theta(\sigma^\theta(a)) = a\sigma^\theta(a^2)\sigma^\theta(\sigma^\theta(a)) \in \sigma^\theta(\mathrm{Nil}(R)) = \mathrm{Nil}(R)$, for all $\theta \in \mathbb{N}^n$, it follows that $a\sigma^\theta(a) \in \mathrm{Nil}(R)$, whence $a \in \mathrm{Nil}(R)$, which means that $\mathrm{Nil}(R)$ is completely semiprime.

Conversely, suppose that $\mathrm{Nil}(R)$ is completely semiprime. Consider an element $r \in R$ such that $a\sigma^\theta(a) \in \mathrm{Nil}(R)$, for all $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{N}^n$. Since $a\sigma^\theta(a)\sigma^{-\theta^{\mathrm{op}}}(a\sigma^\theta(a)) \in \mathrm{Nil}(R)$, where $-\theta^{\mathrm{op}} := (-\theta_n, \dots, -\theta_1)$ (see [51], Remark 4.2), it follows that $a^2 \in \mathrm{Nil}(R)$, and so $a \in \mathrm{Nil}(R)$. Therefore, R is a weak Σ-rigid ring. $\square$

For the next result, Proposition 7, which generalizes [8], Proposition 4, we need some preliminary facts and a proposition (Proposition 6) about quotients of skew PBW extensions: consider $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ a skew PBW extension of a ring R. Let $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$ such as in Proposition 1. Following Lezama et. al. [30], Definition 2.1, if I is an ideal of R, I is called Σ-*invariant* (Δ-*invariant*), if it is invariant under each injective endomorphism σ_i (σ_i-derivation δ_i) of Σ (Δ), that is, $\sigma_i(I) \subseteq I$ ($\delta_i(I) \subseteq I$), for $1 \leqslant i \leqslant n$. If I is both Σ and Δ-invariant ideal, then we say that I is (Σ, Δ)-*invariant*.

Proposition 6 ([30], Proposition 2.6). *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R and I is a (Σ, Δ)-invariant ideal of R, then the following statements hold:*

(i) *IA is an ideal of A and $IA \cap R = I$. IA is a proper ideal of A if and only if I is proper in R. Moreover, if σ_i is bijective and $\sigma_i(I) = I$, for every i, then $IA = AI$.*

(ii) *If I is proper and $\sigma_i(I) = I$, for every $1 \leqslant i \leqslant n$, then A/IA is a skew PBW extension of R/I. In fact, if I is proper and A is bijective, then A/IA is a bijective skew PBW extension of R/I, that is, $A/IA = \widehat{\sigma}(R/IR)\langle x_1, \dots, x_n\rangle$.*

From Proposition 6, we can see that if I is (Σ, Δ)-invariant, then over R/I it is induced a system $(\widehat{\Sigma}, \widehat{\Delta})$ of endomorphisms $\widehat{\Sigma}$ and $\widehat{\Sigma}$-derivations $\widehat{\Delta}$, defined by $\widehat{\sigma_i}(r+I) = \sigma_i(r)+I$ and $\widehat{\delta_i}(r+I) = \delta_i(r)+I$, for $1 \leqslant i \leqslant n$, and every $r \in R$. We keep the variables $x_1, \dots, x_n$ of extension A to the extension A/IA if no confusion arises.

Proposition 7. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a Noetherian ring R which is an algebra over $\mathbb{Q}$. If R is a $\Sigma(*)$-ring and $\delta_i(\sigma_i(r)) = \sigma_i(\delta_i(r))$, for every i and each $r \in R$, then for each $P \in \operatorname{MinSpec}(R)$ we have that PA is a completely prime ideal of A.*

Proof. Consider $P \in \operatorname{MinSpec}(R)$. Remark 4 parts (2), (3) and (4) imply that R is 2-primal, $\sigma_i(P) = P$ and $\delta_i(P) = P$, for every i, and P is completely prime. Having in mind the families of functions $\widehat{\Sigma}$ and $\widehat{\Delta}$ formulated in Proposition 6, it is easy to see that $A/PA \cong \widehat{\sigma}(R/P)\langle x_1, \dots, x_n\rangle$, which shows that PA is a completely prime ideal of A. $\square$

The next theorem extends [8], Theorem 3.

Theorem 8. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a Noetherian ring R which is an algebra over $\mathbb{Q}$. If R is a $\Sigma(*)$-ring and $\delta_i(\sigma_i(r)) = \sigma_i(\delta_i(r))$, for every i and each $r \in R$, then for each $P \in \operatorname{MinSpec}(R)$ we have that $PA \in \operatorname{MinSpec}(A)$.*

Proof. Consider $P \in \operatorname{MinSpec}(R)$. Remark 4 parts (3) and (4) imply that $\sigma_i(P) = P$ and $\delta_i(P) \subseteq P$, for every i. If $PA \notin \operatorname{MinSpec}(A)$, let $P_1 \in \operatorname{MinSpec}(A)$ such that $P_1 \subset PA$. Then $P_1 = (P_1 \cap R)A \subset PA \in \operatorname{MinSpec}(R)$, whence $P_1 \cap R \subset P$, a contradiction, since $P_1 \cap R \in \operatorname{Spec}(R)$. Therefore, $PA \in \operatorname{MinSpec}(A)$. $\square$

The next theorem generalizes [8], Theorem 4.

Theorem 9. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a Noetherian ring R which is an algebra over $\mathbb{Q}$. If R is a $\Sigma(*)$-ring and $\delta_i(\sigma_i(r)) = \sigma_i(\delta_i(r))$, for every i and each $r \in R$, then A is 2-primal if and only if $\operatorname{Nil}_*(R)A = \operatorname{Nil}_*(A)$.*

Proof. Consider A a 2-primal ring. By Theorem 8, $\operatorname{Nil}_*(A) \subseteq \operatorname{Nil}_*(R)A$. Consider an element $f = \sum_{i=0}^{m} a_i X_i$ of $\operatorname{Nil}_*(R)A$. Note that R is a 2-primal ring (Remark 4, part (2)). Hence, every element a_i is nilpotent, and so $a_i \in \operatorname{Nil}(A) = \operatorname{Nil}_*(A)$, for every i, whence $f \in \operatorname{Nil}_*(A)$, and so $\operatorname{Nil}_*(R)A = \operatorname{Nil}_*(A)$.

Conversely, suppose that $\mathrm{Nil}_*(R)A = \mathrm{Nil}_*(A)$ with a fixed monomial order on $\mathrm{Mon}(A)$. Let $g = \sum_{j=0}^{t} b_j Y_j \in A$ with $g^2 \in \mathrm{Nil}_*(A) = \mathrm{Nil}_*(R)A$. The idea is to show that $g \in \mathrm{Nil}_*(A)$. With this aim, note that $\mathrm{lc}(g^2) \in \mathrm{Nil}_*(R) \subseteq P$, for every $P \in \mathrm{MinSpec}(R)$, and since $\sigma_i(P) = P$ and P is completely prime (Theorem 8), it follows that $b_t \in P$, for each $P \in \mathrm{MinSpec}(R)$, i.e., $b_t \in \mathrm{Nil}_*(R)$. Now, Remark 4 part (4) guarantees that $\delta_i(P) \subseteq P$, for all i and every $P \in \mathrm{MinSpec}(R)$, whence we can assert that $\left(\sum_{j=0}^{t-1} b_j Y_j\right) \in \mathrm{Nil}_*(A) = \mathrm{Nil}_*(R)A$, and as before, $b_{t-1} \in \mathrm{Nil}_*(R)$. If we repeat this argument, we can see that $b_j \in \mathrm{Nil}_*(R)$, for all j, and so $g \in \mathrm{Nil}_*(R)A = \mathrm{Nil}_*(A)$. Therefore, $\mathrm{Nil}_*(A)$ is a completely semiprime ideal, that is, A is 2-primal. □

The next theorem extends [8], Proposition 5.

Proposition 8. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over a Noetherian ring R which is 2-primal and an algebra over $\mathbb{Q}$. If R is a $\Sigma(*)$-ring and $\delta_i(\sigma_i(r)) = \sigma_i(\delta_i(r))$, for every $i = 1, \ldots, n$ and each $r \in R$, then* $\mathrm{Nil}(R)A = \mathrm{Nil}(A)$.

Proof. The arguments in the proof are completely similar to those used in the proof of Proposition 4. We only take R to be 2-primal instead of commutative. □

Our Theorem 10 generalizes [8], Theorem 5.

Theorem 10. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over a Noetherian 2-primal weak Σ-rigid ring R which is an algebra over $\mathbb{Q}$, where every $\sigma_i \in \Sigma$ is an automorphism. If the conditions established in Proposition 5 hold, then A is a 2-primal Noetherian weak $\overline{\Sigma}$-rigid ring.*

Proof. From [34], Corollary 2.4, we know that A is a Noetherian ring. By assumption, R is 2-primal weak Σ-rigid, so Theorem 2 implies that R is a $\Sigma(*)$-ring. Now, Theorem 6 guarantees that if $P \in \mathrm{MinSpec}(A)$, then $P \cap R \in \mathrm{MinSpec}(R)$. Theorem 8 shows that $\mathrm{Nil}_*(R)A = \mathrm{Nil}_*(A)$, whence A is 2-primal by Theorem 9. Finally, Theorem 4 implies that A is a weak $\overline{\Sigma}$-rigid ring. Therefore, A is a 2-primal Noetherian weak Σ-rigid ring. □

4. Conclusions and future work

As we said in the Introduction, in [36] and [56] the second author considered the question about the property of being 2-primal and the minimal prime ideals for skew PBW extensions by using a different approach to the established in this paper. More exactly, there, the notions of compatible ring, skew Armendariz ring (see [49], [52] and [53] for more details) and the ascending chain condition on right annihilators (see [58]) on the ring of coefficients were key in the characterization of minimal prime ideals of these extensions. Therefore, the results obtained in this paper are another approach to the study of these ideals of skew PBW extensions.

Last, but not least, we consider as a possible future work to investigate minimal prime ideals and the 2-primal property in a more general context of noncommutative rings than skew PBW extensions such as for example the *semi-graded rings* introduced by Lezama [33] (see also [29] and [32]), or maybe considering a weak notion of compatibility following the ideas presented recently by the second author [57], and of course, in the setting of modules over these extensions, see [36], [42], and [48].

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Contact information

Arturo Niño, Armando Reyes

Department of Mathematics, Faculty of Science, Universidad Nacional de Colombia, Bogotá, D.C., Colombia

E-Mail(s): dieaninotor@unal.edu.co, mareyesv@unal.edu.co

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