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Representations of ordered doppelsemigroups by binary relations*

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Abstract. We extend the study of doppelsemigroups and introduce the notion of an ordered doppelsemigroup. We construct the ordered doppelsemigroup of binary relations on an arbitrary set and prove that every ordered doppelsemigroup is isomorphic to some ordered doppelsemigroup of binary relations. In particular, we obtain an analogue of Cayley's theorem for semigroups in the class of doppelsemigroups. We also describe the representations of ordered doppelsemigroups by binary transitive relations.

1. Introduction

The notion of a doppelalgebra was introduced by B. Richter in [7] in the context of algebraic K-theory. A *doppelalgebra* is a vector space over a field with two binary linear associative operations $\dashv$ and $\vdash$ satisfying the following identities

$$(x \dashv y) \vdash z = x \dashv (y \vdash z), \tag{D1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z) \tag{D2}$$

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An algebraic system which consists of a nonempty set with two binary associative operations $\dashv$ and $\vdash$ satisfying the identities (D_1) and (D_2) is of independent interest. These algebras were first considered in [6] and later they were called as *doppelsemigroups* in [11]. Thus, doppelalgebras are linear analogs of doppelsemigroups. The first results about doppelsemigroups are the descriptions of the free product of doppelsemigroups and such free objects as the free doppelsemigroup, the free commutative doppelsemigroup and the free n-nilpotent doppelsemigroup, and the characterization of the least commutative congruence (the least n-nilpotent congruence) on a free doppelsemigroup [11]. Doppelalgebras and doppelsemigroups have relationships with such algebraic structures as semigroups, duplexes [6] (sets with two associative operations), interassociative semigroups [1, 2], restrictive bisemigroups [8, 9], n-tuple semigroups and n-tuple algebras of associative type [3], dimonoids and dialgebras [4, 12, 17], trioids and trialgebras [5, 13, 18], and other related systems (see, e.g., [19]). If the operations of a doppelsemigroup coincide, then the doppelsemigroup becomes a semigroup. Commutative dimonoids (i.e., dimonoids with both commutative operations) are examples of doppelsemigroups [11]. On the other hand, doppelsemigroups are examples of n-tuple semigroups [3], for $n = 2$. More information on doppelsemigroups can be found, for instance, in [14, 15].

It is well-known that according to Cayley's theorem for semigroups, every semigroup is isomorphic to a semigroup of transformations of some set. For ordered semigroups, a similar statement was proved by K. A. Zaretskiy [10], where in particular, it was shown that every ordered semigroup can be embedded into the ordered semigroup of all binary relations on a suitable set. Also necessary and sufficient conditions under which an ordered semigroup is isomorphic to some ordered semigroup of reflexive (or transitive) binary relations were found. An analogue of Cayley's theorem for the class of dimonoids was obtained in [12] by means the notion of an α-consistent subsemigroup. The concept of an ordered dimonoid was introduced in [20], where the ordered dimonoid of binary relations was defined and an analogue of Zaretskiy's theorem in the class of ordered dimonoids was obtained.

In the present paper, we introduce the concept of an ordered doppelsemigroup and consider the mentioned above problems for it. We give examples of ordered doppelsemigroups and construct the ordered doppelsemigroup of binary relations on an arbitrary nonempty set using the semigroup of binary relations and a particular variant of it (i.e., a sandwich semigroup). The main result of this paper is a representation theorem which shows that every ordered doppelsemigroup can be embedded into the constructed ordered doppelsemigroup of binary relations on a suitable set. As a consequence, we define the transformation doppelsemigroup on an arbitrary nonempty set and show that Cayley's theorem for semigroups has an analogue in the class of doppelsemigroups. Finally, we study the representations of ordered doppelsemigroups by binary transitive relations.

2. Ordered doppelsemigroups and their examples

Let $(D, \dashv, \vdash)$ be an arbitrary doppelsemigroup and let $\leqslant$ be a partial order relation on D. The algebraic system $(D, \dashv, \vdash, \leqslant)$ is called an *ordered doppelsemigroup* if the order relation $\leqslant$ is stable with respect to both operations $\dashv$ and $\vdash$, that is, $x \leqslant y$ implies

$$z * x \leqslant z * y \quad \text{and} \quad x * z \leqslant y * z$$

for all $x, y, z \in D$ and $* \in \{\dashv, \vdash\}$.

Now, we give several examples of ordered doppelsemigroups.

Obviously, every doppelsemigroup can be considered as an ordered doppelsemigroup with respect to the diagonal relation.

Let $(D, \dashv, \vdash)$ be an arbitrary doppelsemigroup and let $\mathscr{P}(D)$ be the set of all subsets of D. Define on $\mathscr{P}(D)$ two binary operations $\dashv'$ and $\vdash'$ by the following rule:

$$A \dashv' B = \{a \dashv b \mid a \in A, b \in B\} \quad \text{and} \quad A \vdash' B = \{a \vdash b \mid a \in A, b \in B\}.$$

Proposition 1. *Let $(D, \dashv, \vdash)$ be a doppelsemigroup. The algebraic system $(\mathscr{P}(D), \dashv', \vdash', \subseteq)$ is an ordered doppelsemigroup with respect to the set-theoretic inclusion $\subseteq$.*

Proof. It is obvious. □

Let $(S, *)$ be an arbitrary semigroup and $a \in S$. Define on S a new binary operation $*_a$ by

$$x *_a y = x * a * y$$

for all $x, y \in S$. Clearly, $(S, *_a)$ is a semigroup, it is called a *variant* of $(S, *)$, or a *sandwich semigroup* of $(S, *)$ with respect to the element a.

Proposition 2. *Let $(S, *, \leqslant)$ be an ordered monoid and $a, b \in S$. The algebraic system $(S, *_a, *_b, \leqslant)$ is an ordered doppelsemigroup.*

Proof. By [11, Lemma 2.1], $(S, *_a, *_b)$ is a doppelsemigroup. The proof of stability of $\leqslant$ with respect to $*_a$ and $*_b$ is obvious. $\square$

Note that if at least one of the elements a or b from Proposition 2 is the identity of $(S, *)$, we obtain the ordered doppelsemigroups $(S, *, *, \leqslant)$, $(S, *, *_b, \leqslant)$, and $(S, *_a, *, \leqslant)$.

Let X be a nonempty set, $\mathrm{F}[X]$ be the free semigroup on X, and T be the free monoid on a fixed two-element set $\{a, b\}$ with the empty word θ. The length of a word $w \in \mathrm{F}[X] \cup T$ is denoted by l_w.

Define two binary operations $\dashv$ and $\vdash$ on the set

$$\mathrm{FDS}(X) = \{(u, v) \in \mathrm{F}[X] \times T \mid l_u - l_v = 1\}$$

by

$$(u_1, v_1) \dashv (u_2, v_2) = (u_1u_2, v_1av_2),$$
$$(u_1, v_1) \vdash (u_2, v_2) = (u_1u_2, v_1bv_2)$$

for all $(u_1, v_1), (u_2, v_2) \in \mathrm{FDS}(X)$.

We fix linear orders $\preceq_1$ and $\preceq_2$ in X and on $\{\theta, a, b\}$, respectively, where θ is the least element with respect to $\preceq_2$. The lexicographic order relations on the free semigroup $\mathrm{F}[X]$ and the free monoid T that are defined by $\preceq_1$ and $\preceq_2$, we will denote as $\preceq_{\mathrm{F}[X]}$ and $\preceq_T$, respectively.

Now we define a binary relation $\preceq$ on $\mathrm{FDS}(X)$ in such way:

$$(u_1, v_1) \preceq (u_2, v_2) \Longleftrightarrow u_1 \preceq_{\mathrm{F}[X]} u_2 \ \& \ v_1 \preceq_T v_2.$$

Proposition 3. *The algebraic system* $(\mathrm{FDS}(X), \dashv, \vdash, \preceq)$ *is an ordered doppelsemigroup.*

Proof. According to [11, Theorem 3.5], $(\mathrm{FDS}(X), \dashv, \vdash)$ is the free doppelsemigroup on X. The rest of the proof is obvious. $\square$

Let $(N, +)$ be the additive semigroup of all natural numbers and $N_{\tilde{2}} = N \cup \{\tilde{2}\}$, where $\tilde{2} \notin N$. Define binary operations $\dashv$ and $\vdash$ on $N_{\tilde{2}}$ by

$$\tilde{2} \dashv \tilde{2} = \tilde{2} \vdash \tilde{2} = 4,$$
$$m \dashv n = m + n,$$
$$m \dashv \tilde{2} = \tilde{2} \dashv m = m \vdash \tilde{2} = \tilde{2} \vdash m = m + 2,$$
$$m \vdash n = \begin{cases} \tilde{2}, & m = n = 1, \\ m + n, & \text{otherwise} \end{cases}$$

for all $m, n \in N$.

We consider the ordinary arithmetic order relation $\leqslant$ on N and extend this relation to $N_{\tilde{2}}$ as follows:

$$1 \leqslant \tilde{2} \leqslant 3 \leqslant 4 \leqslant ...,$$

in addition, the elements 2 and $\tilde{2}$ are not related by $\leqslant$.

Proposition 4. *The algebraic system* $(N_{\tilde{2}}, \dashv, \vdash, \leqslant)$ *is an ordered doppelsemigroup.*

Proof. From [16, Proposition 3] it follows that $(N_{\tilde{2}}, \dashv, \vdash)$ is a commutative dimonoid. As already mentioned in the introduction, any commutative dimonoid is a doppelsemigroup. Therefore, $(N_{\tilde{2}}, \dashv, \vdash)$ is a doppelsemigroup. The proof of the fact that $\leqslant$ is a stable order relation on $N_{\tilde{2}}$ with respect to $\dashv$ and $\vdash$ is obvious. □

One can generalize Proposition 4 in the following way. Let $(I, \preceq)$ be a linearly ordered set of indexes and let

$$S = \{x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4} \,|\, i \in I\}.$$

We define binary operations $\dashv$ and $\vdash$ on S by

$$x_{i,j} \dashv x_{k,l} = \begin{cases} x_{k,4} & \text{if } k \preceq i,\ k \neq i, \\ x_{i,2} & \text{if } i = k,\ j = l = 1, \\ x_{i,4} & \text{otherwise} \end{cases}$$

and

$$x_{i,j} \vdash x_{k,l} = \begin{cases} x_{k,4} & \text{if } k \preceq i,\ k \neq i, \\ x_{i,3} & \text{if } i = k,\ j = l = 1, \\ x_{i,4} & \text{otherwise.} \end{cases}$$

Further, we define a binary relation $\leqslant$ on S by $x_{i,j} \leqslant x_{k,l}$ if $i \preceq k$, $i \neq k$, or $i = k$ and $(j, l) \in \rho$, where $\rho = \{(1,2), (1,3), (1,4), (2,4), (3,4)\} \cup \{(a, a) \,|\, a \in \{1, 2, 3, 4\}\}$.

Proposition 5. *The algebraic system* $(S, \dashv, \vdash, \leqslant)$ *is an ordered doppelsemigroup.*

Proof. In fact, we can easy verify that, for all $x_{i,j}, x_{u,v}, x_{p,q} \in S$ and $*_1, *_2 \in \{\dashv, \vdash\}$,

$$(x_{i,j} *_1 x_{u,v}) *_2 x_{p,q} = x_{m,4} = x_{i,j} *_1 (x_{u,v} *_2 x_{p,q}),$$

where m is the least element of i, u, and p with respect to $\preceq$.

We have still to show that $\leqslant$ is stable with respect to $\dashv$ and $\vdash$. For this, let $x_{i_1,j_1} \leqslant x_{i_2,j_2}$ and $x_{i_3,j_3} \in S$. By k (and l, respectively), we denote the least element of i_1 and i_3 (of i_2 and i_3, respectively) with respect to $\preceq$. Further let $* \in \{\dashv, \vdash\}$. Then there exist $r, s \in \{2,3,4\}$ such that $x_{i_1,j_1} * x_{i_3,j_3} = x_{k,r}$ and $x_{i_2,j_2} * x_{i_3,j_3} = x_{l,s}$. Since $x_{i_1,j_1} \leqslant x_{i_2,j_2}$, we have $i_1 \preceq i_2$ and thus $k \preceq l$. If $k \neq l$ then $x_{k,r} \leqslant x_{l,s}$. Suppose that $k = l$. If $s = 2$ or $s = 3$ then we can calculate that $i_2 = i_3$ and $j_2 = j_3 = 1$. In particular, $i_2 = i_3 = l = k \preceq i_1 \preceq i_2$ provides $i_1 = i_2$, where $(j_1, j_2) = (j_1, 1) \in \rho$. The latter implies $j_1 = 1$, i.e. $x_{i_1,j_1} = x_{i_2,j_2}$, and thus $x_{k,r} = x_{l,s}$. In the case $s = 4$, we have $x_{k,r} \leqslant x_{l,s}$ since $(2,4)$, $(3,4)$, and $(4,4)$ belong to ρ.

Dually, we obtain $x_{i_3,j_3} * x_{i_1,j_1} \leqslant x_{i_3,j_3} * x_{i_2,j_2}$. □

For any nonempty set X, let $\overline{X} = \{\overline{x} \mid x \in X\}$ be a disjoint copy of X. In particular, we have $x_1 = x_2$ if and only if $\overline{x}_1 = \overline{x}_2$ for all $x_1, x_2 \in X$.

Further, we put $\mathscr{X} = X \cup \overline{X}$, and let

$$\triangle_X = \{(x,x) \mid x \in X\} \quad \text{and} \quad \triangle_{\overline{X}} = \{(\overline{x},\overline{x}) \mid x \in X\}.$$

Define two binary operations $\circ_1$ and $\circ_2$ on the set $B(\mathscr{X})$ of all binary relations on $\mathscr{X}$ in such way:

$$\alpha \circ_1 \beta = \triangle_X \circ \alpha \circ \triangle_X \circ \beta,$$

$$\alpha \circ_2 \beta = \triangle_{\overline{X}} \circ \alpha \circ \triangle_{\overline{X}} \circ \beta,$$

where $\circ$ is the ordinary composition of binary relations.

Proposition 6. *The algebraic system $(B(\mathscr{X}), \circ_1, \circ_2, \subseteq)$ is an ordered doppelsemigroup.*

Proof. Taking into account the equalities $\triangle_X \circ \triangle_X = \triangle_X$ and $\triangle_{\overline{X}} \circ \triangle_{\overline{X}} = \triangle_{\overline{X}}$, we immediately obtain associativity of the operations $\circ_1$ and $\circ_2$. Moreover, for these operations doppelsemigroup axioms (D_1) and (D_2) hold since $\triangle_X \circ \triangle_{\overline{X}} = \triangle_{\overline{X}} \circ \triangle_X = \varnothing$. The stability of $\subseteq$ with respect to the both operations $\circ_1$ and $\circ_2$ follows from the stability of $\subseteq$ with respect to the composition $\circ$. □

3. Representations of ordered doppelsemigroups by binary relations

In this section, we show that any ordered doppelsemigroup can be embedded to a suitable ordered doppelsemigroup consisting of binary relations on some set.

Let $(D, \dashv, \vdash, \leqslant)$ and $(D', \dashv', \vdash', \leqslant')$ be arbitrary ordered doppelsemigroups. A bijective mapping $\varphi : D \to D'$ is called an *isomorphism of the ordered doppelsemigroups* if for all $x, y \in D$ and $* \in \{\dashv, \vdash\}$ the following conditions hold:

$$\varphi(x * y) = \varphi(x) *' \varphi(y), \tag{H_1}$$

$$x \leqslant y \Longleftrightarrow \varphi(x) \leqslant' \varphi(y). \tag{H_2}$$

We use the notations from Proposition 6. For any nonempty set X, we put $\mathscr{X} = X \cup \overline{X}$ and let

$$\rho_X = \{(x, \overline{x}), (\overline{x}, \overline{x}) \mid x \in X\}.$$

It is clear that $(B(\mathscr{X}), \circ_{\rho_X})$ is the sandwich semigroup of the semigroup $(B(\mathscr{X}), \circ)$ with respect to the element ρ_X. Moreover, by the remark after Proposition 2, $(B(\mathscr{X}), \circ, \circ_{\rho_X}, \subseteq)$ is an ordered doppelsemigroup.

We call the doppelsemigroup $(B(\mathscr{X}), \circ, \circ_{\rho_X}, \subseteq)$ as the *ordered doppelsemigroup of all binary relations* on $\mathscr{X}$. Subdoppelsemigroups of $(B(\mathscr{X}), \circ, \circ_{\rho_X}, \subseteq)$, we will call *ordered doppelsemigroups of binary relations* on $\mathscr{X}$.

The main result of this paper is the following theorem.

Theorem 1. *Every ordered doppelsemigroup is isomorphic to an ordered doppelsemigroup of binary relations on some set.*

Proof. Let $(D, \dashv, \vdash, \leqslant)$ be an arbitrary ordered doppelsemigroup and D^1 be the set D with externally adjoined element $1 \notin D$ such that $1 \dashv s = 1 \vdash s = s$ for all $s \in D$.

Further for every $s \in D$, we put

$$f_s = \{(s_1, s_2) \in D^1 \times D \mid s_2 \leqslant s_1 \dashv s\} \cup \{(\overline{s}_1, s_2) \in \overline{D^1} \times D \mid s_2 \leqslant s_1 \vdash s\},$$

and let

$$F_D = \{f_s \mid s \in D\}.$$

It is clear that $F_D \subseteq B(\mathscr{D}^1)$. We will prove that

$$f : s \mapsto f_s$$

is an isomorphism of $(D, \dashv, \vdash, \leqslant)$ into $(F_D, \circ, \circ_{\rho_{D^1}}, \subseteq)$. Clearly, by definition of f, the mapping f is surjective. The mapping f is also injective. In fact, let $s_1, s_2 \in D$ with $f_{s_1} = f_{s_2}$. Because of $s_1 \leqslant s_1 = 1 \dashv s_1$, we obtain $(1, s_1) \in f_{s_1} = f_{s_2}$, i.e. $s_1 \leqslant 1 \dashv s_2 = s_2$. Dually, we get $s_2 \leqslant s_1$, thus $s_1 = s_2$.

Show that f satisfies (H_1). First, we consider the operation $\dashv$. Let $(x, y) \in f_{s_1} \circ f_{s_2}$ for some $s_1, s_2 \in D$. Then there exists $z \in D$ such that $(x, z) \in f_{s_1}$ and $(z, y) \in f_{s_2}$. Clearly, $y \in D$. Then $y \leqslant z \dashv s_2$. On the other hand, we have $x \in D^1 \cup \overline{D^1}$. If $x \in D^1$ then $z \leqslant x \dashv s_1$ and using stability and transitivity of $\leqslant$ we obtain $y \leqslant (x \dashv s_1) \dashv s_2 = x \dashv (s_1 \dashv s_2)$, i.e. $(x, y) \in f_{s_1 \dashv s_2}$. If $x \in \overline{D^1}$ then there exists $u \in D$ such that $\overline{u} = x$ and we get $z \leqslant u \vdash s_1$. Similarly as above, this provides $y \leqslant (u \vdash s_1) \dashv s_2 = u \vdash (s_1 \dashv s_2)$ and therefore $(\overline{u}, y) \in f_{s_1 \dashv s_2}$.

Conversely, let $(x, y) \in f_{s_1 \dashv s_2}$, i.e. $y \leqslant x \dashv (s_1 \dashv s_2) = (x \dashv s_1) \dashv s_2$ if $x \in D^1$ and $y \leqslant u \vdash (s_1 \dashv s_2) = (u \vdash s_1) \dashv s_2$ if $u \in D^1$ with $\overline{u} = x$. Then we can conclude that $(x \dashv s_1, y) \in f_{s_2}$ and $(u \vdash s_1, y) \in f_{s_2}$, respectively, where $(x, x \dashv s_1) \in f_{s_1}$ and $(\overline{u}, u \vdash s_1) \in f_{s_1}$ follow from $x \dashv s_1 \leqslant x \dashv s_1$ and $u \vdash s_1 \leqslant u \vdash s_1$, respectively. Hence, $(x, y) \in f_{s_1} \circ f_{s_2}$ and we have shown that $f_{s_1} \circ f_{s_2} = f_{s_1 \dashv s_2}$.

Now we consider the operation $\vdash$. Let $(x, y) \in f_{s_1} \circ_{\rho_{D^1}} f_{s_2} = f_{s_1} \circ \rho_{D^1} \circ f_{s_2}$ for some $s_1, s_2 \in D$. Then there exists $z \in D$ such that $(x, z) \in f_{s_1}$ and $(\overline{z}, y) \in f_{s_2}$. Clearly, $y \in D$ and $y \leqslant z \vdash s_2$. If $x \in D^1$ then $z \leqslant x \dashv s_1$ and we obtain $y \leqslant (x \dashv s_1) \vdash s_2 = x \dashv (s_1 \vdash s_2)$, i.e. $(x, y) \in f_{s_1 \vdash s_2}$. If $x \in \overline{D^1}$ then there exists $u \in D^1$ with $\overline{u} = x$ and $y \leqslant (u \vdash s_1) \vdash s_2 = u \vdash (s_1 \vdash s_2)$, i.e. $(x, y) \in f_{s_1 \vdash s_2}$.

Conversely, let $(x, y) \in f_{s_1 \vdash s_2}$. If $x \in D^1$ then $y \leqslant x \dashv (s_1 \vdash s_2) = (x \dashv s_1) \vdash s_2$ and $x \dashv s_1 \leqslant x \dashv s_1$. This provides $(\overline{x \dashv s_1}, y) \in f_{s_2}$ and $(x, x \dashv s_1) \in f_{s_1}$, i.e. $(x, y) \in f_{s_1} \circ \rho_{D^1} \circ f_{s_2} = f_{s_1} \circ_{\rho_{D^1}} f_{s_2}$. If $x \in \overline{D^1}$ then there exists $u \in D^1$ such that $\overline{u} = x$ and $y \leqslant u \vdash (s_1 \vdash s_2) = (u \vdash s_1) \vdash s_2$ with $u \vdash s_1 \leqslant u \vdash s_1$. It means that $(\overline{u \vdash s_1}, y) \in f_{s_2}$ and $(\overline{u}, u \vdash s_1) \in f_{s_1}$, and thus $(x, y) \in f_{s_1} \circ \rho_{D^1} \circ f_{s_2} = f_{s_1} \circ_{\rho_{D^1}} f_{s_2}$. Consequently, we have shown that $f_{s_1} \circ_{\rho_{D^1}} f_{s_2} = f_{s_1 \vdash s_2}$.

Finally, we show that f satisfies (H_2). Let $s_1, s_2 \in D$ such that $s_1 \leqslant s_2$. Clearly, $1 \dashv s_1 \leqslant 1 \dashv s_2$ and by stability of $\leqslant$, $x \dashv s_1 \leqslant x \dashv s_2$ and $x \vdash s_1 \leqslant x \vdash s_2$ for all $x \in D$. If $(x, y) \in f_{s_1}$ with $x \in D^1$ then $y \leqslant x \dashv s_1 \leqslant x \dashv s_2$, i.e. $(x, y) \in f_{s_2}$. For $(\overline{x}, y) \in f_{s_1}$ with $x \in D^1$, we have $y \leqslant x \vdash s_1 \leqslant x \vdash s_2$, i.e. $(\overline{x}, y) \in f_{s_2}$. This shows $f_{s_1} \subseteq f_{s_2}$. Conversely, let $s_1, s_2 \in D$ with $f_{s_1} \subseteq f_{s_2}$. Since $s_1 \leqslant s_1 = 1 \dashv s_1$, we have $(1, s_1) \in f_{s_1} \subseteq f_{s_2}$, i.e. $s_1 \leqslant 1 \dashv s_2 = s_2$. □

Remark 1. From Theorem 1, it follows that every ordered semigroup is isomorphic to some ordered semigroup of binary relations, that is, Zaretskiy's theorem [10, Theorem of Sect. 5] is a corollary of Theorem 1.

In particular, the representations from Theorem 1 and the mentioned Zaretskiy's theorem are different.

For any nonempty set X, we denote by $T(\mathscr{X})$ the set of all nonempty functional relations (i.e., transformations) on the set $\mathscr{X} = X \cup \overline{X}$. It is clear that $\rho_X \in T(\mathscr{X})$ and $(T(\mathscr{X}), \circ, \circ_{\rho_X})$ is a subdoppelsemigroup of $(B(\mathscr{X}), \circ, \circ_{\rho_X})$. We call this doppelsemigroup $(T(\mathscr{X}), \circ, \circ_{\rho_X})$ as the *doppelsemigroup of all transformations* of $\mathscr{X}$. Subdoppelsemigroups of $(T(\mathscr{X}), \circ, \circ_{\rho_X})$, we call *doppelsemigroups of transformations* of $\mathscr{X}$.

Finally from Theorem 1, we immediately obtain an analogue of Cayley's theorem for semigroups in the class of doppelsemigroups.

Corollary 1. *Every doppelsemigroup is isomorphic to a doppelsemigroup of transformations of some set.*

Proof. Let $(D, \dashv, \vdash)$ be an arbitrary doppelsemigroup and D^1 be the set D with an adjoined element $1 \notin D$ such that for all $s \in D$,

$$1 \dashv s = s \quad \text{and} \quad 1 \vdash s = s.$$

Take the diagonal relation

$$\Delta_D = \{(s, s) \mid s \in D\}$$

as a partial order on D. In this case, we can consider $(D, \dashv, \vdash)$ as an ordered doppelsemigroup $(D, \dashv, \vdash, \Delta_D)$. In addition,

$$f_s = \{(s_1, s_2) \in D^1 \times D \mid s_2 = s_1 \dashv s\} \cup \{(\overline{s}_1, s_2) \in \overline{D^1} \times D \mid s_2 = s_1 \vdash s\}$$

is a functional relation on $\mathscr{D}^1$ for all $s \in D$. By Theorem 1, the ordered doppelsemigroup $(D, \dashv, \vdash, \Delta_D)$ is isomorphic to the ordered doppelsemigroup $(F_D, \circ, \circ_{\rho_{D^1}}, \subseteq)$. In particular, $(D, \dashv, \vdash)$ is isomorphic to $(F_D, \circ, \circ_{\rho_{D^1}})$, where F_D is a subset of $T(\mathscr{D}^1)$. □

Remark 2. On the one hand, Cayley's theorem for semigroups follows from Corollary 1 immediately. However on the other hand, this theorem is not a consequence of Zaretskiy's theorem [10, Theorem of Sect. 5].

At the end of the paper, we will study conditions under which an arbitrary ordered doppelsemigroup is isomorphic to some ordered doppelsemigroup of transitive relations.

Let X be a nonempty set. A doppelsemigroup of binary relations on $\mathscr{X} = X \cup \overline{X}$ we call as a *doppelsemigroup of binary transitive relations* if it consists entirely of binary transitive relations.

Theorem 2. *An arbitrary ordered doppelsemigroup* $(D, \dashv, \vdash, \leqslant)$ *is isomorphic to some ordered doppelsemigroup of binary transitive relations if and only if* $x \dashv x \leqslant x$ *for all* $x \in D$.

Proof. Let $(D, \dashv, \vdash, \leqslant)$ be an ordered doppelsemigroup isomorphic to a subdoppelsemigroup T of $(B(\mathscr{X}), \circ, \circ_{\rho_X}, \subseteq)$ consisting entirely of binary transitive relations defined on a set $\mathscr{X} = X \cup \overline{X}$. In particular, $(D, \dashv, \leqslant)$ and $(T, \circ, \subseteq)$ are isomorphic as ordered semigroups. By [10, Theorem of Sect. 6], we obtain $x \dashv x \leqslant x$ for all $x \in D$.

Conversely, suppose that $x \dashv x \leqslant x$ for every element x of an ordered doppelsemigroup $(D, \dashv, \vdash, \leqslant)$. By Theorem 1, $(D, \dashv, \vdash, \leqslant)$ and $(F_D, \circ, \circ_{\rho_{D^1}}, \subseteq)$ are isomorphic with respect to the isomorphism $f : s \mapsto f_s$. Moreover for all $x \in D$, we get

$$x \dashv x \leqslant x \Longleftrightarrow f(x \dashv x) \subseteq f(x) \Longleftrightarrow f(x) \circ f(x) \subseteq f(x).$$

Thus, all relations of F_D are transitive. □

Observe that Zaretskiy's theorem on representations of ordered semigroups of binary transitive relations [10, Theorem of Sect. 6] follows from Theorem 2.

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