Algebra and Discrete Mathematics
Volume **26** (2018). Number 1, pp. 130–143

RESEARCH ARTICLE

Planarity of a spanning subgraph of the intersection graph of ideals of a commutative ring I, nonquasilocal case

P. Vadhel and S. Visweswaran

Communicated by D. Simson

Abstract. The rings considered in this article are nonzero commutative with identity which are not fields. Let R be a ring. We denote the collection of all proper ideals of R by $\mathbb{I}(R)$ and the collection $\mathbb{I}(R)\setminus\{(0)\}$ by $\mathbb{I}(R)^*$. Recall that the intersection graph of ideals of R, denoted by $G(R)$, is an undirected graph whose vertex set is $\mathbb{I}(R)^*$ and distinct vertices I, J are adjacent if and only if $I\cap J \neq (0)$. In this article, we consider a subgraph of $G(R)$, denoted by $H(R)$, whose vertex set is $\mathbb{I}(R)^*$ and distinct vertices I, J are adjacent in $H(R)$ if and only if $IJ \neq (0)$. The purpose of this article is to characterize rings R with at least two maximal ideals such that $H(R)$ is planar.

1. Introduction

The rings considered in this article are commutative with identity $1 \neq 0$. Let R be a ring. As in [10], we denote the collection of all proper ideals of R by $\mathbb{I}(R)$ and $\mathbb{I}(R) \setminus \{(0)\}$ by $\mathbb{I}(R)^*$. The rings R considered in this article are such that $\mathbb{I}(R)^* \neq \varnothing$. The idea of associating a ring with a graph and studying the interplay between ring-theoretic properties of the ring and the graph-theoretic properties of the graph associated with it was

2010 MSC: 13A15, 05C25.
Key words and phrases: quasilocal ring, special principal ideal ring, clique number of a graph, planar graph.

initiated by I. Beck in [9] and subsequently, a lot of research activity has been carried out by several researchers in this area (see for example, [2, 3, 4, 7, 11, 15]). The study of the intersection graph of ideals of a ring has begun with the work of Chakrabarthy, Ghosh, Mukherjee and Sen [12]. Let R be a ring with identity which is not necessarily commutative. We denote the collection of all proper left ideals of R by $\mathbb{LI}(R)$ and the collection $\mathbb{LI}(R) \setminus \{(0)\}$ by $\mathbb{LI}(R)^*$. Recall from [12] that the *intersection graph of ideals of* R, denoted by $G(R)$, is an undirected graph whose vertex set is $\mathbb{LI}(R)^*$ and distinct vertices I, J are adjacent if and only if $I \cap J \neq (0)$. For any $n \geqslant 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$. In [12], among other results, the planarity of intersection graph of ideals of $\mathbb{Z}_n$ was discussed. Inspired by their work, in [14], S.H. Jaffari and N. Jaffari Rad characterized commutative ring R with identity such that $G(R)$ is planar. An improvement of the results presented in [14] regarding the planarity of $G(R)$ was given in [16]. The intersection graph of ideals of a ring has also been studied by other researchers (see for example, [1, 5, 17]). Inspired by the work done on $G(R)$, with any ring R such that $|\mathbb{I}(R)^*| \geqslant 1$, in [19], we introduced and investigated an undirected graph, denoted by $H(R)$, whose vertex set is $\mathbb{I}(R)^*$ and distinct vertices I, J are adjacent in $H(R)$ if and only if $IJ \neq (0)$. Note that for any ideals I, J of a ring R, $IJ \subseteq I \cap J$. Hence, if $IJ \neq (0)$, then $I \cap J \neq (0)$. Thus distinct $I, J \in \mathbb{I}(R)^*$ are adjacent in $H(R)$, then I and J are adjacent in $G(R)$. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. As $V(H(R)) = V(G(R)) = \mathbb{I}(R)^*$, it follows from the arguments given above that $H(R)$ is a spanning subgraph of $G(R)$. For any set A, we denote the cardinality of A by $|A|$. For any ring R, we denote the set of all maximal ideals of R by $\mathrm{Max}(R)$. Motivated by the work done on the planarity of $G(R)$ in [14, 16], in this article, we focus our study on characterizing rings R with $|\mathrm{Max}(R)| \geqslant 2$ such that $H(R)$ is planar.

It is useful to recall the following results from graph theory. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Recall from [8, Definition 8.1.1] that G is said to be *planar* if G can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G. A graph $G = (V, E)$ is said to be *complete* if any two distinct vertices of G are adjacent in G and for any $n \in \mathbb{N}$, a complete graph on n vertices is denoted by K_n. G is said to be *bipartite* if the vertex set V of G is partitioned into two nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other end in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete* if each element of V_1 is adjacent to all the vertices of V_2. Let $m, n \in \mathbb{N}$. Let

$G = (V, E)$ be a complete bipartite graph with vertex partition V_1 and V_2. If $|V_1| = m$ and $|V_2| = n$, then G is denoted by $K_{m,n}$ [8, Definition 1.1.12].

Recall from [13, page 9] that two adjacent edges of a graph G are said to be *in series* if their common end vertex is of degree two. Two graphs are said to be *homeomorphic* if one graph can be obtained from the other by the insertion of vertices of degree two or by the merger of edges in series [13, page 100]. We note from [13, page 93] that K_5 is referred to as *Kuratowski's first graph* and $K_{3,3}$ is referred to as *Kuratowski's second graph*. The celebrated theorem of Kuratowski states that a finite graph $G = (V, E)$ is planar if and only if G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [13, Theorem 5.9].

Let $G = (V, E)$ be a graph. A *clique* of G is a complete subgraph of G [8, Definition 1.2.2]. Suppose that there exists $k \in \mathbb{N}$ such that any clique of G contains at most k vertices. Then the *clique number* of G, denoted by $\omega(G)$, is defined as the largest positive integer n such that G contains a clique on n vertices [8, page 185]. If G contains a clique on n vertices for all $n \geqslant 1$, then we set $\omega(G) = \infty$.

It is convenient to name the conditions satisfied by a graph $G = (V, E)$:

- (C_1) G does not contain K_5 as a subgraph (that is, equivalently, if $\omega(G) \leqslant 4$);
- (C_2) G does not contain $K_{3,3}$ as a subgraph;
- (C_1^*) G satisfies (C_1) and moreover, G does not contain any subgraph homeomorphic to K_5;
- (C_2^*) G satisfies (C_2) and moreover, G does not contain any subgraph homeomorphic to $K_{3,3}$.

If a graph G is planar, then it follows from Kuratowski's theorem [13, Theorem 5.9] that G satisfies both (C_1^*) and (C_2^*) and hence, G satisfies both (C_1) and (C_2). It is interesting to note that a graph G can be nonplanar, even if it satisfies both (C_1) and (C_2). For an example of this type, refer [13, Figure 5.9(a), page 101] and the graph given in this example does not satisfy (C_2^*). It is not hard to construct an example of a graph G such that G satisfies (C_1) but G does not satisfy (C_1^*).

As is already mentioned in the beginning, the rings considered in this article are commutative with identity $1 \neq 0$. A ring R is said to be *quasilocal* (*semiquasilocal*) if R has only one maximal ideal (respectively, R has only a finite number of maximal ideals). A Noetherian quasilocal (semiquasilocal) ring R is referred to as a *local* (respectively, a *semilocal*) ring. Whenever a set A is a subset of a set B and $A \neq B$, we denote it symbolically by $A \subset B$.

The aim of this article is to characterize rings R with $|\operatorname{Max}(R)| \geqslant 2$ (that is, to characterize nonquasilocal rings R) such that $H(R)$ is planar. Moreover, our aim is to investigate whether the algebraic structure of R plays a role to arrive at the conclusion that $H(R)$ is planar if $H(R)$ satisfies at least one between (C_1) and (C_2).

This article consists of four sections. In Section 2, we state and prove several preliminary results that are needed for proving the main results of this article. Let R be a ring. We denote the nilradical of R by $\operatorname{nil}(R)$ and the Jacobson radical of R by $J(R)$. In Section 2, we assume that R is a ring such that $\mathbb{I}(R)^* \neq \varnothing$ and we do not put any restriction on the number of maximal ideals of R. It is proved in Corollary 2.3 that if $H(R)$ satisfies either (C_1) or (C_2), then $|\operatorname{Max}(R)| \leqslant 3$. If $H(R)$ satisfies either (C_1) or (C_2), then it is verified in Corollary 2.11 that $J(R)$ is nilpotent.

In Section 3, we consider rings R such that $|\operatorname{Max}(R)| = 3$. It is proved in Theorem 3.2 that $H(R)$ satisfies (C_1), if and only if $H(R)$ satisfies (C_2), if and only if $R \cong F_1 \times F_2 \times F_3$ as rings, where F_i is a field for each $i \in \{1, 2, 3\}$, if and only if $H(R)$ is planar.

In Section 4, we consider rings R such that $|\operatorname{Max}(R)| = 2$. The main result proved in Section 4 is Theorem 4.8. It is shown in Theorem 4.8 that $H(R)$ satisfies (C_1), if and only if $H(R)$ satisfies (C_2), if and only if $H(R)$ is planar and moreover, in Theorem 4.8, we characterize up to isomorphism of rings, rings R such that $H(R)$ is planar.

2. Some preliminary results

The aim of this section is to state and prove some preliminary results that are needed for proving the main results of this article. The rings R considered in this section are such that $\mathbb{I}(R)^* \neq \varnothing$.

Lemma 2.1. *Let R be a ring such that $H(R)$ does not contain any infinite clique. Then R is semiquasilocal.*

Proof. Assume that $H(R)$ does not contain any infinite clique. Suppose that $\operatorname{Max}(R)$ is infinite. Then it is possible to find a subset $\{\mathfrak{m}_i | i \in \mathbb{N}\}$ of $\operatorname{Max}(R)$. It is clear that for any distinct $i, j \in \mathbb{N}$, $\mathfrak{m}_i\mathfrak{m}_j \neq (0)$ and so, $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are adjacent in $H(R)$. Therefore, the subgraph of $H(R)$ induced by $\{\mathfrak{m}_i | i \in \mathbb{N}\}$ is an infinite clique. This is a contradiction. Therefore, R is semiquasilocal. □

Lemma 2.2. *Let R be a ring. Let $n \geqslant 4$. If $\omega(H(R)) \leqslant n+1$, then $|\operatorname{Max}(R)| \leqslant n-1$.*

Proof. Assume that $\omega(H(R)) \leqslant n+1$, where $n \geqslant 4$. Suppose that $|\operatorname{Max}(R)| \geqslant n$. Let $\{\mathfrak{m}_i | i \in \{1, 2, \ldots, n\}\} \subseteq \operatorname{Max}(R)$. As $|\operatorname{Max}(R)| \geqslant 4$, it follows that for any three distinct $\mathfrak{m}, \mathfrak{n}, \mathfrak{p} \in \operatorname{Max}(R)$ and for any nonnegative integers i, j, k, $\mathfrak{m}^i \mathfrak{n}^j \mathfrak{p}^k \neq (0)$. Hence, the subgraph of $H(R)$ induced by $\{\mathfrak{m}_i | i \in \{1, 2, \ldots, n\}\} \cup \{\mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_3\}$ is a clique on $n+2$ vertices. This is a contradiction and so, we obtain that $|\operatorname{Max}(R)| \leqslant n-1$. □

Corollary 2.3. *Let R be a ring. If $H(R)$ satisfies either (C_1) or (C_2), then $|\operatorname{Max}(R)| \leqslant 3$.*

Proof. Observe that if $\omega(G) \geqslant 6$ for a graph G, then G neither satisfies (C_1) nor satisfies (C_2). Thus if $H(R)$ satisfies either (C_1) or (C_2), then $\omega(H(R)) \leqslant 5$. Hence, on applying Lemma 2.2 with $n = 4$, it follows that $|\operatorname{Max}(R)| \leqslant 3$. □

It is well-known that for any ring R, $\operatorname{nil}(R) \subseteq J(R)$.

Lemma 2.4. *Let R be a ring. If $a \in J(R) \setminus \operatorname{nil}(R)$, then $Ra^n \neq Ra^m$ for all distinct $n, m \in \mathbb{N}$.*

Proof. Suppose that $Ra^n = Ra^m$ for some distinct $n, m \in \mathbb{N}$. We can assume without loss of generality that $n < m$. Now, $a^n = ra^m$ for some $r \in R$. This implies that $a^n(1 - ra^{m-n}) = 0$. Since $a \in J(R)$, $1 - ra^{m-n}$ is a unit in R, and so, we obtain that $a^n = 0$. This is in contradiction to the assumption that $a \notin \operatorname{nil}(R)$. Therefore, $Ra^n \neq Ra^m$ for all distinct $n, m \in \mathbb{N}$. □

Lemma 2.5. *Let R be a ring such that $H(R)$ does not contain any infinite clique. Then* $\operatorname{nil}(R) = J(R)$.

Proof. Assume that $H(R)$ does not contain any infinite clique. Suppose that $\operatorname{nil}(R) \neq J(R)$. Then there exists $a \in J(R) \setminus \operatorname{nil}(R)$. Then $a^k \neq 0$ for all $k \in \mathbb{N}$. We know from Lemma 2.4 that $Ra^n \neq Ra^m$ for all distinct $n, m \in \mathbb{N}$. Note that the subgraph of $H(R)$ induced by $\{Ra^n | n \in \mathbb{N}\}$ is an infinite clique. This is a contradiction and so, $\operatorname{nil}(R) = J(R)$. □

Lemma 2.6. *Let R be a ring such that $H(R)$ does not contain any infinite clique. Then for any $\mathfrak{m} \in \operatorname{Max}(R)$, $\operatorname{nil}(R_{\mathfrak{m}}) = \mathfrak{m}R_{\mathfrak{m}}$.*

Proof. Assume that $H(R)$ does not contain any infinite clique. We know from Lemma 2.1 that R is semiquasilocal. Let $\{\mathfrak{m}_i | i \in \{1, \ldots, n\}\}$ denote the set of all maximal ideals of R. Hence, $J(R) = \cap_{i=1}^{n} \mathfrak{m}_i$. We know from Lemma 2.5 that $\text{nil}(R) = J(R)$. Let $i \in \{1, \ldots, n\}$. It follows from [6, Corollary 3.12] that $\text{nil}(R_{\mathfrak{m}_i}) = (\text{nil}(R))_{R\setminus\mathfrak{m}_i} = (\cap_{k=1}^{n} \mathfrak{m}_k)_{R\setminus\mathfrak{m}_i} = \cap_{k=1}^{n} \mathfrak{m}_k R_{\mathfrak{m}_i}$. Since $\mathfrak{m}_k R_{\mathfrak{m}_i} = R_{\mathfrak{m}_i}$ for all $k \in \{1, \ldots, n\} \setminus \{i\}$, it follows that $\text{nil}(R_{\mathfrak{m}_i}) = \mathfrak{m}_i R_{\mathfrak{m}_i}$. □

Lemma 2.7. *Let R be a ring such that $H(R)$ does not contain any infinite clique. Then for any $\mathfrak{m} \in \text{Max}(R)$, $R_\mathfrak{m}$ satisfies descending chain condition (d.c.c.) on principal ideals.*

Proof. Assume that $H(R)$ does not contain any infinite clique. Let $\mathfrak{m} \in \text{Max}(R)$. Suppose that $R_\mathfrak{m}$ does not satisfy d.c.c. on principal ideals. Then for each $i \in \mathbb{N}$, there exists $x_i \in R_\mathfrak{m}$ such that $R_\mathfrak{m} x_1 \supset R_\mathfrak{m} x_2 \supset R_\mathfrak{m} x_3 \supset \cdots$ is a strictly descending sequence of principal ideals of $R_\mathfrak{m}$. It is clear that for each $i \in \mathbb{N}$, $x_i \neq \frac{0}{1}$ and $x_{i+1} = y_i x_i$ for some $y_i \in \mathfrak{m} R_\mathfrak{m}$. Hence, for each $i \in \mathbb{N}$, $x_{i+1} = (\prod_{j=1}^{i} y_j) x_i$. Therefore, $\prod_{j=1}^{i} y_j \neq \frac{0}{1}$ for each $i \in \mathbb{N}$. We know from Lemma 2.6 that each element of $\mathfrak{m} R_\mathfrak{m}$ is nilpotent. As $y_i \in \mathfrak{m} R_\mathfrak{m}$ for each $i \in \mathbb{N}$, it follows that there exist integers $1 \leqslant i_1 < i_2 < i_3 < \cdots$ such that $R_\mathfrak{m} y_{i_j} \neq R_\mathfrak{m} y_{i_k}$ for all distinct $j, k \in \mathbb{N}$. Note that for each $i \in \mathbb{N}$, there exist $r_i \in \mathfrak{m}$ and $s_i \in R \setminus \mathfrak{m}$ such that $y_i = \frac{r_i}{s_i}$. As $R_\mathfrak{m} \frac{r_i}{1} = R_\mathfrak{m} y_i$ for each $i \in \mathbb{N}$, it follows from the above discussion that the subgraph of $H(R)$ induced by $\{R r_{i_j} | j \in \mathbb{N}\}$ is an infinite clique. This is in contradiction to the assumption that $H(R)$ does not contain any infinite clique. Therefore, $R_\mathfrak{m}$ satisfies d.c.c. on principal ideals for each $\mathfrak{m} \in \text{Max}(R)$. □

Lemma 2.8. *Let R be a ring such that $H(R)$ does not contain any infinite clique. Then R satisfies d.c.c. on principal ideals.*

Proof. Assume that $H(R)$ does not contain any infinite clique. We know from Lemma 2.1 that R is semiquasilocal. It is shown in Lemma 2.7 that $R_\mathfrak{m}$ satisfies d.c.c. on principal ideals for each $\mathfrak{m} \in \text{Max}(R)$. Therefore, we obtain that R satisfies d.c.c. on principal ideals. □

Lemma 2.9. *Let I be an ideal of a ring R such that $I \subseteq \text{nil}(R)$ and $I = I^2$. If R satisfies d.c.c. on principal ideals, then $I = (0)$.*

Proof. This is [18, Lemma 2.8]. □

Lemma 2.10. *Let R be a ring such that $H(R)$ does not contain any infinite clique. Then $\text{nil}(R)$ is nilpotent.*

Proof. Assume that $H(R)$ does not contain any infinite clique We know from Lemma 2.8 that R satisfies d.c.c. on principal ideals. Suppose that $\mathrm{nil}(R)$ is not nilpotent. If $(\mathrm{nil}(R))^i \neq (\mathrm{nil}(R))^j$ for all distinct $i, j \in \mathbb{N}$, then it follows that the subgraph of $H(R)$ induced by $\{(\mathrm{nil}(R))^i | i \in \mathbb{N}\}$ is an infinite clique. This is impossible and so, there exist $i, j \in \mathbb{N}$ with $i < j$ such that $(\mathrm{nil}(R))^i = (\mathrm{nil}(R))^j$. Let us denote $(\mathrm{nil}(R))^i$ by I. Note that $I \subseteq \mathrm{nil}(R)$ and $I = I^2$ and so, it follows from Lemma 2.9 that $I = (0)$. This proves that $\mathrm{nil}(R)$ is nilpotent. □

Corollary 2.11. *Let R be a ring. If $H(R)$ satisfies either (C_1) or (C_2), then $J(R)$ is nilpotent.*

Proof. Assume that $H(R)$ satisfies either (C_1) or (C_2). Then $\omega(H(R)) \leqslant 5$. We know from Lemma 2.5 that $J(R) = \mathrm{nil}(R)$ and so, we obtain from Lemma 2.10 that $J(R)$ is nilpotent. □

3. The case when R has exactly three maximal ideals

The aim of this section is to characterize rings R with $|\mathrm{Max}(R)| = 3$ such that $H(R)$ is planar.

Lemma 3.1. *Let R be a ring such that $|\mathrm{Max}(R)| = 3$. If $H(R)$ satisfies either (C_1) or (C_2), then $J(R) = (0)$.*

Proof. Let $\{\mathfrak{m}_i | i \in \{1, 2, 3\}\}$ denote the set of all maximal ideals of R. Then $J(R) = \cap_{i=1}^{3} \mathfrak{m}_i$. As distinct maximal ideals of a ring are comaximal, it follows from [6, Proposition 1.10(i)] that $J(R) = \prod_{i=1}^{3} \mathfrak{m}_i$.

Assume that $H(R)$ satisfies (C_1). Suppose that $J(R) \neq (0)$. We consider two cases.

Case 1: $\mathfrak{m}_1 = \mathfrak{m}_1^2$. Note that the subgraph of $H(R)$ induced by $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_3\}$ is a clique on five vertices. This is impossible.

Case 2: $\mathfrak{m}_1 \neq \mathfrak{m}_1^2$. Observe that the subgraph of $H(R)$ induced by $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1^2\}$ is a clique on five vertices. This is impossible.

Thus, if $H(R)$ satisfies (C_1), then $J(R) = (0)$.

Assume that $H(R)$ satisfies (C_2). Suppose that $J(R) \neq (0)$. Let $A = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$ and let $B = \{\mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_3, \mathfrak{m}_2\mathfrak{m}_3\}$. Note that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, $J(R) = (0)$.

This shows that if $H(R)$ satisfies either (C_1) or (C_2), then $J(R) = (0)$. □

Theorem 3.2. *Let R be a ring such that $|\operatorname{Max}(R)| = 3$. Then the following statements are equivalent:*
(i) *$H(R)$ satisfies (C_1).*
(ii) *$R \cong F_1 \times F_2 \times F_3$ as rings, where F_i is a field for each $i \in \{1, 2, 3\}$.*
(iii) *$H(R)$ is planar.*
(iv) *$H(R)$ satisfies (C_2).*
(v) *$H(R)$ satisfies both (C_1^*) and (C_2^*).*

Proof. Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$ denote the set of all maximal ideals of R.

(i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii). We know from Lemma 3.1 that $\cap_{i=1}^{3} \mathfrak{m}_i = (0)$. Hence, we obtain from [6, Proposition 1.10 (ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2} \times \frac{R}{\mathfrak{m}_3}$ defined by $f(r) = (r + \mathfrak{m}_1, r + \mathfrak{m}_2, r + \mathfrak{m}_3)$ is an isomorphism of rings. Let $i \in \{1, 2, 3\}$ and let us denote the field $\frac{R}{\mathfrak{m}_i}$ by F_i. Observe that $R \cong F_1 \times F_2 \times F_3$ as rings.

(ii) $\Rightarrow$ (iii). Assume that $R \cong F_1 \times F_2 \times F_3$ as rings, where F_i is a field for each $i \in \{1, 2, 3\}$. Let us denote the ring $F_1 \times F_2 \times F_3$ by T. Note that $\operatorname{Max}(T) = \{\mathfrak{M}_1 = (0) \times F_2 \times F_3, \mathfrak{M}_2 = F_1 \times (0) \times F_3, \mathfrak{M}_3 = F_1 \times F_2 \times (0)\}$ and $V(H(T)) = \{v_1 = \mathfrak{M}_1, v_2 = \mathfrak{M}_1 \cap \mathfrak{M}_2, v_3 = \mathfrak{M}_2, v_4 = \mathfrak{M}_2 \cap \mathfrak{M}_3, v_5 = \mathfrak{M}_3, v_6 = \mathfrak{M}_1 \cap \mathfrak{M}_3\}$. Observe that $H(T)$ is the union of the cycles $\Gamma_1 : v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_1$ and $\Gamma_2 : v_1 - v_3 - v_5 - v_1$. The cycle Γ_1 can be represented by means of a hexagon. The edges of Γ_2 are three chords of this hexagon, two of them pass through v_1 and the third joins v_3 with v_5. It is clear that Γ_2 can be represented by means of a triangle and it can be drawn inside the hexagon representing Γ_1 in such a way that there are crossing over of the edges. This proves that $H(T)$ is planar. As $R \cong T$ as rings, it follows that $H(R)$ is planar.

(iii) $\Rightarrow$ (v). This follows from Kuratowski's theorem [13, Theorem 5.9].

The statements (v) $\Rightarrow$ (i) and (v) $\Rightarrow$ (iv) are clear. □

4. The case when *R* has exactly two maximal ideals

Our aim in this section is to characterize rings R with $|\operatorname{Max}(R)| = 2$ such that $H(R)$ is planar.

Lemma 4.1. *Let $(R_1, \mathfrak{m}_1)$, $(R_2, \mathfrak{m}_2)$ be quasilocal rings such that $\mathfrak{m}_i$ is nilpotent for each $i \in \{1, 2\}$. Let $R = R_1 \times R_2$. If $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}_i^4 = (0)$ for each $i \in \{1, 2\}$.*

Proof. Let $i \in \{1, 2\}$ and let $n_i \geqslant 1$ be least with the property that $\mathfrak{m}_i^{n_i} = (0)$. Assume that $H(R)$ satisfies either (C_1) or (C_2). First, we show that $\mathfrak{m}_1^4 = (0)$. Suppose that $\mathfrak{m}_1^4 \neq (0)$. Then $n_1 \geqslant 5$ and $\mathfrak{m}_1^i \neq \mathfrak{m}_1^j$ for all distinct $i, j \in \{1, 2, \ldots, n_1\}$. Let $A = \{\mathfrak{m}_1 \times R_2, \mathfrak{m}_1^2 \times R_2, \mathfrak{m}_1^3 \times R_2\}$ and let $B = \{\mathfrak{m}_1^4 \times R_2, (0) \times R_2, R_1 \times (0)\}$. It is clear that $A \cup B \subseteq V(H(R))$ and $A \cap B = \varnothing$. Note that the subgraph of $H(R)$ induced by $A \cup \{\mathfrak{m}_1^4 \times R_2, (0) \times R_2\}$ is a clique on five vertices and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. Thus if $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}_1^4 = (0)$. Similarly, it can be shown that $\mathfrak{m}_2^4 = (0)$. Thus, if $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}_i^4 = (0)$ for each $i \in \{1, 2\}$. □

Lemma 4.2. *Let $R = R_1 \times R_2$, where $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ are as in the statement of Lemma 4.1. If $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}_i$ is principal for each $i \in \{1, 2\}$.*

Proof. Assume that $H(R)$ satisfies either (C_1) or (C_2). Then we know from Lemma 4.1 that $\mathfrak{m}_i^4 = (0)$ for each $i \in \{1, 2\}$. We first prove that $\mathfrak{m}_1$ is principal. Suppose that $\mathfrak{m}_1$ is not principal. Since $\mathfrak{m}_1$ is nilpotent, it follows that $\dim_{\frac{R_1}{\mathfrak{m}_1}}(\frac{\mathfrak{m}_1}{\mathfrak{m}_1^2}) \geqslant 2$. Let $x, y \in \mathfrak{m}_1$ be such that $x + \mathfrak{m}_1^2, y + \mathfrak{m}_1^2$ are linearly independent over $\frac{R_1}{\mathfrak{m}_1}$. Let $A = \{R_1 x \times R_2, R_1 y \times R_2, R_1(x + y) \times R_2\}$ and let $B = \{\mathfrak{m}_1 \times R_2, (0) \times R_2, R_1 \times (0)\}$. Note that $A \cup B \subseteq V(H(R))$, $A \cap B = \varnothing$, the subgraph of $H(R)$ induced by $A \cup \{\mathfrak{m}_1 \times R_2, (0) \times R_2\}$ is a clique on five vertices, and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. Thus, if $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}_1$ is principal. Similarly, it can be shown that $\mathfrak{m}_2$ is principal. This proves that if $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}_i$ is principal for each $i \in \{1, 2\}$. □

Lemma 4.3. *Let $R = R_1 \times R_2$, where $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ are as in the statement of Lemma 4.1. Suppose that $\mathfrak{m}_i \neq (0)$ for each $i \in \{1, 2\}$. If $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1, 2\}$.*

Proof. Assume that $H(R)$ satisfies either (C_1) or (C_2). Let $i \in \{1, 2\}$. As $\mathfrak{m}_i \neq (0)$ and $\mathfrak{m}_i$ is nilpotent, it follows that $\mathfrak{m}_i \neq \mathfrak{m}_i^2$. We first verify that $\mathfrak{m}_1^2 = (0)$. Suppose that $\mathfrak{m}_1^2 \neq (0)$. Let $A = \{\mathfrak{m}_1 \times R_2, \mathfrak{m}_1^2 \times R_2, \mathfrak{m}_1 \times \mathfrak{m}_2\}$ and let $B = \{R_1 \times \mathfrak{m}_2, (0) \times R_2, R_1 \times (0)\}$. Note that $A \cup B \subseteq V(H(R))$, $A \cap B = \varnothing$, the subgraph of $H(R)$ induced by $A \cup \{R_1 \times \mathfrak{m}_2, (0) \times R_2\}$ is a clique on five vertices, and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. Thus, if $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}_1^2 = (0)$. Similarly, it can be shown that $\mathfrak{m}_2^2 = (0)$. This shows that if $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1, 2\}$. □

Remark 4.4. Recall that a principal ideal ring R is said to be a *special principal ideal ring* (SPIR) if R has a unique prime ideal. If $\mathfrak{m}$ is the unique prime ideal of a SPIR R, then $\mathfrak{m}$ is necessarily nilpotent. If R is a SPIR with $\mathfrak{m}$ as its unique prime ideal, then we denote it by mentioning that $(R, \mathfrak{m})$ is a SPIR. Let $(R, \mathfrak{m})$ be a quasilocal ring such that $\mathfrak{m}$ is principal and nilpotent. Let $n \geqslant 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then it follows from the proof of $(iii) \Rightarrow (i)$ of [6, Proposition 8.8] that $\{\mathfrak{m}^i | i \in \{1, \ldots, n-1\}\}$ is the set of all nonzero proper ideals of R and so, $(R, \mathfrak{m})$ is a SPIR. □

Lemma 4.5. *Let $T = F_1 \times F_2$, where F_1 and F_2 are fields. Then $H(T)$ is planar.*

Proof. It is clear that $V(H(T)) = \{(0) \times F_2, F_1 \times (0)\}$ and has no edges. Therefore, $H(T)$ is planar. □

Lemma 4.6. *Let $T = T_1 \times F_2$, where $(T_1, \mathfrak{n}_1)$ is a SPIR with $\mathfrak{n}_1 \neq (0)$ and F_2 is a field. If $\mathfrak{n}_1^4 = (0)$, then $H(T)$ is planar.*

Proof. We consider the following cases.
Case 1: $\mathfrak{n}_1^2 = (0)$. It is clear from Remark 4.4 that $\mathbb{I}(T_1)^* = \{\mathfrak{n}_1\}$. Hence, $V(H(T)) = \{v_1 = (0) \times F_2, v_2 = \mathfrak{n}_1 \times F_2, v_3 = T_1 \times (0), v_4 = \mathfrak{n}_1 \times (0)\}$. Observe that $H(T)$ is the path $v_1 - v_2 - v_3 - v_4$. Therefore, $H(T)$ is planar.
Case 2: $\mathfrak{n}_1^2 \neq (0)$ but $\mathfrak{n}_1^3 = (0)$. Note that it follows from Remark 4.4 that $\mathbb{I}(T_1^*) = \{\mathfrak{n}_1, \mathfrak{n}_1^2\}$. In this case, $V(H(T)) = \{v_1 = (0) \times F_2, v_2 = \mathfrak{n}_1 \times F_2, v_3 = \mathfrak{n}_1 \times (0), v_4 = T_1 \times (0), v_5 = \mathfrak{n}_1^2 \times F_2, v_6 = \mathfrak{n}_1^2 \times (0)\}$. Observe that $H(T)$ is the union of the cycle Γ of length five given by $\Gamma : v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ and the three edges $e_1 : v_2 - v_4, e_2 : v_2 - v_5$, and $e_3 : v_4 - v_6$. Note that Γ can be represented by means of a pentagon. The edges e_1, e_2 are two chords of the pentagon representing Γ through v_2 and they can be drawn inside the pentagon and the edge e_3 which joins the vertex v_4 of the pentagon with the pendant vertex v_6 can be drawn outside this pentagon so that there are no crossing over of the edges. This shows that $H(T)$ is planar.
Case 3: $\mathfrak{n}_1^3 \neq (0)$ but $\mathfrak{n}_1^4 = (0)$. It follows from Remark 4.4 that $\mathbb{I}(T_1)^* = \{\mathfrak{n}_1, \mathfrak{n}_1^2, \mathfrak{n}_1^3\}$. In this case, $V(H(T)) = \{v_1 = (0) \times F_2, v_2 = \mathfrak{n}_1 \times F_2, v_3 = \mathfrak{n}_1^2 \times (0), v_4 = T_1 \times (0), v_5 = \mathfrak{n}_1^3 \times F_2, v_6 = \mathfrak{n}_1 \times (0), v_7 = \mathfrak{n}_1^2 \times F_2, v_8 = \mathfrak{n}_1^3 \times (0)\}$. It is easy to verify that $H(T)$ is the union of cycles $\Gamma_1 : v_1 - v_2 - v_3 - v_4 - v_5 - v_1$, $\Gamma_2 : v_5 - v_4 - v_3 - v_6 - v_7 - v_5$, and the edges $e_1 : v_2 - v_4, e_2 : v_2 - v_5, e_3 : v_4 - v_6, e_4 : v_4 - v_7, e_5 : v_7 - v_1, e_6 : v_7 - v_2, e_7 : v_2 - v_6$, and $e_8 : v_4 - v_8$. Note that the cycles Γ_1 and Γ_2 have exactly two edges in common and

they can be represented by means of two pentagons and they can be drawn side by side without any crossing over of the edges. The edges e_1, e_2 are chords of the pentagon representing Γ_1 and they pass through the vertex v_2. The edges e_3, e_4 are chords of the pentagon representing Γ_2 and they pass through the vertex v_4. The edges e_1, e_2 can be drawn inside the pentagon representing Γ_1 and the edges e_3, e_4 can be drawn inside the pentagon representing Γ_2 without any crossing over of the edges. The edges e_5, e_6, and e_7 can be drawn outside the pentagons representing Γ_1 and Γ_2 and finally the vertex v_8 can be plotted inside the pentagon representing Γ_1 and the edge e_8 which joins v_4 with the pendant vertex v_8 can be drawn inside the pentagon representing Γ_1 in such a way that there are no crossing over of the edges. This proves that $H(T)$ is planar. □

Lemma 4.7. *Let $T = T_1 \times T_2$, where $(T_i, \mathfrak{n}_i)$ is a SPIR for each $i \in \{1, 2\}$. If $\mathfrak{n}_i \neq (0)$ but $\mathfrak{n}_i^2 = (0)$ for each $i \in \{1, 2\}$, then $H(T)$ is planar.*

Proof. It follows from Remark 4.4 that $\mathbb{I}(T_i)^* = \{\mathfrak{n}_i\}$ for each $i \in \{1, 2\}$. Hence, $V(H(T)) = \{v_1 = (0) \times T_2, v_2 = \mathfrak{n}_1 \times \mathfrak{n}_2, v_3 = T_1 \times \mathfrak{n}_2, v_4 = \mathfrak{n}_1 \times (0), v_5 = T_1 \times (0), v_6 = \mathfrak{n}_1 \times T_2, v_7 = (0) \times \mathfrak{n}_2\}$. Note that $H(T)$ is the union of the cycle $\Gamma : v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_1$ and the edges $e_1 : v_3 - v_1, e_2 : v_3 - v_5, e_3 : v_3 - v_6, e_4 : v_2 - v_5, e_5 : v_2 - v_6, e_6 : v_7 - v_1$, and $e_7 : v_7 - v_6$. Observe that the cycle Γ can be represented by means of a hexagon. The edges e_1, e_2, and e_3 are the chords of the hexagon representing Γ through the vertex v_3 and they can be drawn inside the hexagon representing Γ without any crossing over of the edges. The edges e_4 and e_5 can be drawn outside the hexagon representing Γ in such a way that there are no crossing over of the edges. The vertex v_7 can be plotted inside the hexagon representing Γ and the edges e_6 and e_7 can be drawn inside the hexagon representing Γ in such a way that there are no crossing over of the edges. This shows that $H(T)$ is planar. □

Theorem 4.8. *Let R be a ring such that $|\mathrm{Max}(R)| = 2$. The following statements are equivalent:*

(i) *$H(R)$ satisfies (C_1).*

(ii) *R is isomorphic to one of the rings of the following type:*

  (a) *$F_1 \times F_2$, where F_i is a field for each $i \in \{1, 2\}$.*

  (b) *$T_1 \times F_2$, where $(T_1, \mathfrak{n}_1)$ is a SPIR with $\mathfrak{n}_1 \neq (0)$ but $\mathfrak{n}_1^4 = (0)$ and F_2 is a field.*

  (c) *$T_1 \times T_2$, where $(T_i, \mathfrak{n}_i)$ is a SPIR with $\mathfrak{n}_i \neq (0)$ but $\mathfrak{n}_i^2 = (0)$ for each $i \in \{1, 2\}$.*

(iii) *$H(R)$ is planar.*

(iv) $H(R)$ *satisfies* (C_2).
(v) $H(R)$ *satisfies both* (C_1^*) *and* (C_2^*).

Proof. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R.

(i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii). If $H(R)$ satisfies either (C_1) or (C_2), then we know from Corollary 2.11 that $J(R)$ is nilpotent. Let $n \geqslant 1$ be such that $(J(R))^n = (0)$. Hence, $\mathfrak{m}_1^n\mathfrak{m}_2^n = (0)$. As $\mathfrak{m}_1^n$ and $\mathfrak{m}_2^n$ are comaximal, it follows from [6, Proposition 1.10 (i)] that $\mathfrak{m}_1^n \cap \mathfrak{m}_2^n = \mathfrak{m}_1^n\mathfrak{m}_2^n = (0)$. Hence, we obtain from [6, Proposition 1.10 (ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{m}_1^n} \times \frac{R}{\mathfrak{m}_2^n}$ defined by $f(r) = (r + \mathfrak{m}_1^n, r + \mathfrak{m}_2^n)$ is an isomorphism of rings. Let us denote the ring $\frac{R}{\mathfrak{m}_i^n}$ by T_i and $\frac{\mathfrak{m}_i}{\mathfrak{m}_i^n}$ by $\mathfrak{n}_i$ for each $i \in \{1, 2\}$. Note that $(T_i, \mathfrak{n}_i)$ is a quasilocal ring and $\mathfrak{n}_i$ is nilpotent for each $i \in \{1, 2\}$. Let us denote the ring $T_1 \times T_2$ by T. Since $R \cong T$ as rings, it follows that $H(T)$ satisfies either (C_1) or (C_2). Hence, we obtain from Lemma 4.1 that $\mathfrak{n}_i^4 = (0 + \mathfrak{m}_i^n)$ for each $i \in \{1, 2\}$. Moreover, we know from Lemma 4.2 and Remark 4.4 that $(T_i, \mathfrak{n}_i)$ is a SPIR for each $i \in \{1, 2\}$.

If $\mathfrak{n}_i = (0 + \mathfrak{m}_i^n)$ for each $i \in \{1, 2\}$, then we get that T_i is a field for each $i \in \{1, 2\}$ and we obtain that R is isomorphic to a ring of the type mentioned in (ii) (a).

If $\mathfrak{n}_1 \neq (0 + \mathfrak{m}_1^n)$ but $\mathfrak{n}_2 = (0 + \mathfrak{m}_2^n)$. Then $(T_1, \mathfrak{n}_1)$ is a SPIR with $\mathfrak{n}_1 \neq (0 + \mathfrak{m}_1^n)$ but $\mathfrak{n}_1^4 = (0 + \mathfrak{m}_1^n)$ and T_2 is a field. In this case, R is isomorphic to a ring of the type mentioned in (ii) (b).

If $\mathfrak{n}_i \neq (0 + \mathfrak{m}_i^n)$ for each $i \in \{1, 2\}$, then we know from Lemma 4.3 that $\mathfrak{n}_i^2 = (0 + \mathfrak{m}_i^n)$ for each $i \in \{1, 2\}$. Thus in this case, we obtain that $(T_i, \mathfrak{n}_i)$ is a SPIR with $\mathfrak{n}_i \neq (0 + \mathfrak{m}_i^n)$ but $\mathfrak{n}_i^2 = (0 + \mathfrak{m}_i^n)$ for each $i \in \{1, 2\}$ and R is isomorphic to a ring of the type mentioned in $(ii)(c)$.

(ii) $\Rightarrow$ (iii). Let T be a ring. If T is a ring of the form mentioned in (ii) (a), then we know from Lemma 4.5 that $H(T)$ is planar. If T is a ring of the form mentioned in (ii) (b), then we obtain from Lemma 4.6 that $H(T)$ is planar. If T is a ring of the form mentioned in (ii) (c), then from Lemma 4.7, we get that $H(T)$ is planar. As R is isomorphic to one of the rings of the type mentioned in (ii) (a), (ii) (b) or (ii) (c), it follows that $H(R)$ is planar.

(iii) $\Rightarrow$ (v). This follows from Kuratowski's theorem [13, Theorem 5.9].

The statements (v) $\Rightarrow$ (i) and (v) $\Rightarrow$ (iv) are clear. □

Acknowledgement

We are very much thankful to the referee and all the members of the Editorial Board of Algebra and Discrete Mathematics for their suggestions and support. As per the suggestion of the members of the Editorial Board, we shall present the results that we obtained on the planarity of $H(R)$, where R is quasilocal in the article — Planarity of a spanning subgraph of the intersection graph of ideals of a commutative ring II, Quasilocal case.

References

[1] S. Akbari, R. Nikandish and M.J. Nikmehr, Some results on the intersection graphs of ideals of rings, *J. Algebra Appl.* **12**(4) (2013), Article ID: 1250200, 13 pages.

[2] D.F. Anderson, M.C. Axtell and J.A. Stickles, Jr., Zero-divisor graphs in commutative rings, in *Commutative Algebra, Noetherian and Non-Noetherian Perspectives*, eds. M. Fontana, S.E. Kabbaj, B. Olberding and I. Swanson, Springer-Verlag, New York (2011), 23 - 46.

[3] D.F. Anderson and P.S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), 434-447.

[4] D.D. Anderson and M. Naseer, Beck's coloring of a commutative ring, *J. Algebra* **159** (1993), 500-514.

[5] T. Asir and T. Tamilzh Chelvam, The intersection graph of gamma sets in the total graph of a commutative ring II, *J. Algebra Appl.***12** (4) (2013), Article ID: 1250199, 14 pages.

[6] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Massachusetts (1969).

[7] M. Axtell, J. Coykendall and J. Stickles, Zero-divisor graphs of polynomial and power series over commutative rings, *Comm. Algebra* **33** (6) (2005), 2043-2050.

[8] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Universitext, Springer (2000).

[9] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), 208-226.

[10] M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings I, *J. Algebra Appl.* **10**(4) (2011), 727-739.

[11] R. Belshoff and J. Chapman, Planar zero-divisor graphs, *J. Algebra* **316** (2007), 471-480.

[12] I. Chakrabarthy, S. Ghosh, T.K. Mukherjee and M.K. Sen, Intersection graphs of ideals of rings, *Electronic Notes in Disc. Mathematics* **23** (2005), 23-32.

[13] N. Deo, *Graph theory with applications to Engineering and Computer science*, Prentice-Hall of India private limited, New Delhi, (1994).

[14] S.H. Jaffari and N. Jaffari Rad, Planarity of intersection graph of ideals of rings, *Int. Electronic J. Algebra* **8** (2010), 161-166.

[15] T.G. Lucas, The diameter of a zero-divisor graph, *J. Algebra* **301** (2006), 173-193.

[16] Z.S. Pucanovic' and Z.Z. Petrovic', Toroidality of intersection graphs of ideals of commutative rings, *Graphs and Combinatorics* **30** (2014), 707-716.

[17] T. Tamilzh Chelvam and T. Asir, Intersection graph of gamma sets in the total graph of a commutative ring I, *J. Algebra Appl.* **12** (4) (2013), Article ID: 1250198, 18 pages.

[18] S. Visweswaran and Hiren D. Patel, On the clique number of the complement of the annihilating ideal graph of a commutative ring, *Beitr. Algebra Geom.* **57** (2016), 307-320.

[19] S. Visweswaran and Pravin Vadhel, Some results on a spanning subgraph of the intersection graph of ideals of a commutative ring, *Mathematics Today* **31** (2015), 1-20.

Contact information

P. Vadhel, S. Visweswaran Department of Mathematics, Saurashtra University, Rajkot, 360 005 India
E-Mail(s): pravin_2727@yahoo.com, s_visweswaran2006@yahoo.co.in

Received by the editors: 22.09.2015
and in final form 24.08.2018.