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# Normal automorphisms of the metabelian product of free abelian Lie algebras

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Abstract. Let $M$ be the metabelian product of free abelian Lie algebras of finite rank. In this study we prove that every normal automorphism of $M$ is an IA-automorphism and acts identically on $M'$.

## 1. Introduction

Let $L$ be a Lie algebra over a field $K$. An automorphism $\varphi$ of $L$ is called a normal automorphism if $\varphi(I) = I$ for every ideal $I$ of $L$. The set of normal automorphisms of $L$ is a normal subgroup of the automorphism group of $L$.

Automorphisms and more particularly normal automorphisms have a very important place in groups and Lie algebras. Let $G$ be a soluble product of class $n \geqslant 2$ of nontrivial free abelian groups. In [5] it is shown that the subgroup of all normal automorphisms of $G$ coincides with the subgroup of all inner automophisms. In [4] Romankov showed that if $S$ is a free non-abelian soluble group, then the subgroup of normal automorphisms of $S$ is the subgroup of inner automorphisms of $S$. In [1] it is studied normal automorphisms of a free metabelian nilpotent group. Let $L_{m,c}$ be the free $m$-generated metabelian nilpotent of class $c$ Lie algebra over a field of characteristic zero. In [2] it is shown that the group of normal

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automorphisms of $L_{m,c}$ is contained by the group of IA-automorphisms of $L_{m,c}$ for $m \geqslant 3$, $c \geqslant 2$.

For an arbitrary variety of Lie algebras, the metabelian product of Lie algebras $F_i$, $i = 1, \dots, m$ is defined as

$$\left( \prod^{*} F_i \right) / \left( D \cap F'' \right),$$

where $F = \prod^* F_i$ is the free product of the Lie algebras $F_i$ and $D$ is the cartesian subalgebra of $\prod^* F_i$. If the algebras $F_i$ are non-trivial free abelian Lie algebras then the metabelian product of them is isomorphic to $F/F''$, where $F'' = [F', F']$ and $F' = [F, F]$ is the derived subalgebra.

Let $M$ be the metabelian product of free abelian Lie algebras of finite rank. In this study it is shown that every normal automorphism of $M$ is an IA-automorphism and acts identically on $M'$. In proving this result we inspired by the result of Timoshenko in the case of groups [5].

Let $L$ be a Lie algebra and $B$ any subset of $L$. We show that by $\langle B \rangle$ the ideal of $L$ generated by the set $B$.

## 2. Normal automorphisms of metabelian product

Let $A_i$, $i = 1, \dots, m$, be free abelian Lie algebras of finite rank over a field $K$ of characteristic zero and $F = \prod^* A_i$ is the free product of the abelian Lie algebras $A_i$, $i = 1, \dots, m$. If $M$ is the metabelian product of the algebras $A_i$, $M$ is isomorphic to $F/F''$.

**Definition 1.** Let L be a Lie algebra. An automorphism $\varphi$ of $L$ is called a normal automorphism If $\varphi(I) = I$ for evey ideal $I$ of $L$.

**Theorem 1.** *Let $A_i$, $i = 1, \dots, m$, be free abelian Lie algebras of finite rank and let $M$ be their metabelian product. If $\varphi$ is a normal automorphism of $M$ then $\varphi$ is an IA-automorphism.*

*Proof.* Let $\varphi$ be a normal automorphism of $M$. The algebra $M$ can be considered as $M = F/F''$. Let denote by $\widehat{v} = v + F''$, where $v \in F$. Then by [3] there exist $u_i \in F', 1 \leqslant i \leqslant m$, such that

$$\varphi(\widehat{a_i}) = \alpha \widehat{a_i} + \widehat{u_i},$$

where $a_i \in A_i$ and $0 \neq \alpha \in K$. Consider the ideal $\langle \widehat{a_1} \rangle$ of $M$. Since $\varphi$ is normal we have $\varphi(\widehat{a_1}) \in \langle \widehat{a_1} \rangle$ and so $\widetilde{u_1} \in \langle \widehat{a_1} \rangle$ and similarly, for the ideal

$\left\langle \widehat{[a_2, a_3]} \right\rangle$ of $M$ we have $\varphi\left(\widehat{[a_2, a_3]}\right) \in \left\langle \widehat{[a_2, a_3]} \right\rangle$. Then for an element $\widehat{y}$ of $\left\langle \widehat{[a_2, a_3]} \right\rangle$ we have

$$\varphi\left(\widehat{[a_2, a_3]}\right) = \alpha^2 \widehat{[a_2, a_3]} + \widehat{y}.$$

Now consider the ideal $\left\langle \widehat{a_1 + [a_2, a_3]} \right\rangle$ of $M$. Since $\varphi$ is normal we have $\varphi\left(\widehat{a_1 + [a_2, a_3]}\right) \in \left\langle \widehat{a_1 + [a_2, a_3]} \right\rangle$ and for an element $\widehat{z}$ of $\left\langle \widehat{a_1 + [a_2, a_3]} \right\rangle$

$$\varphi\left(\widehat{a_1 + [a_2, a_3]}\right) = c\left(\widehat{a_1 + [a_2, a_3]}\right) + \widehat{z}$$

where $c \in K$. From the last equality we have

$$(\alpha - c)\,\widehat{a_1} + \left(\alpha^2 - c\right)\widehat{[a_2, a_3]} = \widehat{0}.$$

Then we get $c = \alpha$ and $c = \alpha^2$, that is, $\alpha^2 = \alpha$. Hence $\alpha = 1$ and $\varphi$ is an IA-automorphism. □

**Theorem 2.** *Every normal automorphism of $M$ acts identically on $M'$.*

*Proof.* The algebra $M$ can be considered as $M = F/F''$. Let denote by $\widehat{v} = v + F''$, where $v \in F$. Let $\varphi$ be a normal automorphism of $M$. By theorem 1 we have that $\varphi$ is an IA-automorphism. Then there is an element $\widehat{v}$ of $M'$ such that

$$\varphi\left(\widehat{[a_1, a_2]}\right) = \widehat{[a_1, a_2]} + \widehat{v},$$

where $a_1 \in A_1, a_2 \in A_2$. Let $H$ be the ideal of $M'$ generated by the element $\widehat{[a_1, a_2]}$. It is clear that

$$H = \left\{ c\widehat{[a_1, a_2]} : c \in K \right\}.$$

Now suppose that $\widehat{v} \neq \widehat{0}$. Consider the homomorphism $\theta : M' \to M'/H$ which is defined $\theta\left(\widehat{u}\right) = \varphi\left(\widehat{u}\right) + H$ for every element $\widehat{u} \in M'$. Since $\varphi$ is a normal automorphism of $M$ it is clear that $\theta$ is an epimorphism. Let $\widehat{u} \in Ker\theta$. Consider the ideal $\langle \widehat{u} \rangle$ of $M$. Since $\varphi$ is normal we have $\varphi\left(\widehat{u}\right) \in \langle \widehat{u} \rangle$. Then we have $\varphi\left(\widehat{u}\right) = \beta\widehat{u} + \widehat{w}$, where $\widehat{w} \in \langle \widehat{u} \rangle, \beta \in K$. Since $\widehat{u} \in Ker\theta$, we have $\varphi\left(\widehat{u}\right) \in H$, that is,

$$\beta\widehat{u} + \widehat{w} \in \left\{ c\widehat{[a_1, a_2]} : c \in K \right\}.$$

Thus we have $\widehat{u} = d\widehat{[a_1, a_2]}$, where $d \in K$. Then we get

$$\varphi\left(\widehat{u}\right) = d\widehat{[a_1, a_2]} + d\widehat{v} \in H.$$

If $\widehat{v} \neq \widehat{0}$ we get $d = 0$ and $\widehat{u} = \widehat{0}$. Hence we obtain that $\theta$ is an isomorphism. Since $\varphi\left(M'\right) = M'$ and $\widehat{v} \in M'$ there exist an element $\widehat{g}$ of $M'$ such that $\varphi\left(\widehat{g}\right) = \widehat{v}$. By the definition of $\theta$ we have

$$\theta\left(\widehat{g}\right) = \widehat{v} + H.$$

We also have that

$$\theta\left(\widehat{[a_1, a_2]}\right) = \widehat{v} + H.$$

Since $\theta$ is an isomorphism we get

$$\widehat{g} = \widehat{[a_1, a_2]}.$$

Thus we have

$$\varphi\left(\widehat{[a_1, a_2]}\right) = \varphi\left(\widehat{g}\right) = \widehat{v}$$

and

$$\widehat{[a_1, a_2]} + \widehat{v} = \widehat{v}.$$

We obtaian that $\widehat{[a_1, a_2]} = \widehat{0}$. This is a contradiction. Thus we get $\widehat{v} = \widehat{0}$ and

$$\varphi\left(\widehat{[a_1, a_2]}\right) = \widehat{[a_1, a_2]}.$$

Similarly, we obtain that

$$\varphi\left(\widehat{[a_i, a_j]}\right) = \widehat{[a_i, a_j]},$$

where $a_i \in A_i, a_j \in A_j$, $1 \leqslant i < j \leqslant m$. Let $\widehat{u} \in M'$. Then $\widehat{u}$ is a linear combinations of some elemets of $M$ of the form

$$\widehat{[\ldots[[a_{j_1}, a_{j_2}], a_{j_3}], \ldots, a_{j_n}]},$$

where $a_{j_1}, a_{j_2}, \ldots, a_{j_n} \in \bigcup\limits_{i=1}^{m} A_i$, $n \geqslant 2$. we know that

$$\varphi\left(\widehat{[a_{j_1}, a_{j_2}]}\right) = \widehat{[a_{j_1}, a_{j_2}]}.$$

Since $\varphi$ is an IA-automorphism there exist some elements $u_{j_3}, \ldots, u_{j_n} \in F'$ such that

$$\varphi\left(\widehat{a_{j_k}}\right) = \widehat{a_{j_k}} + \widehat{u_{j_k}},\ \ k \geqslant 3.$$

Then

$$\begin{aligned}
&\varphi\big([\ldots[[a_{j_1},\widehat{a_{j_2}]\,,a_{j_3}}]\,,\ldots,a_{j_n}]\big)\\
&\quad=\Big[\ldots\big[\varphi\big(\widehat{[a_{j_1},a_{j_2}]}\big),\varphi\big(\widehat{a_{j_3}}\big)\big],\ldots,\varphi(\widehat{a_{j_n}})\Big]\\
&\quad=\Big[\ldots\big[\widehat{[a_{j_1},a_{j_2}]},\widehat{a_{j_3}}+\widehat{u_{j_3}}\big],\ldots,\widehat{a_{j_n}}+\widehat{u_{j_n}}\Big]\\
&\quad=\Big[\ldots[[a_{j_1},\widehat{a_{j_2}]\,,a_{j_3}}]\,,\ldots,a_{j_n}\Big].
\end{aligned}$$

Hence we get $\varphi(\widehat{u})=\widehat{u}$ for all $\widehat{u}\in M'$. Therefore $\varphi$ acts identically on $M'$. □

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