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RESEARCH ARTICLE

# A filtration on the ring of Laurent polynomials and representations of the general linear Lie algebra

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Abstract. We first present a filtration on the ring $L_n$ of Laurent polynomials such that the direct sum decomposition of its associated graded ring $grL_n$ agrees with the direct sum decomposition of $grL_n$, as a module over the complex general linear Lie algebra $\mathfrak{gl}(n)$, into its simple submodules. Next, generalizing the simple modules occurring in the associated graded ring $grL_n$, we give some explicit constructions of weight multiplicity-free irreducible representations of $\mathfrak{gl}(n)$.

## 1. Introduction

In this section, we give a brief summary of our results.

### 1.1. The ring of polynomials

The ring $P_n = \mathbb{C}[x_1, \dots, x_n]$ of polynomials in $n$ indeterminates over the complex numbers $\mathbb{C}$ is a $\mathbb{Z}$-graded algebra

$$P_n = \bigoplus_{m\in\mathbb{Z}} P_n^{(m)} \tag{1.1}$$

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where $P_n^{(m)}$ is the space of homogeneous polynomials of degree $m$. As a vector space, $P_n$ becomes a module over the complex general linear Lie algebra $\mathfrak{gl}(n) = \mathfrak{gl}_n(\mathbb{C})$ under the action

$$A \cdot f = \sum_{ij} a_{ij} x_i \frac{\partial f}{\partial x_j} \quad \text{for } A = (a_{ij}) \in \mathfrak{gl}(n) \text{ and } f \in P_n. \tag{1.2}$$

Then, the direct sum decomposition (1.1) of $P_n$ as a graded ring agrees with the decomposition of $P_n$ as a $\mathfrak{gl}(n)$-module into its simple submodules $P_n^{(m)}$. They are the finite dimensional representations of $\mathfrak{gl}(n)$ labeled by Young diagrams with single rows.

## 1.2. The ring of Laurent polynomials

The first goal of this paper is to obtain an analogous result of the above observation for the ring of Laurent polynomials

$$L_n = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}].$$

It turns out that a filtration and its associated graded structure give us an answer. Note that (1.1) can be seen as the graded ring associated with the $\mathbb{Z}$-filtration of $P_n$ given by degree.

We will define a filtration on $L_n$ by a partially ordered monoid constructed from integers and subsets of $\{1, 2, \dots, n\}$

$$L_n = \bigcup_{(m,J) \in \mathbb{Z} \times \mathscr{P}_n} L_n^{\leqslant (m,J)}$$

and show that the direct sum decomposition of its associated graded ring

$$grL_n = \bigoplus_{(m,J) \in \mathbb{Z} \times \mathscr{P}_n} L_n^{\leqslant (m,J)} / L_n^{< (m,J)}$$

provides the decomposition of $grL_n$, as a $\mathfrak{gl}(n)$-module, into its simple submodules.

Extending the space with the action (1.2) of $\mathfrak{gl}(n)$ from $P_n$ to $L_n$, we identify Laurent monomials $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ with integral points $\mathbf{k} = (k_1, k_2 \dots, k_n)$ in $\mathbb{R}^n$. Note that they are weight vectors with respect to the Cartan subalgebra of $\mathfrak{gl}(n)$ consisting of diagonal matrices. Since this action preserves the degree of monomials, we can focus on integral points on the hyperplane $k_1 + \cdots + k_n = m$ for each $m \in \mathbb{Z}$.

One of main difficulties in studying the $\mathfrak{gl}(n)$-module structure of $L_n$ is that the symmetric behavior of raising and lowering operators we had

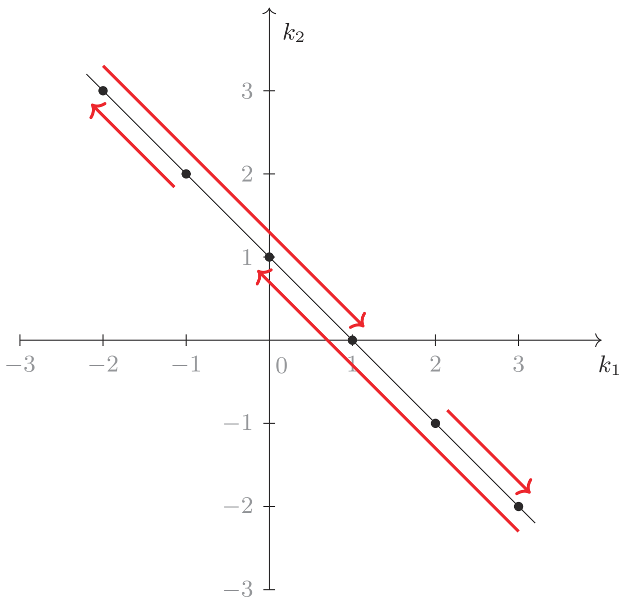


FIGURE 1. The action of $\mathfrak{gl}(2)$ on $x_1^{k_1}x_2^{k_2}$ with $k_1+k_2=1$.

when working with $P_n$ is not trivial anymore. For example, when $n=2$ as in Figure 1,

$$\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\cdot x_1^{k_1}x_2^{m-k_1} = k_1 x_1^{k_1-1}x_2^{m+1-k_1},$$
$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\cdot x_1^{m-k_2}x_2^{k_2} = k_2 x_1^{m+1-k_2}x_2^{k_2-1}.$$

The cases $k_1=0$ and $k_2=0$ divide the line $k_1+k_2=m$ into three parts. The monomials with $k_1\geqslant 0$ and $k_2\geqslant 0$ can be obtained by applying some elements of $\mathfrak{gl}(2)$ to monomials with $k_1k_2<0$. However, monomials with $k_1k_2<0$ cannot be obtained from the ones with $k_1\geqslant 0$ and $k_2\geqslant 0$.

More generally, the planes $k_j=0$ divide the hyperplane $k_1+k_2+\cdots+k_n=m$ into regions labeled by the signs of the coordinates $k_i$. Then, for each $i$, we can obtain weight vectors $\mathbf{x}^{\mathbf{k}}$ with $k_i\geqslant 0$ starting from the ones with $k_i<0$ by successively applying some elements of $\mathfrak{gl}(n)$, but the opposite way is not possible. See Figure 2.

Therefore, our indecomposable submodules in $L_n$ and simple modules obtained from their quotients are labeled by degree $m$ of $\mathbf{x}^{\mathbf{k}}$ and subsets $J$ of $\{1,2,\ldots,n\}$ indicating the position of possible negative components

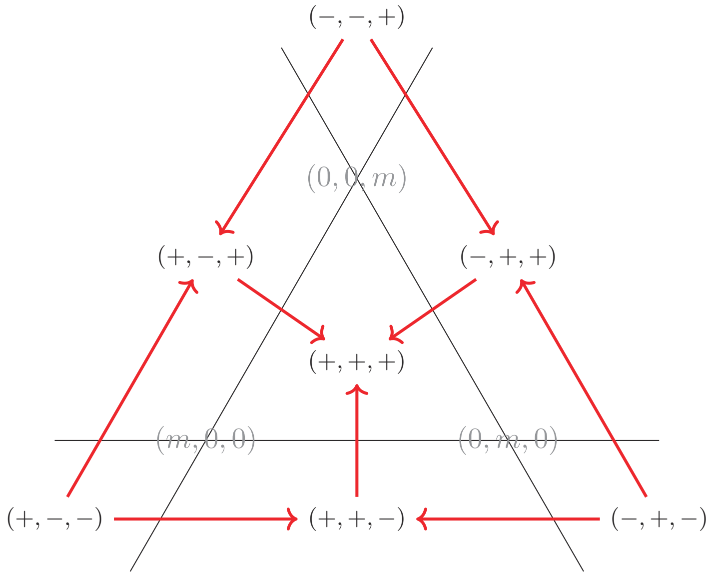


FIGURE 2. The action of $\mathfrak{gl}(3)$ on $x_1^{k_1}x_2^{k_2}x_3^{k_3}$ with $k_1 + k_2 + k_3 = m$ $(m > 0)$.

in $\mathbf{k}$. Their structures depend heavily on $m$ and the cardinality of $J$. We will give a clear case-by-case analysis of them.

## 1.3. Representations of the general linear Lie algebra

Our next goal is to provide explicit constructions of weight multiplicity-free irreducible representations of $\mathfrak{gl}(n)$ obtained by twisting the action (1.2). For a general theory on weight multiplicity-free representations of simple Lie algebras, see [1] and references therein.

Motivated by works on weight modules of the Lie algebra of diffeomorphisms of the $n$-dimensional torus (see, for example, [3,4,6]), for each $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, we will define a representation $L_n^{\boldsymbol{\alpha}}$ of $\mathfrak{gl}(n)$ on the vector space $L_n$ (see Definition 3.2). Then, we investigate two families of its submodules, $L_n^{\boldsymbol{\alpha}}(m, j)$ and $V_n^{\boldsymbol{\alpha}}(m, J)$, parameterized by integers $m$, $j$, and subsets $J$ of $\{i : \alpha_i = 0\}$. We can obtain explicit simple $\mathfrak{gl}(n)$-modules from the decomposition of the quotient modules

$$L_n^{\boldsymbol{\alpha}}(m, j)/\, L_n^{\boldsymbol{\alpha}}(m, j-1) = \bigoplus_{J:|J|=j} W_n^{\boldsymbol{\alpha}}(m, J)$$

where $W_n^{\boldsymbol{\alpha}}(m, J)$ are simple modules defined by

$$W_n^{\boldsymbol{\alpha}}(m, J) = \left(V_n^{\boldsymbol{\alpha}}(m, J) + L_n^{\boldsymbol{\alpha}}(m, j-1)\right)/L_n^{\boldsymbol{\alpha}}(m, j-1).$$

Among these simple modules, there are highest weight modules with highest weights of the form $\psi^\lambda \in \mathfrak{h}^*$ where

$$\lambda = (-1, \dots, -1, z, 0, \dots, 0) \in \mathbb{C}^n$$

including the finite dimensional ones having integral dominant weights with $\lambda = (k, 0, \dots, 0)$ and $(-1, \dots, -1, \ell)$ for $k \geqslant 0$ and $\ell \leqslant -1$.

## 2. A filtration on $L_n$ and simple modules in $grL_n$

In this section, we impose a filtration on the ring

$$L_n = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

of Laurent polynomials in $n$ indeterminates over the complex numbers $\mathbb{C}$, and then show that the graded structure of its associated graded ring is compatible with the module structure of $L_n$ over the complex general linear Lie algebra $\mathfrak{gl}(n) = \mathfrak{gl}_n(\mathbb{C})$.

Recall that $\mathfrak{gl}(n)$ is the Lie algebra of $n \times n$ complex matrices with the usual matrix addition and the Lie bracket given by the commutator of two matrices. We will write $\mathcal{U}_n = \mathcal{U}(\mathfrak{gl}(n))$ for the universal enveloping algebra of $\mathfrak{gl}(n)$.

### 2.1. Submodules of $L_n$

The complex vector space $L_n$ is spanned by monomials

$$\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

for $\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$. We define some subspaces of $L_n$.

**Definition 2.1.** Let $m$ be an integer, $j$ be an integer with $0 \leqslant j \leqslant n$, and $J$ be a subset of $\{1, 2, \dots, n\}$.

1) Let $V_n(m, J)$ be the subspace of $L_n$ spanned by all the monomials $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ such that

$$\sum_{i=1}^{n} k_i = m \quad \text{and} \quad \{i : k_i < 0\} \subseteq J.$$

2) Let $L_n(m, j)$ be the sum of the subspaces $V_n(m, J)$ of $L_n$

$$L_n(m, j) = \sum_{J : |J| = j} V_n(m, J)$$

over all subsets $J$ of $\{1, 2, \dots, n\}$ having $j$ elements.

It follows directly from the definition that

$$V_n(m, J_1) \subseteq V_n(m, J_2) \quad \text{for all } J_1 \subseteq J_2.$$

Then, we have

$$L_n(m, j-1) \subseteq L_n(m, j)$$

and their quotient can be expressed as

$$\begin{aligned} L_n(m,j)/L_n(m,j-1) &= \Bigg(\sum_{J:|J|=j} V_n(m,J)\Bigg)/L_n(m,j-1) \\ &= \sum_{J:|J|=j} \big(V_n(m,J) + L_n(m,j-1)\big)/L_n(m,j-1). \end{aligned}$$

**Definition 2.2.** For an integer $m$ and a subset $J$ of $\{1, 2, \ldots, n\}$ having $j$ elements, we let $W_n(m, J)$ denote the subspace

$$W_n(m,J) = \left(V_n(m,J) + L_n(m,j-1)\right)/L_n(m,j-1)$$

of the quotient space $L_n(m, j)/L_n(m, j-1)$.

The spaces $V_n(m, J)$, $L_n(m, j)$, and $W_n(m, J)$ are special cases of the ones defined in Definition 3.5 and Definition 6.1 with $\boldsymbol{\alpha} = \mathbf{0}$ and $I_{\boldsymbol{\alpha}} = \{1, 2, \ldots, n\}$, and they are modules over $\mathcal{U}_n$ with respect to

$$A \cdot f = \sum_{ij} a_{ij} x_i \frac{\partial f}{\partial x_j}$$

for $A = (a_{ij}) \in \mathfrak{gl}(n)$ and $f \in L_n$. See Theorems 4.4, 4.5, 5.4, and 5.5. Moreover, $W_n(m, J)$ are simple modules. See Theorem 6.2.

**Lemma 2.3.** For $m \in \mathbb{Z}$ and a subset $J$ of $\{1, 2, \ldots, n\}$ having $j$ elements, as a $\mathcal{U}_n$-module,

$$W_n(m,J) \;\cong\; V_n(m,J)/\sum_{J'} V_n(m,J')$$

where the summation is over all subsets $J'$ of $J$ having $j - 1$ elements

*Proof.* Note that

$$\begin{aligned} W_n(m,J) &= \left(V_n(m,J) + L_n(m,j-1)\right)/L_n(m,j-1) \\ &\cong V_n(m,J)/\left(V_n(m,J) \cap L_n(m,j-1)\right). \end{aligned}$$

Then, the statement follows from the following observation

$$V_n(m,J) \cap L_n(m,j-1) = \sum_{J'} V_n(m,J')$$

where the sum is over all subsets $J'$ of $J$ having $j - 1$ elements. □

### 2.2. Filtration by a partially ordered monoid

Let $\mathscr{P}_n$ be the set of all subsets of $\{1, 2, \dots, n\}$. On the set $\mathbb{Z} \times \mathscr{P}_n$, we define the partial order $(m_1, J_1) \leqslant (m_2, J_2)$ if $m_1 \leqslant m_2$ and $J_1 \subseteq J_2$, and the multiplication

$$(m_1, J_1) * (m_2, J_2) = (m_1 + m_2, J_1 \cup J_2).$$

With $\leqslant$ and $*$, $\mathbb{Z} \times \mathscr{P}_n$ becomes a partially ordered monoid with the identity $(0, \varnothing)$, and using this monoid we want to impose a filtration on $L_n$. For basic properties of a filtration of a ring given by a partially ordered monoid, we refer to [2, §I.12].

**Definition 2.4.** For each $(m, J) \in \mathbb{Z} \times \mathscr{P}_n$, we define

$$L_n^{\leqslant(m,J)} = \sum_{(m_1,J_1)} V_n(m_1, J_1) \quad \text{and} \quad L_n^{<(m,J)} = \sum_{(m_2,J_2)} V_n(m_2, J_2)$$

where the first summation is over all $(m_1, J_1)$ such that

$$(m_1, J_1) \leqslant (m, J),$$

and the second summation is over all $(m_2, J_2)$ such that

$$(m_2, J_2) \leqslant (m, J) \text{ but } (m_2, J_2) \neq (m, J).$$

**Proposition 2.5.** The family $\{L_n^{\leqslant(m,J)} : (m, J) \in \mathbb{Z} \times \mathscr{P}_n\}$ of subspace of $L_n$ defines a filtration on $L_n$ by the partially ordered monoid $\mathbb{Z} \times \mathscr{P}_n$.

*Proof.* We need to check the following conditions (see [2, §I.12]).

1) $1 \in L_n^{\leqslant(0,\varnothing)}$,
2) for all $(m_1, J_1) \leqslant (m_2, J_2)$,
$$L_n^{\leqslant(m_1,J_1)} \subseteq L_n^{\leqslant(m_2,J_2)},$$
3) for all $(m_1, J_1)$ and $(m_2, J_2)$,
$$L_n^{\leqslant(m_1,J_1)} L_n^{\leqslant(m_2,J_2)} \subseteq L_n^{\leqslant(m_1,J_1)*(m_2,J_2)},$$
4) the union of all such subspaces is equal to $L_n$
$$\bigcup_{(m,J)\in\mathbb{Z}\times\mathscr{P}_n} L_n^{\leqslant(m,J)} = L_n.$$

All of them follow directly from the definitions of $L_n^{\leqslant(m,J)}$ and $V_n(m, J)$. □

For $(m_1, J_1)$ and $(m_2, J_2)$ in $\mathbb{Z} \times \mathscr{P}_n$, we have

$$L_n^{\leqslant(m_1,J_1)} L_n^{<(m_2,J_2)} + L_n^{<(m_1,J_1)} L_n^{\leqslant(m_2,J_2)} \subseteq L_n^{<(m_1,J_1)*(m_2,J_2)}$$

and therefore the quotient spaces $L_n^{\leqslant(m,J)}/L_n^{<(m,J)}$ form the homogeneous components of the associated graded ring of $L_n$.

**Theorem 2.6.** With the filtration $\{L_n^{\leqslant(m,J)} : (m,J) \in \mathbb{Z} \times \mathscr{P}_n\}$ imposed on $L_n$, the direct sum decomposition of the associated graded ring

$$grL_n = \bigoplus_{(m,J)\in\mathbb{Z}\times\mathscr{P}_n} L_n^{\leqslant(m,J)}/L_n^{<(m,J)}$$

agrees with the decomposition of $grL_n$ into its simple submodules over $\mathcal{U}_n$. In particular, for each $(m,J) \in \mathbb{Z} \times \mathscr{P}_n$, as a $\mathcal{U}_n$-module,

$$L_n^{\leqslant(m,J)}/L_n^{<(m,J)} \cong W_n(m,J).$$

*Proof.* For an integer $m$ and a subset $J$ of $\{1, 2, \ldots, n\}$ having $j$ elements, let us define

$$S = V_n(m,J) \quad \text{and} \quad T = \sum_{J_1 \subset J} V_n(m,J_1) + \sum_{m_1<m} V_n(m_1,J)$$

where the first summation for $T$ is over all proper subsets $J_1$ of $J$, and the second summation is over all $m_1$ strictly less than $m$. Then, we have

$$\begin{aligned} S \cap T &= V_n(m,J) \cap \left( \sum_{J_1 \subset J} V_n(m,J_1) + \sum_{m_1<m} V_n(m_1,J) \right) \\ &= V_n(m,J) \cap \left( \sum_{J_1 \subset J} V_n(m,J_1) \right) \\ &= \sum_{J'} V_n(m,J') \end{aligned}$$

where the last summation is over all subsets $J'$ of $J$ having $j-1$ elements. Note that

$$S + T = L_n^{\leqslant(m,J)} \quad \text{and} \quad T = L_n^{<(m,J)}$$

and then, by the usual module isomorphism theorem $(S+T)/T \cong S/(S \cap T)$, we obtain

$$L_n^{\leqslant(m,J)}/L_n^{<(m,J)} \cong V_n(m,J)/\sum_{J'} V_n(m,J') \tag{2.1}$$

where the summation is over all subsets $J'$ of $J$ having $j-1$ elements. Now, using the realization of $W_n(m,J)$ given in Lemma 2.3, we have

$$L_n^{\leqslant(m,J)}/L_n^{<(m,J)} \cong W_n(m,J). \qquad \square$$

In § 6, we will investigate the simple modules $W_n(m,J)$ in a more general setting.

## 3. Modules $L_n^{\boldsymbol{\alpha}}(m,j)$ and $V_n^{\boldsymbol{\alpha}}(m,J)$

In this section, generalizing $L_n(m,J)$ and $V_n(m,J)$ discussed in the previous section, we define some submodules of $L_n$ over the universal enveloping algebra $\mathcal{U}_n$ of $\mathfrak{gl}(n)$ parameterized by $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\in\mathbb{C}^n$.

**Notation 3.1.** 1) For a finite set $S$, we will write $|S|$ for the cardinality of $S$.

2) For $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\in\mathbb{C}^n$, we write $\boldsymbol{\alpha}[\ell]$ for the $\ell$th component $\alpha_\ell$ of $\boldsymbol{\alpha}$. Then, for $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{C}^n$, we let $\boldsymbol{\alpha}\pm\boldsymbol{\beta}$ be the elements in $\mathbb{C}^n$ such that
$$(\boldsymbol{\alpha}\pm\boldsymbol{\beta})[\ell]=\boldsymbol{\alpha}[\ell]\pm\boldsymbol{\beta}[\ell] \quad \text{for } 1\leqslant\ell\leqslant n.$$

3) We write $\mathbf{e}_j$ for the element in $\mathbb{Z}^n$ whose $j$th entry is one and all the other entries are zero.
$$\mathbf{e}_j[\ell]=\begin{cases}1 & \text{if } \ell=j,\\ 0 & \text{otherwise.}\end{cases}$$

4) For $\mathbf{k}\in\mathbb{Z}^n$, let $\mathbf{x}^{\mathbf{k}}$ be the monomial
$$\mathbf{x}^{\mathbf{k}}=x_1^{\mathbf{k}[1]}x_2^{\mathbf{k}[2]}\cdots x_n^{\mathbf{k}[n]}$$
in $L_n$. In this setting, we denote the negative part of $\mathbf{k}$ by
$$\mathbf{k}^{neg}=\{\ell:\mathbf{k}[\ell]<0\}.$$

We let $E_{ab}\in\mathfrak{gl}(n)$ be the $n\times n$ matrix with one in $(a,b)$ and zero elsewhere.

**Definition 3.2.** For each $\boldsymbol{\alpha}\in\mathbb{C}^n$, $E_{ab}$ acts on the monomials $\mathbf{x}^{\mathbf{k}}$ in the algebra $L_n$ of Laurent polynomials as
$$E_{ab}\cdot\mathbf{x}^{\mathbf{k}} \;=\; (\mathbf{k}[b]+\boldsymbol{\alpha}[b])\,\mathbf{x}^{\mathbf{k}+\mathbf{e}_a-\mathbf{e}_b} \qquad \text{for } 1\leqslant a,b\leqslant n.$$
With this action, the space $L_n$ gives rise to a $\mathcal{U}_n$-module, which we will denote by $L_n^{\boldsymbol{\alpha}}$, and for $f\in L_n$ we write $\langle f\rangle$ for the cyclic submodule of $L_n^{\boldsymbol{\alpha}}$ generated by $f$.

Informally, we may think of the above action as

$$E_{ab} \cdot f \;=\; \mathbf{x}^{-\boldsymbol{\alpha}}\, x_a \frac{\partial}{\partial x_b}\, \mathbf{x}^{\boldsymbol{\alpha}} f \quad \text{for } f \in L_n$$

and then the action in the definition can be considered a generalization of the action (1.2) of $\mathfrak{gl}(n)$ on the polynomial ring which provides all the finite dimensional representations of $\mathfrak{gl}(n)$ labeled by Young diagrams with single rows. See Theorem 5.4 (2).

**Lemma 3.3.** For $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} \in \mathbb{C}^n$, if $\boldsymbol{\alpha} - \boldsymbol{\beta} \in \mathbb{Z}^n$ then, as a $\mathcal{U}_n$-module, $L_n^{\boldsymbol{\alpha}}$ is isomorphic to $L_n^{\boldsymbol{\beta}}$.

*Proof.* It is enough to show that the linear map $\psi$ from $L_n$ to $L_n$ sending $\mathbf{x}^{\mathbf{k}}$ to $\mathbf{x}^{\mathbf{k}+\boldsymbol{\alpha}-\boldsymbol{\beta}}$ for $\mathbf{k} \in \mathbb{Z}^n$ gives a $\mathcal{U}_n$-module map from $L_n^{\boldsymbol{\alpha}}$ to $L_n^{\boldsymbol{\beta}}$. It follows from

$$\begin{aligned}
\psi(E_{ab} \cdot \mathbf{x}^{\mathbf{k}}) &= (\mathbf{k}[b] + \boldsymbol{\alpha}[b])\, \mathbf{x}^{\mathbf{k}+\mathbf{e}_a-\mathbf{e}_b} \times \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \\
&= \{(\mathbf{k} + \boldsymbol{\alpha} - \boldsymbol{\beta})[b] + \boldsymbol{\beta}[b]\}\, \mathbf{x}^{(\mathbf{k}+\boldsymbol{\alpha}-\boldsymbol{\beta})+\mathbf{e}_a-\mathbf{e}_b} \\
&= E_{ab} \cdot \mathbf{x}^{(\mathbf{k}+\boldsymbol{\alpha}-\boldsymbol{\beta})} = E_{ab} \cdot \psi(\mathbf{x}^{\mathbf{k}})
\end{aligned}$$

for all $1 \leqslant a, b \leqslant n$. □

With this lemma, we can focus on the following choice of $\boldsymbol{\alpha}$.

**Notation 3.4.** Once and for all, we fix $\boldsymbol{\alpha} \in \mathbb{C}^n$ the entries of whose real parts satisfy

$$0 \leqslant \mathrm{Re}(\boldsymbol{\alpha}[\ell]) < 1 \qquad \text{for all } 1 \leqslant \ell \leqslant n,$$

and the following subset of $\{1, 2, \dots, n\}$

$$I_{\boldsymbol{\alpha}} = \{\ell : \boldsymbol{\alpha}[\ell] = 0\}.$$

With this choice of $\boldsymbol{\alpha}$, we note that for $\mathbf{k} \in \mathbb{Z}^n$,

$$\mathbf{k}[\ell] + \boldsymbol{\alpha}[\ell] = 0 \quad \text{for some } \ell \tag{3.1}$$

only when $\mathbf{k}[\ell] = \boldsymbol{\alpha}[\ell] = 0$.

**Definition 3.5** (Submodules $L_n^{\boldsymbol{\alpha}}(m, j)$ and $V_n^{\boldsymbol{\alpha}}(m, J)$)**.**

1) For integers $m$ and $j$ with $0 \leqslant j \leqslant |I_{\boldsymbol{\alpha}}|$, we let $L_n^{\boldsymbol{\alpha}}(m, j)$ be the subspace of $L_n^{\boldsymbol{\alpha}}$ spanned by all the monomials $\mathbf{x}^{\mathbf{k}}$ such that

$$\sum_{\ell=1}^{n} \mathbf{k}[\ell] = m \quad \text{and} \quad |\mathbf{k}^{neg} \cap I_{\boldsymbol{\alpha}}| \leqslant j.$$

2) For a subset $J$ of $I_{\boldsymbol{\alpha}}$ with $|J| = j$, we let $V_n^{\boldsymbol{\alpha}}(m, J)$ be the subspace of $L_n^{\boldsymbol{\alpha}}(m, j)$ spanned by all the monomials $\mathbf{x}^{\mathbf{k}}$ in $L_n^{\boldsymbol{\alpha}}(m, j)$ such that

$$(\mathbf{k}^{neg} \cap I_{\boldsymbol{\alpha}}) \subseteq J.$$

**Example 3.6.** Let $n = 4$, $\boldsymbol{\alpha} = (0, 1/2, 0, 0)$, and therefore $I_{\boldsymbol{\alpha}} = \{1, 3, 4\}$.

1) Let $j = 2$. From the condition $|\mathbf{k}^{neg} \cap \{1, 3, 4\}| \leqslant 2$, the space $L_n^{\boldsymbol{\alpha}}(m, j)$ is spanned by all the monomials $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4}$ in $L_n$ such that

$$i)\ k_1 + k_2 + k_3 + k_4 = m \quad \text{and} \quad ii)\ k_1 \geqslant 0 \text{ or } k_3 \geqslant 0 \text{ or } k_4 \geqslant 0.$$

2) Let $J = \{1, 4\}$. From the condition $(\mathbf{k}^{neg} \cap \{1, 3, 4\}) \subset \{1, 4\}$, $V_n^{\boldsymbol{\alpha}}(m, J)$ is the subspace of $L_n^{\boldsymbol{\alpha}}(m, j)$ spanned by all the monomials of degree $m$ with $k_3 \geqslant 0$.

See more examples in Example 4.2. The following is easy to check.

**Lemma 3.7.** *If $J = \varnothing$ (therefore $j = 0$) or $J = I_{\boldsymbol{\alpha}}$ (therefore $j = |I_{\boldsymbol{\alpha}}|$), then*

$$V_n^{\boldsymbol{\alpha}}(m, J) = L_n^{\boldsymbol{\alpha}}(m, j).$$

From Definition 3.5, it immediately follows that

$$L_n^{\boldsymbol{\alpha}}(m, j-1) \subseteq L_n^{\boldsymbol{\alpha}}(m, j) \quad \text{and} \quad L_n^{\boldsymbol{\alpha}}(m, j) = \sum_{J : |J| = j} V_n^{\boldsymbol{\alpha}}(m, J) \tag{3.2}$$

where the summation runs over all subsets $J$ of $I_{\boldsymbol{\alpha}}$ with $|J| = j$. Also, for two subsets $J_1$ and $J_2$ of $I_{\boldsymbol{\alpha}}$, we have

$$\begin{aligned} &V_n^{\boldsymbol{\alpha}}(m, J_1) \subset V_n^{\boldsymbol{\alpha}}(m, J_2) \quad \text{for } J_1 \subset J_2; \\ &V_n^{\boldsymbol{\alpha}}(m, J_1 \cap J_2) = V_n^{\boldsymbol{\alpha}}(m, J_1) \cap V_n^{\boldsymbol{\alpha}}(m, J_2). \end{aligned} \tag{3.3}$$

Now we show that $V_n^{\boldsymbol{\alpha}}(m, J)$ and $L_n^{\boldsymbol{\alpha}}(m, j)$ are indeed modules over $\mathcal{U}_n$.

**Proposition 3.8.** *$V_n^{\boldsymbol{\alpha}}(m, J)$ and $L_n^{\boldsymbol{\alpha}}(m, j)$ are $\mathcal{U}_n$-submodules of $L_n^{\boldsymbol{\alpha}}$.*

*Proof.* For a monomial $\mathbf{x}^{\mathbf{p}} \in V_n^{\boldsymbol{\alpha}}(m, J)$, we need to show

$$E_{ab} \cdot \mathbf{x}^{\mathbf{p}} = (\mathbf{p}[b] + \boldsymbol{\alpha}[b])\, \mathbf{x}^{\mathbf{p} + \mathbf{e}_a - \mathbf{e}_b}$$

are in $V_n^{\boldsymbol{\alpha}}(m, J)$ for all $1 \leqslant a, b \leqslant n$. Since the action of $E_{ab}$ preserves the degree of monomials, writing $\mathbf{q} = (\mathbf{p} + \mathbf{e}_a - \mathbf{e}_b)$, it is enough to show that for $a \neq b$ if $(\mathbf{p}^{neg} \cap I_{\boldsymbol{\alpha}}) \subseteq J$, then $(\mathbf{q}^{neg} \cap I_{\boldsymbol{\alpha}}) \subseteq J$ or the coefficient $(\mathbf{p}[b] + \boldsymbol{\alpha}[b])$ is zero.

For this, because the only element which is not in $\mathbf{p}^{neg}$ but can possibly appear in $\mathbf{q}^{neg}$ is $b$, it is enough to consider the case

$$b \notin \mathbf{p}^{neg} \cap I_{\boldsymbol{\alpha}} \quad \text{and} \quad b \in \mathbf{q}^{neg} \cap I_{\boldsymbol{\alpha}}.$$

This happens only when $\mathbf{p}[b] = 0$ and in this case, since $b \in I_{\boldsymbol{\alpha}}$, we have $\boldsymbol{\alpha}[b] = 0$. Therefore, the coefficient $(\mathbf{p}[b] + \boldsymbol{\alpha}[b])$ is zero. Consequently, we have $E_{ab} \cdot \mathbf{x}^{\mathbf{p}} \in V_n^{\boldsymbol{\alpha}}(m, J)$ for all $1 \leqslant a, b \leqslant n$, and $V_n^{\boldsymbol{\alpha}}(m, J)$ is a submodule of $L_n^{\boldsymbol{\alpha}}$. Now from (3.2), $L_n^{\boldsymbol{\alpha}}(m, j)$ is a submodule of $L_n^{\boldsymbol{\alpha}}$. □

## 4. Structure of $V_n^{\boldsymbol{\alpha}}(m, J)$

In this section, we investigate the structure of $V_n^{\boldsymbol{\alpha}}(m, J)$. We first give a technical lemma.

**Lemma 4.1.** For two distinct monomials $\mathbf{x}^{\mathbf{p}}$ and $\mathbf{x}^{\mathbf{q}}$ in $L_n^{\boldsymbol{\alpha}}(m, j)$ such that

$$(\mathbf{q}^{neg} \cap I_{\boldsymbol{\alpha}}) \subseteq (\mathbf{p}^{neg} \cap I_{\boldsymbol{\alpha}}),$$

there exists $X \in \mathcal{U}_n$ such that $X \cdot \mathbf{x}^{\mathbf{p}} = \mathbf{x}^{\mathbf{q}}$.

*Proof.* For simplicity, we let $\mathbf{p}[\ell] = p_\ell$, $\mathbf{q}[\ell] = q_\ell$, and $\boldsymbol{\alpha}[\ell] = \alpha_\ell$ for all $\ell$. Consider the difference $\mathbf{p} - \mathbf{q} = (p_1 - q_1, \ldots, p_n - q_n) \in \mathbb{Z}^n$. Since $\mathbf{x}^{\mathbf{p}}$ and $\mathbf{x}^{\mathbf{q}}$ have the same degree $m$, we have $\sum_{\ell=1}^{n} (p_\ell - q_\ell) = 0$ and therefore we can separate the positive and negative parts of $\mathbf{p} - \mathbf{q}$

$$r = \sum_{\ell : p_\ell - q_\ell > 0} (p_\ell - q_\ell) = \sum_{\ell : p_\ell - q_\ell < 0} (q_\ell - p_\ell).$$

With Notation 3.1, we define $1 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_r \leqslant n$ and $1 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_r \leqslant n$ so that

$$\sum_{k=1}^{r} \mathbf{e}_{s_k} = \sum_{\ell : p_\ell - q_\ell > 0} (p_\ell - q_\ell)\, \mathbf{e}_\ell \quad \text{and} \quad \sum_{k=1}^{r} \mathbf{e}_{t_k} = \sum_{\ell : p_\ell - q_\ell < 0} (q_\ell - p_\ell)\, \mathbf{e}_\ell$$

where the summations are over $\ell$ such that $p_\ell - q_\ell > 0$ and $p_\ell - q_\ell < 0$ respectively.

Setting $\mathbf{p}_0 = \mathbf{p}$ and $\mathbf{p}_k = \mathbf{p}_{k-1} + \mathbf{e}_{t_k} - \mathbf{e}_{s_k}$ for $1 \leqslant k \leqslant r$, we have $E_{t_k s_k} \cdot \mathbf{x}^{\mathbf{p}_{k-1}} = w_k \mathbf{x}^{\mathbf{p}_k}$. Furthermore, from

$$\begin{aligned}\mathbf{p}_r &= \mathbf{p}_0 + \sum_{k=1}^{r}(\mathbf{e}_{t_k} - \mathbf{e}_{s_k}) \\ &= \mathbf{p}_0 + \sum_{\ell : p_\ell - q_\ell < 0} (q_\ell - p_\ell)\mathbf{e}_\ell - \sum_{\ell : p_\ell - q_\ell > 0} (p_\ell - q_\ell)\mathbf{e}_\ell \\ &= \mathbf{p}_0 + (\mathbf{q} - \mathbf{p}) = \mathbf{q},\end{aligned}$$

by setting $Y = \prod_{k=1}^{r} E_{t_k s_k} \in \mathcal{U}_n$, we obtain

$$\begin{aligned}Y \cdot \mathbf{x}^{\mathbf{p}} &= \mathbf{x}^{\mathbf{p}}(w_1 x_{t_1} x_{s_1}^{-1})(w_2 x_{t_2} x_{s_2}^{-1}) \cdots (w_r x_{t_r} x_{s_r}^{-1}) \\ &= \left(\prod_{k=1}^{r} w_k\right) \mathbf{x}^{\mathbf{p}} \mathbf{x}^{\mathbf{q}-\mathbf{p}} = \left(\prod_{k=1}^{r} w_k\right) \mathbf{x}^{\mathbf{q}}\end{aligned}$$

where the coefficient is

$$\prod_{k=1}^{r} w_k = \prod_{\ell : p_\ell - q_\ell > 0} (p_\ell + \alpha_\ell)(p_\ell - 1 + \alpha_\ell) \cdots (p_\ell - (p_\ell - q_\ell - 1) + \alpha_\ell).$$

For each $\ell$ in the above product, if $\ell \notin I_{\boldsymbol{\alpha}}$ then $\alpha_\ell \neq 0$, and therefore the corresponding factor is not zero by (3.1). Now let $\ell \in I_{\boldsymbol{\alpha}}$ and therefore $\alpha_\ell = 0$. To derive a contradiction, suppose $(p_\ell - c)$ in the $\ell$th factor of the above product is zero for some $0 \leqslant c \leqslant (p_\ell - q_\ell - 1)$. Then, from $p_\ell = c$, we have

$$0 \leqslant p_\ell \leqslant (p_\ell - q_\ell - 1) \tag{4.1}$$

and therefore $q_\ell \leqslant -1$. On the other hand, from the given hypothesis $(\mathbf{q}^{neg} \cap I_{\boldsymbol{\alpha}}) \subseteq (\mathbf{p}^{neg} \cap I_{\boldsymbol{\alpha}})$, we know that $p_\ell < 0$ whenever $q_\ell < 0$, which contradicts to (4.1). Therefore, we have $\prod_{k=1}^{r} w_k \neq 0$ and with the element

$$X = \prod_{k=1}^{r} w_k^{-1} E_{t_k s_k}$$

we see that $X \cdot \mathbf{x}^{\mathbf{p}} = \mathbf{x}^{\mathbf{q}}$. □

**Example 4.2.** 1) Let $\boldsymbol{\alpha} = (1/2, i, 0)$, $I_{\boldsymbol{\alpha}} = \{3\}$, and $J = \varnothing$. If $m = 4$ then from the condition

$$\mathbf{k}^{neg} \cap \{3\} \subseteq \varnothing,$$

$V_n^{\boldsymbol{\alpha}}(m, J)$ is spanned by all the monomials $x_1^{k_1} x_2^{k_2} x_3^{k_3}$ of degree 4 with $k_3 \geqslant 0$.

Note that $\mathbf{x}^{\mathbf{p}} = x_1^4$ and $\mathbf{x}^{\mathbf{q}} = x_1 x_2^{-2} x_3^5$ in $V_n^{\boldsymbol{\alpha}}(m, J)$ satisfy the condition in Lemma 4.1. From

$$\mathbf{p} - \mathbf{q} = (3, 2, -5) = (3, 2, 0) + (0, 0, -5)$$

we define the element $Y = E_{31} \cdot E_{31} \cdot E_{31} \cdot E_{32} \cdot E_{32} \in \mathcal{U}_n$ to obtain

$$Y \cdot \mathbf{x}^{\mathbf{p}} = (2 + 1/2)(3 + 1/2)(4 + 1/2)(-1 + i)(0 + i)\, \mathbf{x}^{\mathbf{q}}.$$

Therefore there is $X \in \mathcal{U}_n$ such that $X \cdot \mathbf{x}^{\mathbf{p}} = \mathbf{x}^{\mathbf{q}}$.

2) Let $\alpha = (0, 0, 0)$, $I_{\boldsymbol{\alpha}} = \{1, 2, 3\}$, and $J = \{1, 3\}$. If $m = -2$ then from the condition

$$\mathbf{k}^{neg} \cap \{1, 2, 3\} \subseteq \{1, 3\},$$

$V_n^{\boldsymbol{\alpha}}(m, J)$ is spanned by all the monomials $x_1^{k_1} x_2^{k_2} x_3^{k_3}$ of degree $-2$ with $k_2 \geqslant 0$.

Note that $\mathbf{x}^{\mathbf{p}} = x_1^{-1} x_3^{-1}$ and $\mathbf{x}^{\mathbf{q}} = x_1 x_2^2 x_3^{-5}$ in $V_n^{\boldsymbol{\alpha}}(m, J)$ satisfy the condition in Lemma 4.1. From

$$\mathbf{p} - \mathbf{q} = (-2, -2, 4) = (0, 0, 4) + (-2, -2, 0)$$

we define $Y = E_{13} \cdot E_{13} \cdot E_{23} \cdot E_{23} \in \mathcal{U}_n$ to obtain

$$Y \cdot \mathbf{x}^{\mathbf{p}} = (-4 + 0)(-3 + 0)(-2 + 0)(-1 + 0)\, \mathbf{x}^{\mathbf{q}}.$$

Therefore, there is $X \in \mathcal{U}_n$ such that $X \cdot \mathbf{x}^{\mathbf{p}} = \mathbf{x}^{\mathbf{q}}$.

Now we investigate the structure of $V_n^{\boldsymbol{\alpha}}(m, J)$ for $\boldsymbol{\alpha} \neq \mathbf{0}$.

**Theorem 4.3** (Structure of $V_n^{\boldsymbol{\alpha}}(m, J)$ with nonzero $\boldsymbol{\alpha}$)**.**
Let $\boldsymbol{\alpha} \neq \mathbf{0}$ and $J = \{\ell_1, \ldots, \ell_j\} \subseteq I_{\boldsymbol{\alpha}}$. Then, $V_n^{\boldsymbol{\alpha}}(m, J)$ is the cyclic submodule of $L_n^{\boldsymbol{\alpha}}$ generated by

$$\mathbf{x}_{J,t} = x_{\ell_1}^{-1} x_{\ell_2}^{-1} \cdots x_{\ell_j}^{-1} x_t^{m+j} \quad \text{for some } t \in \{1, 2, \ldots, n\} \setminus I_{\boldsymbol{\alpha}}.$$

*Proof.* Since $\boldsymbol{\alpha} \neq \mathbf{0}$, there exists $t$ such that $\boldsymbol{\alpha}[t] \neq 0$. Write $\mathbf{x}^{\mathbf{p}}$ for $\mathbf{x}_{J,t}$ and let $\mathbf{x}^{\mathbf{q}}$ be an arbitrary monomial in $V_n^{\boldsymbol{\alpha}}(m, J)$. Since $(\mathbf{q}^{neg} \cap I_{\boldsymbol{\alpha}}) \subseteq (\mathbf{p}^{neg} \cap I_{\boldsymbol{\alpha}}) = J$, we can apply Lemma 4.1 to obtain $X \in \mathcal{U}_n$ such that $X \cdot \mathbf{x}^{\mathbf{p}} = \mathbf{x}^{\mathbf{q}}$. Therefore, $\mathbf{x}^{\mathbf{p}}$ generates the module $V_n^{\boldsymbol{\alpha}}(m, J)$. □

Next we consider the other cases with $\boldsymbol{\alpha} = \mathbf{0}$. Let us fix

$$J = \{\ell_1, \ldots, \ell_j\} \subseteq I_{\boldsymbol{\alpha}} = \{1, 2, \ldots, n\}.$$

**Theorem 4.4** (Structure of $V_n^{\mathbf{0}}(m,J)$ with nonnegative degree $m$)**.**

1) If $m \geqslant 0$ and $0 \leqslant j \leqslant n-1$, then $V_n^{\mathbf{0}}(m,J)$ is the cyclic submodule of $L_n^{\mathbf{0}}(m,j)$ generated by

$$\mathbf{x}_{J,t} = x_{\ell_1}^{-1}x_{\ell_2}^{-1}\cdots x_{\ell_j}^{-1}x_t^{m+j} \quad \text{for some } t \in \{1,2,\dots,n\}\setminus J.$$

2) If $m \geqslant 0$ and $j = n$ (therefore, $J = \{1,2,\dots,n\}$), then

$$\begin{aligned} V_n^{\mathbf{0}}(m,\{1,2,\dots,n\}) &= L_n^{\mathbf{0}}(m,n) = L_n^{\mathbf{0}}(m,n-1) \\ &= \sum_{J'} V_n^{\mathbf{0}}(m,J') \end{aligned}$$

where the summation is over all the subsets $J'$ of $\{1,2,\dots,n\}$ with $|J'| = n-1$.

*Proof.* For Statement (1), we first note that $\mathbf{x}_{J,t} \in V_n^{\mathbf{0}}(m,J)$. Write $\mathbf{x}^{\mathbf{p}}$ for $\mathbf{x}_{J,t}$ and let $\mathbf{x}^{\mathbf{q}}$ be an arbitrary monomial in $V_n^{\mathbf{0}}(m,J)$. Since $\mathbf{q}^{neg} \subseteq \mathbf{p}^{neg} = J$, applying Lemma 4.1, we see that there exists $X \in \mathcal{U}_n$ such that $X \cdot \mathbf{x}^{\mathbf{p}} = \mathbf{x}^{\mathbf{q}}$. Therefore, $\mathbf{x}^{\mathbf{q}}$ belongs to the module generated by $\mathbf{x}_{J,t}$ and we have $V_n^{\mathbf{0}}(m,J) = \langle \mathbf{x}_{J,t}\rangle$.

For Statement (2), if $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$ is a monomial in $L_n^{\mathbf{0}}(m,n)$, then not all $k_j$'s can be negative, because the degree $m$ is nonnegative. Therefore, $L_n^{\mathbf{0}}(m,n) = L_n^{\mathbf{0}}(m,n-1)$. The other equalities follow from Lemma 3.7 and (3.2). □

**Theorem 4.5** (Structure of $V_n^{\mathbf{0}}(-m,J)$ with negative degree $-m$)**.**

Let $1 \leqslant m < n$.

1a) If $j = 0$ (therefore, $J = \varnothing$), then

$$V_n^{\mathbf{0}}(-m,\varnothing) = L_n^{\mathbf{0}}(-m,0) = \{0\}.$$

1b) If $1 \leqslant j \leqslant m$, then $V_n^{\mathbf{0}}(-m,J)$ is the cyclic submodule of $L_n^{\mathbf{0}}(-m,j)$ generated by

$$\mathbf{x}_J = x_{\ell_1}^{-1}x_{\ell_2}^{-1}\cdots x_{\ell_{j-1}}^{-1}x_{\ell_j}^{j-1-m}.$$

1c) If $m+1 \leqslant j \leqslant n-1$, then $V_n^{\mathbf{0}}(-m,J)$ is the cyclic submodule of $L_n^{\mathbf{0}}(-m,j)$ generated by

$$\mathbf{x}_{J,t} = x_{\ell_1}^{-1}x_{\ell_2}^{-1}\cdots x_{\ell_{j-1}}^{-1}x_{\ell_j}^{-1}x_t^{j-m} \quad \text{for some } t \in \{1,2,\dots,n\}\setminus J.$$

1d) If $j = n$ (therefore, $J = \{1, 2, \ldots, n\}$), then

$$V_n^{\mathbf{0}}(-m, \{1, 2, \ldots, n\}) = L_n^{\mathbf{0}}(-m, n) = L_n^{\mathbf{0}}(-m, n-1) \\ = \sum_{J'} V_n^{\mathbf{0}}(-m, J')$$

where the summation is over all $J' \subset \{1, 2, \ldots, n\}$ with $|J'| = n - 1$.

Now let $m \geqslant n$.

2a) If $j = 0$ (therefore, $J = \varnothing$), then

$$V_n^{\mathbf{0}}(-m, \varnothing) = L_n^{\mathbf{0}}(-m, 0) = \{0\}.$$

2b) If $1 \leqslant j \leqslant n$, then $V_n^{\mathbf{0}}(-m, J)$ is the cyclic submodule of $L_n^{\mathbf{0}}(-m, j)$ generated by

$$\mathbf{x}_J = x_{\ell_1}^{-1} x_{\ell_2}^{-1} \cdots x_{\ell_{j-1}}^{-1} x_{\ell_j}^{j-1-m}.$$

*Proof.* Statements 1a) and 2a) follow directly from Definition 3.5. For Statement 1d), note that if $\mathbf{x}^{\mathbf{k}} \in L_n^{\mathbf{0}}(-m, n)$ then $\sum_\ell \mathbf{k}[\ell] = -m > -n$ and therefore there should be at least one $\ell$ with $\mathbf{k}[\ell] \geqslant 0$. This implies that $L_n^{\mathbf{0}}(-m, n) = L_n^{\mathbf{0}}(-m, n-1)$. Now the statements follows from Lemma 3.7 and (3.2). The other statements can be shown similarly to Theorem 4.3 and Theorem 4.4 (1). □

**Remark 4.6.** Let $\mathbf{x}^{\mathbf{p}}$ be the generators $\mathbf{x}_J$ or $\mathbf{x}_{J,t}$ of the cyclic modules $V_n^{\boldsymbol{\alpha}}(m, J)$ given in Theorem 4.3, Theorem 4.4, and Theorem 4.5. We remark that these generators are not unique. This is because, when applying Lemma 4.1, if

$$(\mathbf{q}^{neg} \cap I_{\boldsymbol{\alpha}}) \ = \ (\mathbf{p}^{neg} \cap I_{\boldsymbol{\alpha}}) \,, \tag{4.2}$$

then we can exchange the roles of $\mathbf{x}^{\mathbf{p}}$ and $\mathbf{x}^{\mathbf{q}}$. Therefore, every monomial $\mathbf{x}^{\mathbf{q}} \in V_n^{\boldsymbol{\alpha}}(m, J)$ satisfying (4.2) can also generate the module $V_n^{\boldsymbol{\alpha}}(m, J)$.

## 5. Structure of $L_n^{\boldsymbol{\alpha}}(m, j)$

In this section, we investigate the structure of $L_n^{\boldsymbol{\alpha}}(m, j)$. Let us begin with another technical lemma.

**Lemma 5.1.** For an element $f = \sum_{i=1}^r c_i \mathbf{x}^{\mathbf{k}_i}$ of $L_n^{\boldsymbol{\alpha}}$ with distinct monomials and nonzero coefficients, the cyclic module generated by $f$ includes the cyclic modules generated by the terms of $f$

$$\langle \mathbf{x}^{\mathbf{k}_i} \rangle \subseteq \langle f \rangle \quad \text{for all } i.$$

*Proof.* We want to prove the statement by induction on the number $r$ of the terms of $f$. If $r = 1$ then we have nothing to prove. Suppose $r \geqslant 2$. We first note that, since $\mathbf{x}^{\mathbf{k}_i}$ are distinct, they are weight vectors with different weights under the action of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(n)$ spanned by $E_{aa}$ for $1 \leqslant a \leqslant n$. Let $w_i$ be the weight for the monomial $\mathbf{x}^{\mathbf{k}_i}$. Then, there is an element $H \in \mathfrak{h}$ such that

$$g = w_1(H)f - H \cdot f = \sum_{i:w_i(H)\neq w_1(H)} (w_1(H) - w_i(H))\, c_i \mathbf{x}^{\mathbf{k}_i}$$

is a non-zero element in $\langle f \rangle$. Also, since the number of terms in $g$ is less than $r$, by the induction hypothesis the cyclic modules generated by the terms of $g$ are included in $\langle g \rangle$. This shows that for $i$ with $w_i(H) \neq w_1(H)$ we have $\langle \mathbf{x}^{\mathbf{k}_i} \rangle \subseteq \langle f \rangle$.

Now we note that

$$h = f - \sum_{i:w_i(H)\neq w_1(H)} c_i \mathbf{x}^{\mathbf{k}_i} = \sum_{i:w_i(H)= w_1(H)} c_i \mathbf{x}^{\mathbf{k}_i}.$$

is a nonzero element in $\langle f \rangle$. Since the number of the terms of $h$ is less than $r$, again by the induction hypothesis, the cyclic modules generated by the terms of $h$ are included in $\langle h \rangle$. Therefore, $\langle f \rangle$ contains $\langle \mathbf{x}^{\mathbf{k}_i} \rangle$ for $i$ with $w_i(H) = w_1(H)$ as well. □

As an immediate consequence of the above lemma, we obtain the following result for the special case of Theorem 4.5 with $J = \{\ell\}$ for $1 \leqslant \ell \leqslant n$.

**Proposition 5.2.** For $m \geqslant 1$ and $1 \leqslant \ell \leqslant n$, $V_n^{\mathbf{0}}(-m, \{\ell\})$ is a simple submodule of $L_n^{\mathbf{0}}(-m, 1)$ generated by $x_\ell^{-m}$.

*Proof.* From Theorem 4.5 1b) and 2b),

$$V_n^{\mathbf{0}}(-m, J) = \langle x_\ell^{-m} \rangle \subseteq L_n^{\mathbf{0}}(-m, 1).$$

For a nonzero $f \in V_n^{\mathbf{0}}(-m, J)$, writing $f = \sum_i c_i \mathbf{x}^{\mathbf{k}_i}$, let us consider the submodule of $V_n^{\mathbf{0}}(-m, J)$ generated by $f$. By Lemma 5.1, it contains the cyclic submodules generated by $\mathbf{x}^{\mathbf{k}_i}$.

$$\langle \mathbf{x}^{\mathbf{k}_i} \rangle \subseteq \langle f \rangle \subseteq V_n^{\mathbf{0}}(-m, J).$$

On the other hand, for each $i$, since $(\mathbf{k}_i^{neg} \cap \{1, 2, \dots, n\}) \subseteq \{\ell\}$ and the degree of $\mathbf{x}^{\mathbf{k}_i}$ should be $-m < 0$, we have $\mathbf{k}_i[\ell] < 0$ and $\mathbf{k}_i[\ell'] \geqslant 0$ for $\ell' \neq \ell$. By Remark 4.6, each of these monomials can generate the whole module $V_n^{\mathbf{0}}(-m, J)$. Therefore, we have $\langle f \rangle = V_n^{\mathbf{0}}(-m, J)$ and conclude that $V_n^{\mathbf{0}}(-m, \{\ell\})$ has no nonzero proper submodules. □

Now, we investigate the structure of $L_n^{\boldsymbol{\alpha}}(m,j)$ for $\boldsymbol{\alpha} \neq \mathbf{0}$.

**Theorem 5.3** (Structure of $L_n^{\boldsymbol{\alpha}}(m,j)$ with nonzero $\boldsymbol{\alpha}$)**.**

1) If $\boldsymbol{\alpha} \neq \mathbf{0}$ and $0 \leqslant j \leqslant |I_{\boldsymbol{\alpha}}|$, then $L_n^{\boldsymbol{\alpha}}(m,j)$ is indecomposable.
2) In particular, if $\boldsymbol{\alpha} \neq \mathbf{0}$ and $j = 0$, then $L_n^{\boldsymbol{\alpha}}(m,0) = V_n^{\boldsymbol{\alpha}}(m,\varnothing)$ is a nonzero simple module over $\mathcal{U}_n$.

*Proof.* For Statement (2), from Lemma 3.7, $L_n^{\boldsymbol{\alpha}}(m,0) = V_n^{\boldsymbol{\alpha}}(m,\varnothing)$, and by Theorem 4.3 it is generated by $\mathbf{x}_{\varnothing,t} = x_t^m$ for some $t \in \{1,2,\ldots,n\} \setminus I_{\boldsymbol{\alpha}}$. Let $f$ be a nonzero element of $V_n^{\boldsymbol{\alpha}}(m,\varnothing)$, then we can write $f = \sum_{i=1}^r c_i \mathbf{x}^{\mathbf{k}_i}$ with nonzero coefficients such that $\mathbf{k}_i^{neg} \cap I_{\boldsymbol{\alpha}} = \varnothing$ for all $i$. By Lemma 5.1, the monomials $\mathbf{x}^{\mathbf{k}_i}$ belong to $\langle f \rangle$. On the other hand, by Remark 4.6, each of these monomials generates $V_n^{\boldsymbol{\alpha}}(m,\varnothing)$. Therefore, $\langle f \rangle = V_n^{\boldsymbol{\alpha}}(m,\varnothing)$. This shows that $L_n^{\boldsymbol{\alpha}}(m,0) = V_n^{\boldsymbol{\alpha}}(m,\varnothing)$ is simple.

For Statement (1), we will show that every nonzero submodule $M$ of $L_n^{\boldsymbol{\alpha}}(m,j)$ contains $V_n^{\boldsymbol{\alpha}}(m,\varnothing)$, which is nonzero by Statement (2). For a nonzero $f \in M$, let $c\mathbf{x}^{\mathbf{p}}$ be a nonzero term of $f$. Then, by Lemma 5.1, $\langle f \rangle$ includes $\langle \mathbf{x}^{\mathbf{p}} \rangle$. On the other hand, since every monomial $\mathbf{x}^{\mathbf{q}} = x_t^m$ for $t \notin I_{\boldsymbol{\alpha}}$ satisfies the condition

$$\varnothing = (\mathbf{q}^{neg} \cap I_{\boldsymbol{\alpha}}) \subseteq (\mathbf{p}^{neg} \cap I_{\boldsymbol{\alpha}}),$$

we can apply Lemma 4.1 to obtain $X \in \mathcal{U}_n$ such that $X \cdot \mathbf{x}^{\mathbf{p}} = \mathbf{x}^{\mathbf{q}}$. Therefore, we have $\mathbf{x}^{\mathbf{q}} \in \langle \mathbf{x}^{\mathbf{p}} \rangle \subseteq \langle f \rangle \subseteq M$, and therefore $V_n^{\boldsymbol{\alpha}}(m,\varnothing) \subseteq M$. This shows that $L_n^{\boldsymbol{\alpha}}(m,j)$ cannot be written as a direct sum of its proper nonzero submodules. □

Next, we consider the other cases with $\boldsymbol{\alpha} = \mathbf{0}$.

**Theorem 5.4** (Structure of $L_n^{\mathbf{0}}(m,j)$ with nonnegative degree $m$)**.**

1) If $m \geqslant 0$ and $1 \leqslant j \leqslant n$, then $L_n^{\mathbf{0}}(m,j)$ is indecomposable.
2) In particular, if $m \geqslant 0$ and $j = 0$, then $L_n^{\mathbf{0}}(m,0) = V_n^{\mathbf{0}}(m,\varnothing)$ is the cyclic module generated by

$$\mathbf{x}_{\varnothing,1} = x_1^m.$$

   It is a finite dimensional simple module over $\mathfrak{gl}(n)$ of dimension $(m+1)$.

*Proof.* For Statement (2), observe that for $m \geqslant 0$, $L_n^{\mathbf{0}}(m,0)$ is the space of homogeneous polynomials of degree $m$,

$$L_n^{\mathbf{0}}(m,0) \cong \mathrm{Sym}^m(\mathbb{C}^n),$$

and that $x_1^m$ is the highest weight vector with respect to the standard Borel subalgebra of upper triangular matrices in $\mathfrak{gl}(n)$.

For Statement (1), if $j = n$, then $L_n^{\mathbf{0}}(m, n) = L_n^{\mathbf{0}}(m, n-1)$ by Theorem 4.4 (2) and therefore we can assume $0 \leqslant j \leqslant n-1$. This case can be shown similarly to Theorem 5.3 (1). □

**Theorem 5.5** (Structure of $L_n^{\mathbf{0}}(-m, j)$ with negative degree $-m$)**.**

1) If $m \geqslant 1$ and $j = 0$, then $L_n^{\mathbf{0}}(-m, j) = V_n^{\mathbf{0}}(-m, \varnothing) = \{0\}$.
2) If $m \geqslant 1$ and $j = 1$, then $L_n^{\mathbf{0}}(-m, j)$ decomposes into simple submodules
$$L_n^{\mathbf{0}}(-m, 1) = \bigoplus_{\ell=1}^{n} V_n^{\mathbf{0}}(-m, \{\ell\}).$$
3) If $m \geqslant 1$ and $2 \leqslant j \leqslant n$, then $L_n^{\mathbf{0}}(-m, j)$ is indecomposable.

*Proof.* Statement (1) is straightforward to check. For Statement (2), from (3.2) and Theorem 4.5 1b) and 2b), the module $L_n^{\mathbf{0}}(-m, 1)$ is the sum of the cyclic modules $V_n^{\mathbf{0}}(-m, \{\ell\}) = \langle x_\ell^{-m} \rangle$, which are simple by Proposition 5.2. Also, for $\ell \neq \ell'$, by (3.3),
$$V_n^{\mathbf{0}}(-m, \{\ell\}) \cap V_n^{\mathbf{0}}(-m, \{\ell'\}) = V_n^{\mathbf{0}}(-m, \{\ell\} \cap \{\ell'\})$$
which is $V_n^{\mathbf{0}}(-m, \varnothing) = \{0\}$ by Statement (1). Hence we obtain the direct sum expression.

For Statement (3), we first consider the case $2 \leqslant j \leqslant n-1$. In order to derive a contradiction, suppose $L_n^{\mathbf{0}}(-m, j) = M \bigoplus N$ for some submodules $M$ and $N$. From (3.2), $L_n^{\mathbf{0}}(-m, j)$ is generated by the generators of $V_n^{\mathbf{0}}(-m, J)$ given in Theorem 4.5. We denote these generators $\mathbf{x}_J$ or $\mathbf{x}_{J,t}$ by $g_J$. For each of these monomials $g_J \in L_n^{\mathbf{0}}(-m, j)$, if $g_J = h_1 + h_2$ with $h_1 \in M$ and $h_2 \in N$, then $g_J$ appears in $h_1$ or $h_2$. Therefore, by Lemma 5.1, $g_J$ belongs to $\langle h_1 \rangle \subseteq M$ or $\langle h_2 \rangle \subseteq N$.

If all these $g_J$ are in $M$ (or $N$), then $L_n^{\mathbf{0}}(-m, j) = M$ (or $N$). If some of them are in $M$ and some of them are in $N$, then we claim that there are $J$ and $J'$ such that $g_J$ is an element in $M$, $g_{J'}$ is an element in $N$, and $J \cap J' \neq \varnothing$. Suppose there are not such $J$ and $J'$. Then we can partition $\{1, 2, \ldots, n\}$ into two nontrivial parts $S_M$ and $S_N$ with $n_1$ elements and $n_2$ elements such that we have the disjoint union
$$\{J \subset \{1, 2, \ldots, n\} : |J| = j\} = \{J \subset S_M : |J| = j\} \cup \{J \subset S_N : |J| = j\}$$
where $J \subset S_M$ if and only if $g_J$ belongs to $M$; $J \subset S_N$ if and only if $g_J$ belongs to $N$. Note that it contradicts to $\binom{n}{j} > \binom{n_1}{j} + \binom{n_2}{j}$ for

$2 \leqslant j \leqslant n-1$. Therefore, we conclude that there are $J$ and $J'$ such that $J \cap J' \neq \varnothing$. Now from (3.3) we have

$$V_n^{\mathbf{0}}(-m, J \cap J') \subset V_n^{\mathbf{0}}(-m, J) \subset M$$

and

$$V_n^{\mathbf{0}}(-m, J \cap J') \subset V_n^{\mathbf{0}}(-m, J') \subset N.$$

Therefore, $V_n^{\mathbf{0}}(-m, J \cap J') \neq \{0\}$ and $M \cap N$ contains a non-trivial element. Hence, $L_n^{\mathbf{0}}(-m, j)$ is indecomposable.

Next, let us consider the case $j = n$. First, if $1 \leqslant m < n$ then from Theorem 4.5 1d) we have $L_n^{\mathbf{0}}(-m, n) = L_n^{\mathbf{0}}(-m, n-1)$ and therefore it goes back to the previous case. Second, if $m \geqslant n$ then $L_n^{\mathbf{0}}(-m, n) = V_n^{\mathbf{0}}(-m, \{1, 2, \ldots, n\})$ by Lemma 3.7 and its generator is $\mathbf{x}_{\{1,2,\ldots,n\}} = x_1^{-1} \cdots x_{n-1}^{-1} x_n^{n-1-m}$ by Theorem 4.5 2b). If $L_n^{\mathbf{0}}(-m, n) = M \oplus N$, then $\mathbf{x}_{\{1,2,\ldots,n\}} = h_1 + h_2$ for some $h_1 \in M$ and $h_2 \in N$, and the monomial $\mathbf{x}_{\{1,2,\ldots,n\}}$ appears in $h_1$ or $h_2$. By Lemma 5.1, $\langle \mathbf{x}_{\{1,2,\ldots,n\}} \rangle \subseteq \langle h_1 \rangle \subseteq M$ or $\langle \mathbf{x}_{\{1,2,\ldots,n\}} \rangle \subseteq \langle h_2 \rangle \subseteq N$. This shows that $M$ or $N$ should be equal to $L_n^{\mathbf{0}}(-m, n)$. Therefore, $L_n^{\mathbf{0}}(-m, j)$ is indecomposable. □

## 6. Simple modules $W_n^{\boldsymbol{\alpha}}(m, J)$

In this section, we investigate some submodules of the quotients

$$L_n^{\boldsymbol{\alpha}}(m, j) / L_n^{\boldsymbol{\alpha}}(m, j-1).$$

We will assume $L_n^{\boldsymbol{\alpha}}(m, j) = \{0\}$ for $j \leqslant -1$.

**Definition 6.1.** For $m \in \mathbb{Z}$ and a subset $J$ of $I_{\boldsymbol{\alpha}}$ with cardinality $j$, we define the following submodule of the quotient $L_n^{\boldsymbol{\alpha}}(m, j)/L_n^{\boldsymbol{\alpha}}(m, j-1)$

$$W_n^{\boldsymbol{\alpha}}(m, J) = \left(V_n^{\boldsymbol{\alpha}}(m, J) + L_n^{\boldsymbol{\alpha}}(m, j-1)\right) / L_n^{\boldsymbol{\alpha}}(m, j-1).$$

We note that

$$W_n^{\boldsymbol{\alpha}}(m, J) \cong V_n^{\boldsymbol{\alpha}}(m, J) / \left(V_n^{\boldsymbol{\alpha}}(m, J) \cap L_n^{\boldsymbol{\alpha}}(m, j-1)\right)$$

and

$$V_n^{\boldsymbol{\alpha}}(m, J) \cap L_n^{\boldsymbol{\alpha}}(m, j-1) = \sum_{J'} V_n^{\boldsymbol{\alpha}}(m, J')$$

where the summation runs over all $J' \subset J$ with $|J'| = j-1$.

In § 5, we saw that nontrivial modules

$$L_n^{\boldsymbol{\alpha}}(m,0)/L_n^{\boldsymbol{\alpha}}(m,-1) \cong L_n^{\boldsymbol{\alpha}}(m,0) = V_n^{\boldsymbol{\alpha}}(m,\varnothing)$$

are simple, and that for $m \geqslant 1$ the following quotient decomposes into simple submodules:

$$L_n^{\mathbf{0}}(-m,1)/L_n^{\mathbf{0}}(-m,0) \cong \bigoplus_{\ell=1}^{n} V_n^{\mathbf{0}}(-m,\{\ell\}).$$

Let us generalize these observations.

**Theorem 6.2.** Let $m \in \mathbb{Z}$.

1) For $J \subset I_{\boldsymbol{\alpha}}$, the module $W_n^{\boldsymbol{\alpha}}(m,J)$ is simple.
2) For $1 \leqslant j \leqslant |I_{\boldsymbol{\alpha}}|$, the quotient module $L_n^{\boldsymbol{\alpha}}(m,j)/\, L_n^{\boldsymbol{\alpha}}(m,j-1)$ decomposes as

$$L_n^{\boldsymbol{\alpha}}(m,j)/\, L_n^{\boldsymbol{\alpha}}(m,j-1) = \bigoplus_{J:|J|=j} W_n^{\boldsymbol{\alpha}}(m,J)$$

where the direct sum is taken over all subsets $J$ of $I_{\boldsymbol{\alpha}}$ with cardinality $j$.

*Proof.* For Statement (1), for any nonzero element $\bar{f} \in W_n^{\boldsymbol{\alpha}}(m,J)$, we want to show that $\langle \bar{f} \rangle = W_n^{\boldsymbol{\alpha}}(m,J)$. From the definition of $W_n^{\boldsymbol{\alpha}}(m,J)$, we can assume that

$$\bar{f} = f + L_n^{\boldsymbol{\alpha}}(m,j-1)$$

where $f = \sum_{i=1}^{r} c_i \mathbf{x}^{\mathbf{k}_i} \in V_n^{\boldsymbol{\alpha}}(m,J)$ having distinct monomials $\mathbf{x}^{\mathbf{k}_i}$ in $V_n^{\boldsymbol{\alpha}}(m,J)$ with $\mathbf{k}_i^{neg} \cap I_{\boldsymbol{\alpha}} = J$. From Lemma 5.1, $\langle f \rangle$ includes the cyclic modules $\langle \mathbf{x}^{\mathbf{k}_i} \rangle$. On the other hand, by Theorem 4.3, Theorem 4.4, Theorem 4.5, and Remark 4.6, each $\mathbf{x}^{\mathbf{k}_i}$ generates the module $V_n^{\boldsymbol{\alpha}}(m,J)$. This shows that $\langle \bar{f} \rangle = W_n^{\boldsymbol{\alpha}}(m,J)$.

For Statement (2), with (3.2) we see that

$$L_n^{\boldsymbol{\alpha}}(m,j)/L_n^{\boldsymbol{\alpha}}(m,j-1) = \Bigg(\sum_{J:|J|=j} V_n^{\boldsymbol{\alpha}}(m,J)\Bigg)/L_n^{\boldsymbol{\alpha}}(m,j-1)$$

$$= \Bigg(\sum_{J:|J|=j} V_n^{\boldsymbol{\alpha}}(m,J) + L_n^{\boldsymbol{\alpha}}(m,j-1)\Bigg)/L_n^{\boldsymbol{\alpha}}(m,j-1).$$

Therefore, we have

$$L_n^{\boldsymbol{\alpha}}(m,j)/L_n^{\boldsymbol{\alpha}}(m,j-1) = \sum_{J:|J|=j} W_n^{\boldsymbol{\alpha}}(m,J)$$

where the summation is over $J \subset I_{\boldsymbol{\alpha}}$ with $|J| = j$. Now, suppose we have

$$\bar{f} \in W_n^{\boldsymbol{\alpha}}(m, J_1) \cap W_n^{\boldsymbol{\alpha}}(m, J_2)$$

with distinct subsets $J_1$ and $J_2$ of $I_{\boldsymbol{\alpha}}$. Then, we can assume that $\bar{f} = f + L_n^{\boldsymbol{\alpha}}(m, j-1)$ where $f = \sum_{i=1}^r c_i \mathbf{x}^{\mathbf{k}_i}$ with distinct monomials $\mathbf{x}^{\mathbf{k}_i}$ such that $\mathbf{k}_i^{neg} \cap I_{\boldsymbol{\alpha}} \subseteq J_1 \cap J_2$ for all $i$. Since $|J_1 \cap J_2| < j$, this shows that $f \in L_n^{\boldsymbol{\alpha}}(m, j-1)$ and therefore $\bar{f}$ is zero in the quotient $L_n^{\boldsymbol{\alpha}}(m, j)/L_n^{\boldsymbol{\alpha}}(m, j-1)$. Therefore, we obtain the direct sum expression in the statement. □

Next we investigate the cases when $W_n^{\boldsymbol{\alpha}}(m, J)$ are highest weight modules.

**Theorem 6.3** (Highest weight vector in $W_n^{\boldsymbol{\alpha}}(m, J)$)**.**

1) For an integer $1 \leqslant \ell \leqslant n$, if $\boldsymbol{\alpha} \in \mathbb{C}^n$ is such that $\boldsymbol{\alpha}[\ell] = c$ is nonzero and $\boldsymbol{\alpha}[\ell'] = 0$ for all $\ell' \neq \ell$ and $J = \{1, 2, \ldots, \ell - 1\} \subseteq I_{\boldsymbol{\alpha}}$, then for every $m \in \mathbb{Z}$ the module $W_n^{\boldsymbol{\alpha}}(m, J)$ is a highest weight module having a highest weight vector

$$(x_1^{-1} x_2^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1}) + L_n^{\boldsymbol{\alpha}}(m, \ell - 2)$$

with highest weight

$$(-1, -1, \ldots, -1, m + \ell - 1 + c, 0, \ldots, 0).$$

2) Let $\boldsymbol{\alpha} = \mathbf{0}$ and therefore $I_{\boldsymbol{\alpha}} = \{1, 2, \ldots, n\}$. For $1 \leqslant \ell \leqslant n$, if $J = \{1, 2, \ldots, \ell - 1\}$ then for $m \in \mathbb{Z}$ such that $m + \ell - 1 \geqslant 0$ the module $W_n^{\mathbf{0}}(m, J)$ is a highest weight module having a highest weight vector

$$(x_1^{-1} x_2^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1}) + L_n^{\mathbf{0}}(m, \ell - 2)$$

with highest weight

$$(-1, -1, \ldots, -1, m + \ell - 1, 0, \ldots, 0).$$

In particular, if $\ell = 1$ and $J = \varnothing$ then for $m \geqslant 0$, the module $W_n^{\mathbf{0}}(m, \varnothing)$ is a $(m+1)$-dimensional module with highest weight $(m, 0, \ldots, 0)$.

3) Let $\boldsymbol{\alpha} = \mathbf{0}$ and therefore $I_{\boldsymbol{\alpha}} = \{1, 2, \ldots, n\}$. For $1 \leqslant \ell \leqslant n$, if $J = \{1, 2, \cdots, \ell\}$ then for $m \in \mathbb{Z}$ such that $m + \ell - 1 < 0$ the module $W_n^{\mathbf{0}}(m, J)$ is a highest weight module having a highest weight vector

$$(x_1^{-1} x_2^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1}) + L_n^{\mathbf{0}}(m, \ell - 1)$$

with highest weight

$$(-1, -1, \ldots, -1, m+\ell-1, 0, \ldots, 0).$$

In particular, if $\ell = n$ and $J = \{1, 2, \ldots, n\}$, then for $m \leqslant -n$ the module $W_n^{\mathbf{0}}(m, \{1, 2, \ldots, n\})$ is a finite dimensional module with highest weight

$$(-1, -1, \ldots, -1, m+n-1).$$

*Proof.* We first notice that the given elements $\mathbf{x}^{\mathbf{k}} + L_n^{\boldsymbol{\alpha}}(m, j-1)$ generate $W_n^{\boldsymbol{\alpha}}(m, J)$ where $j = |J|$ (see Theorem 4.3, Theorem 4.4, and Theorem 4.5). It is straightforward to verify their weights under the action of the Cartan subalgebra of $\mathfrak{gl}(n)$ generated by $E_{aa}$ for $1 \leqslant a \leqslant n$. Therefore, now it is enough to show that

$$\begin{aligned} E_{ab} \cdot (\mathbf{x}^{\mathbf{k}} + L_n^{\boldsymbol{\alpha}}(m, j-1)) &= (\mathbf{k}[b] + \boldsymbol{\alpha}[b])(x_a x_b^{-1})\mathbf{x}^{\mathbf{k}} + L_n^{\boldsymbol{\alpha}}(m, j-1) \\ &= L_n^{\boldsymbol{\alpha}}(m, j-1) \end{aligned} \tag{6.1}$$

in $W_n^{\boldsymbol{\alpha}}(m, J)$ for all $1 \leqslant a < b \leqslant n$.

For Statement (1), if $a < b$ and $b \geqslant \ell + 1$, then since $\mathbf{k}[b] = \boldsymbol{\alpha}[b] = 0$ we have

$$E_{ab} \cdot (x_1^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1}) = (0+0)(x_a^1 x_b^{-1})(x_1^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1}) = 0.$$

If $a < b$ and $b \leqslant \ell$, then $a \leqslant \ell - 1$ and

$$E_{ab} \cdot (x_1^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1}) = (\mathbf{k}[b] + \boldsymbol{\alpha}[b])\,(x_a^1 x_b^{-1})(x_1^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1})$$

where $\mathbf{k}[b] = -1$ and $\boldsymbol{\alpha}[b] = 0$ if $b \leqslant \ell - 1$; and $\mathbf{k}[b] = m + \ell - 1$ and $\boldsymbol{\alpha}[b] = c$ if $b = \ell$. Writing $\mathbf{x}^{\mathbf{q}}$ for the monomial in the right hand side, we see that $\mathbf{q}[a] = 0$ because $a \leqslant \ell - 1$ and therefore $|\mathbf{q}^{neg} \cap I_{\boldsymbol{\alpha}}| < \ell - 1$. This shows that $\mathbf{x}^{\mathbf{q}} \in L_n^{\boldsymbol{\alpha}}(m, \ell-2)$ and therefore (6.1) is true.

For Statement (2), the first part can be shown similarly to the previous case. The second part with the conditions $\ell = 1$ and $J = \varnothing$ follows directly from Definition 6.1 with $L_n^{\mathbf{0}}(m, -1) = \{0\}$ and Theorem 5.4 (2).

For Statement (3), if $a < b$ and $b \geqslant \ell + 1$, then since $\mathbf{k}[b] = \boldsymbol{\alpha}[b] = 0$ we have

$$E_{ab} \cdot (x_1^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1}) = (0+0)(x_a^1 x_b^{-1})(x_1^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1} x_b^{-1}) = 0.$$

If $a < b$ and $b \leqslant \ell$, then since $\boldsymbol{\alpha}[b] = 0$ we have

$$E_{ab} \cdot (x_1^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1}) = \mathbf{k}[b](x_a^1 x_b^{-1})(x_1^{-1} \cdots x_{\ell-1}^{-1} x_\ell^{m+\ell-1})$$

where $\mathbf{k}[b] = -1$ if $b \leqslant \ell - 1$ and $\mathbf{k}[b] = m + \ell - 1$ if $b = \ell$. Again, by denoting the monomial in the right hand side by $\mathbf{x}^{\mathbf{q}}$, we see that $\mathbf{q}[a] = 0$ because $a \leqslant \ell - 1$, and therefore $|\mathbf{q}^{neg} \cap I_{\boldsymbol{\alpha}}| < \ell$. This shows that $\mathbf{x}^{\mathbf{q}} \in L_n^{\boldsymbol{\alpha}}(m, \ell - 1)$ and therefore (6.1) is true. □

We note that the highest weights of $W_n^{\boldsymbol{\alpha}}(m, J)$ given in Theorem 6.3 are integral dominant (see, for example, [5, §3]) only when

i) $\boldsymbol{\alpha} = \mathbf{0}$, $J = \varnothing$, and $m \geqslant 0$;

ii) $\boldsymbol{\alpha} = \mathbf{0}$, $J = \{1, 2, \ldots, n\}$, and $m \leqslant -n$.

Indeed, one can easily check that these are the only cases when the modules $W_n^{\boldsymbol{\alpha}}(m, J)$ are finite dimensional.

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